On bounded Q/QS-algebras

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Abstract. In this article we discuss not only the newly established properties of bounded Q/QS-algebras but also some of their substructures such as, for example, (incomplete) sub-algebras and ideals. Additionally, it was shown that on a (bounded) QS-algebra the natural congruence can be determined so that the corresponding quotient-algebra is a (bounded) BCH-algebra.

1. Introduction

Considering the properties of BCK-algebras in 1979, K. Iseki raised the question of the existence of non-commutative BCK-algebras that satisfy the so-called double negation condition ([**13**]). Such logical algebras, i.e. bounded logical algebras that, in addition, satisfy the double negation condition, are called involutive algebras. The study of various bounded (and involutive) algebras has been the focus of several researchers. So, for example, bounded BCK-algebras are studied in [**11, 12**] by K. Iséki. Bounded and involutive BE-algebras are studied in [**8**] by R. Borzooei et al. Bounded GE-algebras were discussed in [**7**] by R. K. Bandaru et al. The internal architecture of involutive WE-algebras was the focus of a paper [**24**] written by A. Walendziak. This author participated in the consideration of the properties of involutive WE-algebras by the article [**17**]. The boundedness property of logical algebras has been the focus of this author for a long time. This author introduced and analyzed the concepts of bounded and involutive BI/QI/BH-algebras ([**18, 19, 20**]). It seems that these aforementioned studies of bounded WE/BI/QI/BH-algebras can serve as a justification for our strong interest in studying the boundedness property in other algebras as well.

In 2001, Neggers et al. defined ([**16**]) a generalization of BCH/BCI/BCK-algebras as a new notion, called Q-algebra. (The definition of the concepts BCH / BCI / BCK-algebras can be found, for example, in [**10**].) Also, authors looked at the validity of some of the properties expressed about BCH/BCI/BCK-algebras now in a new environment. This class of logical algebras has been the subject of study by several researchers. The concept of QS-algebras, as a subclass of the class

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Q-algebras, was introduced in [**3**] by S. S. Ahn and H. S. Kim. The study of ideals ([**1**] by H. K. Abdullah and H. K. Jawad.) and filters ([**2**] by H. K. Abdullah and H. S. Salman) in Q-algebras, led to the concept of 'bounded Q-algebras'.

In this article, which is, in a literal sense, a continuation of the research started in [**1, 2**], we focus on examining the internal architecture of bounded Q-algebras as well as the properties of their substructures. Besides analyzing the properties of standard substructures in bounded Q-algebra, we introduce and analyze some new substructures in bounded Q-algebra such as, for example, 'incomplete sub-algebra'. Thus, it was shown that every ideal in a bounded Q-algebra is an incomplete sub-algebra and that the converse need not hold. Also, it is shown that in a (bounded) QS-algebra $\mathfrak{A}$ there exists a natural congruence relation $\equiv$ and that the corresponding quotient-algebra $\mathfrak{A}/\equiv$ is a (bounded) BCH-algebra.

2. Preliminaries

In this section, the necessary notions and notations and some of their interrelationships, mostly taken from paper [**10, 17**], are listed in the order to enable a reader to comfortably follow the presentation in this report. It should be pointed out here that the notations for logical conjunction, logical implication and others have a literal meaning. The notation =: in the formula $A =: B$ serves to indicate that A in it is the abbreviation for the formula B.

The concept of Q-algebra first appeared in 1999 in [**3**] based on the article [**16**] written by J. Neggers, S. S. Ahn and H. S. Kim, but which appeared three years later.

Definition 2.1. ([**16**], pp. 749) A Q-algebra is a non-empty set A with a constant 0 and a binary operation "$*$" satisfying axioms:

(Re) $(\forall x \in A)(x * x = 0)$,

(M) $(\forall x \in A)(x * 0 = x)$,

(Ex) $(\forall x, y, z \in A)((x * y) * z = (x * z) * y)$.

We denote this axiomatic system by $\mathbf{Q}$ and the corresponding algebra $\mathfrak{A} =: (A, *, 0)$ by Q-algebra.

Remark 2.1. The concept of Q-algebra, defined here, should not be confused with the term 'Q-algebra' described, for example, in the text [**23**] in the following sense: "A commutative Banach algebra A is called Q-algebra if it is isomorphic to a quotient algebra B/J where B is a uniform algebra and J is a closed ideal in B."

Remark 2.2. This class of logical algebras is also known as RME-algebra (see, for example, [**10**], Definition 4.6(6)).

Remark 2.3. The concept of CI-algebra was introduced in 2009 in [**14**], Definition 3.1, by B. L. Meng as dual Q-algebra. A CI-algebra is an algebra $\mathfrak{A} =: (A, *, 1)$ of type (2,0) satisfying the condition (Re) and the following axioms:

(M_L) $(\forall x \in A)(1 * x = x)$

(Ex_L) $(\forall x, y, z \in A)(x * (y * z) = y * (x * z))$.

For any Q-algebra $\mathfrak{A} =: (A, *, 0)$, the set $B(X) =: \{x \in A : 0 * x = 0\}$ is called the p-radical of $\mathfrak{A}$ (see, [**16**], pp. 752). If $B(A) = \{0\}$, then we say that $\mathfrak{A}$ is a p-semisimple Q-algebra. Also, the 'G-part' of a Q-algebra $\mathfrak{A} =: (A, *, 0)$ is determined as follows $G(A) =: \{x \in A : 0 * x = x\}$.

However, not every logical algebra has to be a Q-algebra (see, for example, the following example).

Example 2.1. Let $A = \{0, a, b, c\}$ a set and the operation $*$ given by the following table

$*_2$	0	a	b	c
0	0	b	a	0
a	a	0	0	0
b	b	0	0	0
c	c	c	c	0

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	0	0	0
c	c	c	c	0

Then $\mathfrak{A} =: (A, *, 0)$ is a Q-algebra ([**16**], Example 2.2) bit the structure $(A, *_2, 0)$ is not a Q-algebras because, for example, we have $(a *_2 b) *_2 c = 0 *_2 c = 0$ and $(a *_2 c) *_2 b = 0 *_2 b = a$. $\square$

S. S. Ahn and H. S. Kim introduced ([**3**], Definition 2.1) the notion of QS-algebras. A Q-algebra $\mathfrak{A} =: (A, *, 0)$ is said to be a QS-algebra if it satisfies the additional relation:

(QS) $(\forall x, y, z \in A)((x * y) * (x * z) = z * y)$.

Example 2.2. ([**15**], Example 5.2) Let $A = \{0, a, b, c\}$ a set and the operation $*$ given by the following table

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Then $\mathfrak{A} =: (A, *, 0)$ is a QS-algebras. $\square$

Example 2.3. ([**16**], Example 4.3) Let $\mathfrak{G} =: GF(p^n)$ be a Galois field. Define $x * y =: x - y + e$, where $e \in \mathfrak{G}$. Then $(\mathfrak{G}, *, e)$ is a quadratic Q-algebra. (For the definition of the concept 'the quadratic Q-algebra', see [**16**], Section 4.)

Let G be a field with $|G| \geqslant 3$. Then every quadratic Q-algebra on G is a (quadratic) QS-algebra ([**16**], Theorem 4.4). $\square$

The properties of this class of logical algebras are summarized in the following proposition.

Proposition 2.1. *Let $\mathfrak{A} =: (A, *, 0)$ be a Q-algebra. Then:*

(a) ([**16**], Lemma 3.1) $(\forall x, y, z \in A)(x * y = x * z \implies 0 * y = 0 * z)$.

(b) ([**5**], Lemma 2.4) $(\forall x, y \in A)(0 * (x * y) = (0 * x) * (0 * y))$.

Definition 2.2. ([**5**], Definition 2.1) Let $\mathfrak{A} =: (A, *, 0)$ be a Q-algebra. A nonempty subset S in A is called a sub-algebra in $\mathfrak{A}$ if the following holds:

(S1) $(\forall x \in A)((x \in S \wedge y \in S) \implies x * y \in S)$.

We denote the family of all sub-algebras in the Q-algebra $\mathfrak{A}$ by $\mathfrak{S}(A)$.

It can be shown without difficulty that every sub-algebra S in a Q-algebra $\mathfrak{A}$ satisfies the condition

(S0) $0 \in S$.

Indeed, since S is not empty, there exists at least some $x \in A$ such that $x \in S$. Now we have $0 = x * x \in S$ according to (S1) with respect to (Re).

Definition 2.3. ([**16**], Definition 3.6) Let $\mathfrak{A} =: (A, *, 0)$ be a Q-algebra. A nonempty subset J in A is called an ideal in $\mathfrak{A}$ if the following holds:

(J0) $0 \in J$.

(J1) $(\forall x, y \in A)((x * y \in J \wedge y \in J) \implies x \in J)$.

We denote the family of all ideals in the Q-algebra $\mathfrak{A}$ by $\mathfrak{J}(A)$.

Definition 2.4. ([**2**], Definition (2.2)) A Q-algebra $\mathfrak{A} =: (A, *, 0)$ is called a bounded Q-algebra if there exists an element $1 \in A$ which, additionally, satisfies the condition

(F) $(\forall x \in A)(x * 1 = 0)$.

The element $1 \in A$, which satisfies the condition (F), is called the unit in $\mathfrak{A}$. We denote the bounded Q-algebra by $(A, *, 0, 1)$.

Remark 2.4. The concept of bounded Q-algebras was introduced in 2018 in the paper [**1**] written by H. K. Abdullah and H. K. Jawad. However, this article is not available to the public. That's why we took the determination of this concept from the available article [**2**] written by H. K. Abdullah and H. S. Salman.

Example 2.4. Let $A = \{0, a, b, c\}$ a set and the operation $*$ given by the table as in Example 2.1 Then ([**2**], Example (2.1)) $\mathfrak{A} = (A, *, 0)$ is a bounded Q-algebra with the unit c. □

Remark 2.5. The unit in a bounded Q-algebra not be a unique as explain in the following example: Let $A = \{0, a, b\}$ a set and the operation $*$ given by the following table

$*$	0	a	b
0	0	0	0
a	a	0	0
b	b	0	0

Then $\mathfrak{A} = (A, *, 0)$ is a bounded Q-algebra with two the units a and b. It should be said here that this algebra is a bounded QS-algebra. □

Let $\mathfrak{A} =: (A, *, 0, 1)$ be a bounded Q-algebra. We will put $y^- =: 1 * y$ for arbitrary $y \in A$. It is clear that $0^- = 1*0 = 1$ according to (M), and $1^- = 1*1 = 0$ according to (Re).

3. The main results

3.1. A bit more about bounded Q-algebras. In this article, we will consider bounded Q-algebras that have only one the unit. The following lemma gives an important property of bounded Q-algebras:

LEMMA 3.1. *Let* $\mathfrak{A} =: (A, *, 0, 1)$ *be a bounded Q-algebra. Then*

(L) $(\forall y \in A)(0 * y = 0)$.

PROOF. If we put $z = 1$ in (Ex), we get $(x * y) * 1 = (x * 1) * y$ for arbitrary $x, y \in A$. From here it follows (L), according to (F) and with respect to (M). □

Additionally, for every bounded Q-algebra $\mathfrak{A} =: (A, *, 0, 1)$, we have $B(A) = A$ and $G(A) = \{0\}$.

As a consequence of the previous lemma, we have:

COROLLARY 3.1. *Let* $\mathfrak{A} =: (A, *, 0, 1)$ *be a bounded Q-algebra. Then*

(1) $(\forall x, y \in A)((x * y) * x = 0)$.

PROOF. According to (Ex), (Re) and (L), for arbitrary $x, y \in A$, we have $(x * y) * x = (x * x) * y = 0 * y = 0$. □

REMARK 3.1. Let us recall (see, for example, [**9**]) that the algebra $(A, *, 0)$ of type (2,0) is a BCH-algebra if it satisfies the conditions (Re), (Ex) and the following axiom

(An) $(\forall x, y \in A)((x * y = 0 \wedge y * x = p) \implies x = y)$.

In any BCH-algebra $\mathfrak{A}$, the condition (M) holds. Therefore, every BCH-algebra is a Q-algebra.

Let $\mathfrak{A} =: (A, *, 0, c)$ be a bounded Q-algebra as in Example 2.1, but it is not a BCH/BCI/BCK-algebra since, in the general case, it does not satisfy the condition (An).

The following two propositions give some important properties of bounded Q/QS-algebras.

PROPOSITION 3.1. *Let* $\mathfrak{A} =: (A, *, 0, 1)$ *be a bounded Q-algebra. Then*

$$(\forall y, z \in A)(y^- * z = z^- * y).$$

PROOF. This is a valid formula in every bounded Q-algebra $\mathfrak{A}$ since it can be obtained by putting $x = 1$ in (Ex). □

PROPOSITION 3.2. *Let* $\mathfrak{A} =: (A, *, 0, 1)$ *be a bounded QS-algebra. Then*

(a) $(\forall y, z \in A)(y^- * z^- = z * y)$.

(b) $(\forall x, y \in A)(x * y = y^-)$.

PROOF. (a): The validity of formula (a) is obtained from the presence of formula (QS) by putting $x = 1$.

(b): The validity of the formula (b) is obtained from the validity of the formula (QS) by setting $z = 1$ and taking into account (M): $x * y = (x * y) * 0 = (x * y) * (x * 1) = 1 * y = y^-$. □

In the following example we illustrate the appearance of a bounded Q-algebra, taking into account property (L).

EXAMPLE 3.1. ([**2**], Example (2.3)) Let $A = \{0, a, b, c, 1\}$ a set and the operation $*$ given by the following table

$*$	0	a	b	c	1
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	0
c	c	b	a	0	0
1	1	0	0	0	0

Then $\mathfrak{A} = (A, *, 0, 1)$ is a bounded Q-algebra with the units c and 1. □

However, not every Q-algebra has to be a bounded Q-algebra nor can every Q-algebra be extended to a bounded Q-algebra as the following example shows.

EXAMPLE 3.2. Let $A = \{0, a, b, c, d\}$ a set and the operation $*$ given by the following table

$*$	0	a	b	c	d
0	0	0	c	b	c
a	a	0	c	b	c
b	b	b	0	c	0
c	c	c	b	0	b
d	d	b	a	c	0

Then $\mathfrak{A} = (A, *, 0, 1)$ is a Q-algebra ([**6**], Example 3.3) which is not a bounded Q-algebra nor can it be extended to a bounded Q-algebra. Indeed, in order for a Q/QS-algebra to be extended to a bounded Q/QS-algebra, it must satisfy the condition (L), which, in the general case, is not present. □

In what follows, we deal with the creation of the direct product bounded Q-algebras. Let $\{(A_i, *_i, 0_i, 1_i) : i \in I\}$ be a family of bounded Q-algebras. If on the set

$$\prod_{i \in I} A_i =: \{f : I \longrightarrow \cup_{i \in I} A_i \mid (\forall i \in I)(f(i) \in A_i)\},$$

we define the operation $\odot$ as follows

$$(\forall f, g \in \prod_{i \in I} A_i)(\forall \in I)((f \odot g)(i) =: f(i) *_i g(i)),$$

we created the structure $(\prod_{i \in I} A_i, \odot, f_0, f_1)$, where f_0 and f_1 were chosen as follows

$$(\forall i \in I)(f_0(i) =: 0_i) \text{ and}$$
$$(\forall i \in I)(f_1(i) =: 1_i).$$

Before we start working with direct products of bounded Q-algebras, we say that the operation, determined in this way, is well-defined. If a priori we accept conditions that ensure the existence of non-empty direct product, we can prove the following theorem.

THEOREM 3.1. *The direct product of any family of bounded Q-algebras, determined as above, is a bounded Q-algebra.*

PROOF. According to [**6**], Proposition 5, structure $(\prod_{i\in I} A_i, \odot, f_0, f_1)$ is a Q-algebra since it satisfies all its axioms. It remains to show that this structure is a bounded Q-algebra. We have $(f \odot f_1)(i) = f(i) *_i f_1(i) = f(i) *_i 1_i = 0_i = f_0(i)$ by (F) in $(A_i, *_i, 0_i, 1_i)$. Hence, $f \odot f_1 = f_0$.

Therefore, the structure $(\prod_{i\in I} A_i, \odot, f_0, f_1)$ is a bounded Q-algebra with the unit f_1. □

The previous theorem is a necessary predecessor of Theorem 3.6.

EXAMPLE 3.3. Let $\mathfrak{A} =: (A, *, 0, 1)$ be a bounded Q-algebra as in the Example 3.1. Then the product $\mathfrak{A} \times \mathfrak{A} = (A \times A, \odot, (0, 0), (1, 1))$ is a bounded Q-algebra, where the operation $\odot$ is determined by

$$(\forall x, y, u, v \in A)((x, y) \odot (u, v) =: (x * u, y * v)).$$

□

3.2. Sub-algebras and ideals. The concept of sub-algebra in a bounced Q-algebra is introduced by the following way:

DEFINITION 3.1. Let $\mathfrak{A} =: (A, *, 0, 1)$ be a bounded Q-algebra. A nonempty subset of K in A is called a sub-algebra in $\mathfrak{A}$ if

(K1) $1 \in K$.

(S1) $(\forall x, y \in A)((x \in K \wedge y \in K) \implies x * y \in K)$.

We denote the family of all sub-algebras in the bounded Q-algebra $\mathfrak{A}$ by $\mathfrak{K}(A)$.

As can be seen from the previous definition, the concept of a sub-algebra in a bounded Q-algebra is somewhat different from the concept of a sub-algebra in Q-algebras in the general case. (Compare this definition with Definition 2.2.)

PROPOSITION 3.3. *If K is a sub-algebra in a bounded Q-algebra $\mathfrak{A} =: (A, *, 0, 1)$, then holds*

(S0) $0 \in K$.

PROOF. Since the sub-algebra K is not empty, there exists at least some $x \in A$ such that $x \in K$. For that $x \in K$, we have $0 = x * x \in K$ according to (S1) and with respect to (Re). □

EXAMPLE 3.4. Let $\mathfrak{A} =: (A, *, 0, 1)$ be a bounded Q-algebra as in the Example 3.1.

Subsets $K_0 =: \{0, 1\}$, $K_1 =: \{0, 1, a\}$, $K_2 =: \{0, 1, b\}$, $K_3 =: \{0, 1, c\}$, $K_4 =: \{0, 1, a, b\}$ are sub-algebras in $\mathfrak{A}$. The subset $K_5 =: \{0, 1, a, c\}$ is not a sub-algebra in $\mathfrak{A}$ because, for example, we have $c*a = b \notin K_5$. Also, the subset $K_6 =: \{0, 1, b, c\}$ is not a sub-algebra in $\mathfrak{A}$ because, for example, we have $c * b = a \notin K_6$. □

In the previous example, the following subsets $S_0 =: \{0\}$, $S_1 =: \{0, a\}$, $S_2 =: \{0, b\}$, $S_3 =: \{0, c\}$, $S_4 =: \{0, a, b\}$ and $S_7 =: \{0, a, b, c\}$ in A although satisfy the condition (S1), but they do not satisfy the condition (K0). This justifies the introduction of a new concept in bounded Q-algebras:

Definition 3.2. Let $\mathfrak{A} =: (A, *, 0, 1)$ be a bounded Q-algebras. A nonempty subset S of A that satisfies (S1) and the following condition

(S01) $1 \notin S$

is called an incomplete sub-algebra of $\mathfrak{A}$. We denote the family of all incomplete sub-algebras in $\mathfrak{A}$ by $\mathfrak{S}(A)$.

The family $\mathfrak{S}(A)$ is not empty since $S_0 = \{0\} \in \mathfrak{S}(A)$. However, $A \notin \mathfrak{S}(A)$ and $\mathfrak{S}(A) \cap \mathfrak{K}(A) = \emptyset$.

The concept of ideal in bounded Q-algebra is introduced by means of Definition 2.3. For an ideal J in a bounded Q-algebra $\mathfrak{A}$ we say that it is a nontrivial ideal in $\mathfrak{A}$ if holds $J \neq A$. We denote the family of all ideals in a bounded Q-algebra $\mathfrak{A} =: (A, *, 0, 1)$ by $\mathfrak{J}(A)$. Additionally, we write $\mathfrak{J}_p(A) =: \mathfrak{J}(A) \smallsetminus A$.

In any bounded Q-algebra $\mathfrak{A} =: (A, *, 0, 1)$ we define a binary relation $\preccurlyeq$ by $x \preccurlyeq y$ if and only if $x * y = 0$ for arbitrary elements $x, y \in A$.

Proposition 3.4. *Let J be an ideal in a bounded Q-algebra $\mathfrak{A} =: (A, *, 0, 1)$. Then*

(2) $(\forall x, y \in A)((x \preccurlyeq y \wedge y \in J) \Longrightarrow x \in J)$.

(3) $(\forall x, y \in A)(x \in J \Longrightarrow x * y \in J)$.

Proof. (2): Let $x, y \in A$ be such that $x \preccurlyeq y$ and $y \in J$. This means $x * y = 0 \in J$ and $y \in J$. Thus $x \in J$ by (J1).

(3): Let $x, y \in J$ be arbitrary elements. Then $x * y \preccurlyeq x$ by (1). Thus $x * y \in J$ according to (2). □

Proposition 3.5. *Let J be a nontrivial ideal in a bounded Q-algebra $\mathfrak{A} =: (A, *, 0, 1)$. Then*

(4) $1 \notin J$.

Proof. If it were $1 \in J$, we would have $x * 1 = 0 \in J$, by (F), from which it follows that $x \in J$ according to (J1) for arbitrary $x \in A$, which is impossible because J is not a trivial ideal in $\mathfrak{A}$. The resulting contradiction breaks the assumption $1 \in J$. □

Now, we have:

Theorem 3.2. *Every nontrivial ideal in a bounded Q-algebra $\mathfrak{A}$ is an incomplete sub-algebra in $\mathfrak{A}$ This means $\mathfrak{J}_p(A) \subseteq \mathfrak{S}(A)$.*

Proof. Let J be a nontrivial ideal in a bounded Q-algebra $\mathfrak{A} =: (A, *, 0, 1)$ and let $x, y \in A$ be sych that $x \in J$ and $y \in J$. Then $x * y \in J$ in accordance with (3). so, the ideal J is an incomplete sub-algebra in $\mathfrak{A}$ since $1 \notin J$ according to (4). □

EXAMPLE 3.5. Let $A = \{0, a, b, c, 1\}$ a set and the operation $*$ given by the following table

$*$	0	a	b	c	1
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	b	0
c	c	c	c	0	0
1	1	1	c	b	0

Then $\mathfrak{A} = (A, *, 0, 1)$ is a bounded Q-algebra with the unit 1 ([**2**], Example (2.5)).

Subsets $K_0 =: \{0, 1\}$, $K_1 =: \{0, 1, a\}$, $K_2 =: \{0, 1, b\}$, $K_4 =: \{0, 1, a, b\}$ and $K_6 =: \{0, 1, b, c\}$ are sub-algebras in $\mathfrak{A}$. Subsets $K_3 =: \{0, 1, c\}$ and $K_5 =: \{0, 1, a, c\}$ are not sub-algebras in $\mathfrak{A}$.

Subsets $S_0 = \{0\}$, $S_1 = \{0, a\}$, $S_2 = \{0, b\}$, $S_3 = \{0, c\}$, $S_4 = \{0, a, b\}$, $S_5 = \{0, a, c\}$ and $S_6 = \{0, b, c\}$ are incomplete sub-algebras in $\mathfrak{A}$.

Subsets $J_0 = \{0\}$, $J_1 = \{0, a\}$, $J_2 = \{0, b\}$ and $J_4 = \{0, a, b\}$ are ideals in $\mathfrak{A}$. Subset $J_3 = \{0, c\}$ is not an ideal in $\mathfrak{A}$ because, for example, we have $a * c = 0 \in J_3$ but $a \notin J_3$. Subsets $J_5 = \{0, a, c\}$ and $J_6 = \{0, b, c\}$ are not ideals in $\mathfrak{A}$ either. Indeed, for J_5 we have $1 * b = c \in J_5$ but $1 \notin J_5$. Similarly, for J_6, we have $a * c = 0 \in J_6$ but $a \notin J_6$. □

REMARK 3.2. An incomplete sub-algebra in a bounded Q-algebra $\mathfrak{A}$ does not have to be an ideal in $\mathfrak{A}$ as shown in the previous example: Incomplete sub-algebras S_3, S_5 and S_6 in $\mathfrak{A}$ are not ideals in $\mathfrak{A}$. So, $\mathfrak{J}_p(A) \subsetneqq \mathfrak{S}(A)$.

Since the family $\mathfrak{K}(A)/\mathfrak{S}(A)/\mathfrak{J}(A)$ is not empty, it can be proved:

THEOREM 3.3. *Let $\mathfrak{A} = (A, *, 0, 1)$ be a bounded Q-algebra. Then the family $\mathfrak{K}(A)/\mathfrak{S}(A)/\mathfrak{J}_p(A)/\mathfrak{J}(A)$ forms a complete lattice.*

PROOF. (a) Let $\{S_i\}_{i\in I}$ be a family of (incomplete) sub-algebras / ideals in a bounded Q-algebra $\mathfrak{A} = (A, *, 0, 1)$. Then $0 \in \cap_{i\in I} S_i$ since each of the aforementioned substructures contains the element 0.

(i) Let $\{S_i\}_{i\in I}$ be a family of (incomplete) sub-algebras in $\mathfrak{A}$ and let $x, y \in A$ be such that $x \in \cap_{i\in I} S_i$ and $y \in \cap_{i\in I} S_i$. Then $x \in S_i$ and $y \in S_i$, for each $i \in I$. Thus $x * y \in S_i$ since S_i is a (an incomplete) sub-algebra in $\mathfrak{A}$ for all $i \in I$. Hence $x * y \in \cap_{i\in I} S_i$. So, $\cap_{i\in I} S_i$ is a (an incomplete) sub-algebra in $\mathfrak{A}$.

(ii) Let Let $\{S_i\}_{i\in I}$ be a family of (non-trivial) ideals in $\mathfrak{A}$ and let $x, y \in A$ be such that $x * y \in \cap_{i\in I} S_i$ and $y \in \cap_{i\in I} S_i$. Then $x * y \in S_i$ and $y \in S_i$ for each $i \in I$. Thus $x \in S_i$ because S_i is an (non-trivial) ideal in $\mathfrak{A}$ for each $i \in I$. Hence $x \in \cap_{i\in I} S_i$. So, $\cap_{i\in I} S_i$ is (a non-trivial) an ideal in $\mathfrak{A}$.

(iii) Let $\{S_i\}_{i\in I}$ be a family of sub-algebras in $\mathfrak{A}$. Then $1 \in \cap_{i\in I} S_i$ since each of the aforementioned substructures contains the element 1. This family also satisfies the condition (S1) as shown in (i) of this proof. So, $\cap_{i\in I} S_i$ is a sub-algebra in $\mathfrak{A}$.

(b) Let $\mathcal{Z}$ be the family of all incomplete sub-algebras/(non-trivial) ideals/sub-algebras in $\mathfrak{A}$ that contain $\cup_{i\in I} S_i$. Then $\cap \mathcal{Z}$ is an incomplete sub-algebra / (a non-trivial) an ideal / a sub-algebra in $\mathfrak{A}$, respectively, according to the first part of the proof of this theorem.

(c) If we put $\sqcap_{I\in I}S_i = \cap_{i\in I}S_i$ and $\sqcup_{i\in I}S_i = \cap\mathcal{Z}$, then

$$(\mathfrak{S}(A), \sqcap, \sqcup),\ (\mathfrak{J}(A), \sqcap, \sqcup),\ (\mathfrak{J}_p(A), \sqcap, \sqcup) \text{ and } (\mathfrak{K}(A), \sqcap, \sqcup)$$

are complete lattices, respectively. □

Corollary 3.2. *Let $\mathfrak{A} =: (A, *, 0, 1)$ be a bounded Q-algebra and $x \in A$. Then there is a smallest incomplete sub-algebra / (non-trivial) ideal / sub-algebra S_x in $\mathfrak{A}$ that contains x.*

Proof. Let $\mathcal{Z}$ be the family of all incomplete sub-algebras / ideals / sub-algebras in $\mathfrak{A}$ that contain the element x. Then, by the previous theorem, $S_x =: \cap\mathcal{Z}$ is an incomplete sub-algebra / an ideal / a sub-algebra in $\mathfrak{A}$ that contains x.

Let Y be an incomplete sub-algebra / an ideal / a sub-algebra in $\mathfrak{A}$ which contains x. Then $Y \in \mathcal{Z}$, so, therefore, $S_x \subseteq Y$. Therefore, S_x is the smallest incomplete subalgebra/ideal/sub-algebra in $\mathfrak{A}$ containing x. □

The following theorem gives a criterion for recognizing ideals in bounded Q-algebras.

Theorem 3.4. *Let $\mathfrak{A} =: (A, *, 0, 1)$ be a bounded Q-algebra and let J be a subset in A that satisfies the condition (J0). Then J is an ideal in $\mathfrak{A}$ if and only if it holds*

(J2) $(\forall x, y, z \in A)(((x * y) * z \in J \wedge y \in J) \implies x * z \in J)$.

Proof. Let J be an ideal in $\mathfrak{A}$ and let $x, y, z \in A$ such that $(x * y) * z \in J$ and $y \in J$. Then $(x * z) * y = (x * y) * z \in J$ and $y \in J$ in accordance with (Ex). This $x * z \in J$ according to (J1) since J is an ideal in $\mathfrak{A}$. Therefore, the formula (J2) is valid.

Conversely, suppose that the subset J satisfies the condition (J2). If we put $z = 0$ in (J2), we get (J1) according to (M). □

Here it should be said that condition (J2), together with condition (J0), determine the concept of strong ideal in an algebra (see, for example [**21**], Definition 2.3). Thus, in a bounded Q-algebra $\mathfrak{A}$, every ideal in $\mathfrak{A}$ is a strong ideal in $\mathfrak{A}$.

On the other hand, we have the determination of the concept of weak ideal (see, for example, [**22**], Definition 3.1) in a bounded Q-algebra as follows:

Definition 3.3. Let $\mathfrak{A} =: (A, *, 0, 1)$ be a bounded Q-algebra. A nonempty subset J in A is called a weak ideal in $\mathfrak{A}$ if, in addition to the conditions (J0), it also satisfies the condition

(Jw) $(\forall x, y, z \in A)((x * (y * z) \in J \wedge y \in J) \implies x * z \in J)$.

We denote the family of all weak ideals in a bounded Q-algebra $\mathfrak{A}$ by $\mathfrak{J}_w(A)$.

Remark 3.3. This concept, the concept of weak ideals in Q-algebras, is sometimes called a 'T-ideal' (see, for example, [**5**], Definition 3.13).

Proposition 3.6. *Let $\mathfrak{A} =: (A, *, 0, 1)$ be a bounded Q-algebra and J a weak ideal in $\mathfrak{A}$. Then J also satisfies the condition (J0).*

Proof. Since J is a nonempty subset of A, there exists at least some $x \in A$ such that $x \in J$. Now, from $x = x * 0 = x * (x * x) \in J$ and $x \in J$, according to (Jw), it follows $x * x = 0 \in J$. □

Proposition 3.7. *Every weak ideal in a bounded Q-algebra $\mathfrak{A}$ is an ideal in $\mathfrak{A}$. This means $\mathfrak{J}_w \subseteq \mathfrak{J}(A)$.*

Proof. If we put $z = 0$ in (Jw), we get (J1) with respect to (M). □

However, we can prove that the reverse is also true.

Theorem 3.5. *Let $\mathfrak{A} =: (A, *, 0, 1)$ be a bounded Q-algebra. A nonempty subset J in A is a weak ideal in $\mathfrak{A}$ if and only if it is an ideal in $\mathfrak{A}$. This means $\mathfrak{J}_w(A) = \mathfrak{J}(A)$.*

Proof. If J is a weak ideal in $\mathfrak{A}$, then, according to Proposition 3.7, J is an ideal in $\mathfrak{A}$.

Let J be an ideal in A and let $x, y, z \in A$ be such that $x * (y * z) \in J$ and $y \in J$. Then $(x * (y * z)) * z \in J$ and $y * z \in J$ with respect to (3). Thus $(x * z) * (y * z) \in J$ and $y * z \in J$ with the use of (Ex). Hence $x * z \in J$ according (J1). □

Further on, we have:

Theorem 3.6. *Let $\{(A_i, *_i, 0_i) : i \in I\}$ be a family of (bounded) Q-algebras, K be a subset of I and let J_i be an ideal in $(A_i, *_i, 0_i)$ for each $i \in K$. Then $\prod_{i \in I} T_i$, where $T_i = J_i$ for $i \in K$ and $T_i = A_i$ for $i \in I \backslash K$, is an ideal in the (bounded) Q-algebra $\prod_{i \in I} A_i$.*

Proof. First, it is clear that $f_0 \in \prod_{i \in I} T_i$.

If $K = \emptyset$, then $\prod_{i \in I} T_i = \prod_{i \in I} A_i$, so $\prod_{i \in I} T_i$ is certainly an ideal in $\prod_{i \in I} A_i$. Assume, therefore, that $K \neq \emptyset$.

Let $x, y \in \prod_{i \in I} A_i$ be such that $x \odot y \in \prod_{i \in I} T_i$ and $y \in \prod_{i \in I} T_i$. This means $(x \odot y)(i) = x(i) *_i y(i) \in J_i$ and $y(i) \in J_i$ for each $i \in K$. Then $x(i) \in J_i$ since J_i is an ideal in $(A_i, *_i, 0_i)$ for each $i \in K$. Hence $x \in \prod_{i \in I} T_i$.

As shown, $\prod_{i \in I} T_i$ is an ideal in $\prod_{i \in I} A_i$. □

This theorem is an extension of Proposition 6 in [**6**].

Example 3.6. Let $\mathfrak{A} =: (A, *, 0, 1)$ be a bounded Q-algebra as in Example 3.5. Then $\mathfrak{A} \times \mathfrak{A} =: (A \times A, \odot, (0, 0), (1, 1))$ is a bounded Q-algebra according to Theorem 3.1, where the operation is $\odot$ determined as follows

$$(\forall x, y, u, v \in A)((x, y) \odot (u, v) =: (x * u, y * v)).$$

The subset $J_4 =: \{0, a, b\}$ is an ideal in $\mathfrak{A}$ as shown in Example 3.5. Now, according to the previous theorem, the subsets $J_4 \times A$, $J_4 \times J_4$ and $J_4 \times A$ of the set $A \times A$ are ideals in $\mathfrak{A} \times \mathfrak{A}$. □

3.3. (Left, right) congruence and corresponding substructures. Let $\mathfrak{A} =: (A, *, 0, 1)$ be a bounded Q-algebra. For an equivalence relation ρ on A we say that it is a left congruence on $\mathfrak{A}$ if holds

$$(\forall x, y, z \in A)((x, y) \in \rho \implies (z * x, z * y) \in \rho).$$

Right congruence can be defined analogously. For an equivalence of ρ on A, we say that a is a congruence on $\mathfrak{A}$ if it is both a left and a right congruence on $\mathfrak{A}$. We denote the family of all (left, right) congruences on $\mathfrak{A}$ by ($\mathfrak{Q}_l(A)$, $\mathfrak{Q}_r(A)$) $\mathfrak{Q}(A)$ respectively. Using analogous technology as in Theorem 3.3, it can be proven that the family $\mathfrak{Q}(A)$ ($\mathfrak{Q}_l(A)$, $\mathfrak{Q}_r(A)$, respectively) forms a complete lattice.

For the sake of consistency in the presentation of the material in this report, we state the previous result in the form of a lemma:

LEMMA 3.2. *Let $\mathfrak{A} =: (A, *, 0, 1)$ be a bounded Q-algebra and $\mathfrak{Q}_l(A)$ ($\mathfrak{Q}_r(A)$, $\mathfrak{Q}(A)$) be the family of all (left, right) congruence on $\mathfrak{A}$ respectively. Each of the aforementioned families forms complete lattice.*

PROOF. Let $\{\rho_i\}_{i\in I}$ be a family of (left, right) congruences on $\mathfrak{A}$.

(a) If we take arbitrary elements $x, y, z \in A$, then we have:

(i) Since every ρ_i for arbitrary $i \in I$ is a reflexive relation, we have $(x, x) \in \rho_i$ for each $i \in I$ and for every $x \in A$. So $(x, x) \in \cap_{i\in I}\rho_i$ for every $x \in A$. This shows that $\cap_{i\in I}\rho_i$ is a reflexive relation on $\mathfrak{A}$.

(ii) Let $(x, y) \in \cap_{i\in I}\rho_i$ for some $x, y \in A$. This means that $(x, y) \in \rho_i$ for every $i \in I$. Then $(y, x) \in \rho_i$ since ρ_i is a symmetric relation on A for every $i \in I$. Thus $(y, x) \in \cap_{i\in I}\rho_i$. This shows that $\cap_{i\in I}\rho_i$ is a symmetric relation on $\mathfrak{A}$.

(iii) Let $(x, y) \in \cap_{i\in I}\rho_i$ and $(y, z) \in \cap_{i\in I}\rho_i$ for some $x, y, z \in A$. This means that $(x, y) \in \rho_i$ and $(y, z) \in \rho_i$ for every $i \in I$. Then $(x, z) \in \rho_i$ for every $i \in I$ since ρ_i is a transitive relation for every $i \in I$. Thus $(x, z) \in \cap_{i\in I}\rho_i$. This shows that $\cap_{i\in I}\rho_i$ is a transitive relation on A.

(iv) Let $(x, y) \in \cap_{i\in I}\rho_i$ for some $x, y \in A$. This means $(x, y) \in \rho_i$ for every $i \in I$. Then $(z * x, z * y) \in \rho_i$ ($(x * z, y * z) \in \rho_i$) for every $i \in I$ since ρ_i is compatible with the left side (res. with the right side) with the operation in $\mathfrak{A}$. Thus $(z * x, z * y) \in \cap_{i\in I}\rho_i$ (res. $(x * z, y * z) \in \cap_{i\in I}\rho_i$).

(b) Let $\mathcal{Z}$ be the family of all (left, right) congruences on $\mathfrak{A}$ containing $\cup_{i\in I}\rho_i$. Then, according to the first part of this proof, $\cap\mathcal{Z}$ is a (left, right) congruence on $\mathfrak{A}$ containing $\cup_{i\in I}\rho_i$.

(c) If we put $\sqcap_{I\in I}\rho_i = \cap_{i\in I}\rho_i$ and $\sqcup_{i\in I}\rho_i = \cap\mathcal{Z}$, then

$$(\mathfrak{Q}_l(A), \sqcap, \sqcup),\ (\mathfrak{Q}_r(A), \sqcap, \sqcup) \text{ and } (\mathfrak{Q}(A), \sqcap, \sqcup)$$

are complete lattices, respectively. □

In the next three propositions we will consider the kernel $[0] =: \{x \in A : (x, 0) \in \rho\}$ of the (left, right) congruence ρ on a bounded Q-algebra $\mathfrak{A} = (A, *, 0, 1)$. The knowledge we gain in this process allows us to establish correspondences between the families $\mathfrak{Q}_l(A)$, $\mathfrak{Q}_r(A)$, $\mathfrak{Q}(A)$ with the corresponding substructures in that algebra.

PROPOSITION 3.8. *Let ρ be a left congruence on a bounded Q-algebra $\mathfrak{A} =: (A, *, 0, 1)$. Then*

(5) *The class $[0] =: \{x \in A : (x, 0) \in \rho\}$ is an ideal in $\mathfrak{A}$.*

(6) $(\forall x, y \in A)((x \in [0] \wedge y \in [0]) \implies (y * (y * x) \in [0] \wedge x * (x * y) \in [0]))$.

PROOF. (5): Let $x, y \in A$ be such that $x * y \in [0]$ and $y \in [0]$. This means $(x*y, 0) \in \rho$ and $(y, 0) \in \rho$. Then $(x*y, 0) \in \rho$ and $(x*y, x*0) \in \rho$ since ρ is a left congruence on $\mathfrak{A}$. Thus $(x * y, 0) \in \rho$ and $(x * y, x) \in \rho$ according to (M). Hence $(x, 0) \in \rho$ by transitivity of ρ. So, $x \in [0]$. Therefore, $[0]$ is an ideal in $\mathfrak{A}$.

(6): Let $x, y \in A$ be such that $x \in [0]$ and $y \in [0]$. This means $(x, 0) \in \rho$ and $(y, 0) \in \rho$. Then $(y * x, y * 0) = (y * x, y) \in \rho$ and $(y, 0) \in \rho$ since ρ is a left congruence on $\mathfrak{A}$. Thus $(y*x, 0) \in \rho$ by transitivity of ρ. Hence $(y*(y*x), y*0) = (y*(y*x), y) \in \rho$ and $(y, 0) \in \rho$. Finally, we get $(y*(y*x), 0) \in \rho$. So, $y*(y*x) \in [0]$. The second part of the implication (6) can be obtained analogously. □

REMARK 3.4. The previous proposition illustrates the correspondence

$$\mathfrak{Q}_l(A) \longrightarrow \mathfrak{J}_p(A).$$

The previous proposition is a justification for introducing the following concept:

DEFINITION 3.4. Let $\mathfrak{A} =: (A, *, 0, 1)$ be a bounded Q-algebra. For a subset H of A we say that it is of class $\mathcal{H}$ in $\mathfrak{A}$ if it holds

(H1) $(\forall x, y \in A)((x \in H \wedge y \in H) \implies (y * (y * x) \in H \wedge x * (x * y) \in H))$.

Let $\mathfrak{A} =: (A, *, 0, 1)$ be a bounded Q-algebra. Consider the family $\mathcal{H}(A)$ of subsets of the set A that satisfy the condition (H1). First, this family is not empty, because every singleton $\{x\}$ for arbitrary $x \in A$, is a member of this family since $x * (x * x) = x * 0 = x$. Therefore, $\{x\} \in \mathcal{H}(A)$. Also, all subsets of the set A of the form $\{0, x\}$ are members of the family $\mathcal{H}(A)$ since $0 * (0 * x) = 0$ and $x * (x * 0) = x$. Therefore, there are some members of the family $\mathcal{H}(A)$ in a bounded Q-algebra $\mathfrak{A}$ that do not have to be (incomplete) sub-algebras in $\mathfrak{A}$.

Furthermore:

PROPOSITION 3.9. *Let $\mathfrak{A} =: (A, *, 0, 1)$ be a bounded Q-algebra. All incomplete sub-algebras and all sub-algebras in $\mathfrak{A}$ are members of the family $\mathcal{H}(A)$.*

However, not all subsets of the set A in a bounded Q-algebra $\mathfrak{A} =: (A, *, 0, 1)$ need to be members of the family $\mathcal{H}(A)$. For illustration:

EXAMPLE 3.7. In Example 3.5, for a subset $T =: \{a, c\}$ we have $a * (a * c) = a * 0 = a$ but $c * (c * a) = c * c = 0 \notin T$. Therefore, $T \notin \mathcal{H}(A)$. □

REMARK 3.5. The Proposition 3.8 also illustrates the correspondence

$$\mathfrak{Q}_l(A) \longrightarrow \mathcal{H}(A).$$

Proposition 3.10. *Let ρ be a right congruence on a bounded Q-algebra $\mathfrak{A} =: (A, *, 0, 1)$. Then*

(7) $(\forall x, y \in A)((x \in [0] \wedge y \in [0]) \implies ((x * y) * y \in [0] \wedge (y * x) * x \in [0]))$.

Proof. Let $x, y \in A$ be such that $x \in [0]$ and $y \in [0]$. This means $(x, 0) \in \rho$ and $(y, 0) \in \rho$. Then $(x * y, 0 * y) = (x * y, 0) \in \rho$ since ρ is a right congruence on $\mathfrak{A}$ and with respect to (L). Further on, for the same reasons we have again $((x * y) * y, 0 * y) = ((x * y) * y, 0) \in \rho$. Hence, $(x * y) * y \in [0]$. The second part of the implication can be obtained analogously. □

The previous proposition is a justification for introducing the following concept:

Definition 3.5. Let $\mathfrak{A} =: (A, *, 0, 1)$ be a bounded Q-algebra. For a subset G of A we say that it is of class $\mathcal{G}$ in $\mathfrak{A}$ if it holds

(G1) $(\forall x, y \in A)((x \in G \wedge y \in G) \implies ((y * x) * x \in G \wedge (x * y) * y \in G))$.

Let $\mathfrak{A} =: (A, *, 0, 1)$ be a bounded Q-algebra. Consider the family $\mathcal{G}(A)$ of subsets of the set A that satisfy the condition (G1). This family is not empty because $\{0\} \in \mathcal{G}(A)$. However, no singleton belongs to this family because, for arbitrary $x \in A$, we have $(x * x) * x = 0 * x = 0$ according to (L). So, $\{x\} \notin \mathcal{G}(A)$. On the other hand, every subset of the form $\{0, x\}$, for arbitrary $x \in A$ belongs to the family $\mathcal{G}(A)$ because we have $(x * 0) * 0 = 0 * 0 = 0$ and $(0 * x) * x = 0 * x = 0$ according to (L). Therefore, $\{0, x\} \in \mathcal{G}(A)$.

Proposition 3.11. *Let $\mathfrak{A} =: (A, *, 0, 1)$ be a bounded Q-algebra. All incomplete sub-algebras and all sub-algebras in $\mathfrak{A}$ are members of the family $\mathcal{G}(A)$.*

Further on, we have:

Example 3.8. In Example 3.1, the subset $T =: \{a, c\}$ does not belong to the family $\mathcal{G}(A)$ because we have $(a * c) * c = 0 * c = 0$ but $(c * a) * a = b * a = b \notin T$. Therefore, $T \notin \mathcal{G}(A)$. □

Remark 3.6. The Proposition 3.10 illustrates the correspondence

$$\mathfrak{Q}_r(A) \longrightarrow \mathcal{G}(A).$$

Proposition 3.12. *Let $\mathfrak{A} =: (A, *, 0, 1)$ be a bounded Q-algebra. If ρ is a congruence on $\mathfrak{A}$, then:*

(8) $(\forall x, y, z \in A)((x * (y * z) \in [0] \wedge y \in [0]) \implies x * z \in [0])$.

Proof. Let $x, y, z \in A$ be such that $(x * y) * z \in [0]$ and $y \in [0]$. This means $(x * (y * z), 0) \in \rho$ and $(y, 0) \in \rho$. Then $(y * z, 0 * z) = (y * z, 0) \in \rho$ since ρ is a right congruence on $\mathfrak{A}$ and with respect to (L). Firther on, we have $(x * (y * z), x * 0) = (x * (y * z), x) \in \rho$ since ρ is a left congruence on A with respect to (M). From here, due to the transitivity of the relation ρ, we get $(x, 0) \in \rho$. Thus, $x \in [0]$. Finally, according to (3), we get $x * z \in [0]$ since $[0]$ is an ideal in $\mathfrak{A}$ by (5). □

Summarizing the previous facts, we conclude that there is a correspondence

$$\mathfrak{Q}(A) \longrightarrow \mathfrak{I}(A) \cap \mathcal{H}(A) \cap \mathcal{G}(A).$$

EXAMPLE 3.9. Let $\mathfrak{A} =: (A, *_A, 0_A, 1_A)$ and $\mathfrak{B} =: (B, *_B, 0_B, 1_A)$ be two Q-algebras. A mapping $f : A \longrightarrow B$ is called a homomorphism between Q-algebras if the following holds:

(f1) $f(1_A) = 1_B$.

(f2) $(\forall x, y \in A)(f(x *_A y) = f(x) *_B f(y))$.

It is easy to see that it is valid

$f(0_A) = 0_B$.

We denote this homomorphism by $f : \mathfrak{A} \longrightarrow \mathfrak{B}$. For any $x, y \in A$, we define $(x, y) \in \rho_f$ if and only if $f(x) = f(y)$. Then, according to [**4**], Lemma 2.6, the relation ρ_f is a congruence on $\mathfrak{A}$. □

In what follows we need the following lemma.

LEMMA 3.3 ([**3**], Proposition 2.5). *Let $\mathfrak{A} =: (A, *, 0, 1)$ be a bounded QS-algebra. Then*

(9) $(\forall x, y, z \in A)((x \preccurlyeq y \wedge y \preccurlyeq z) \Longrightarrow x \preccurlyeq z)$.

(10) $(\forall x, y, z \in A)(x \preccurlyeq y \Longrightarrow z * y \preccurlyeq z * x)$

(11) $(\forall x, y, z \in A)(x \preccurlyeq y \Longrightarrow x * z \preccurlyeq y * z)$.

As can be concluded from the previous lemma, the relation $\preccurlyeq$ determined in this way, is a quasi-order on the set A (a reflexive and transitive relation) right compatible and left reverse compatible with the operation in $\mathfrak{A}$.

THEOREM 3.7. *Let us define the relation $\equiv$ on the (bounded) QS-algebra $\mathfrak{A} =: (A, *, 0, 1)$ as follows*

$$(\forall x, y \in A)(x \equiv y \Longleftrightarrow (x \preccurlyeq y \wedge y \preccurlyeq x)).$$

Then $\equiv$ is a congruence on $\mathfrak{A}$.

PROOF. (i) Let $x \in A$ be an arbitrary element. It is clear that $x \equiv x$ holds due to the reflexivity of the relation $\preccurlyeq$.

Let $x, y, z \in A$ be such that $x \equiv y$ and $y \equiv z$. This means $x \preccurlyeq y$, $y \preccurlyeq x$, $y \preccurlyeq z$ and $z \preccurlyeq y$. Then $x \preccurlyeq z$ and $z \preccurlyeq x$ in accordance with (9). Hence, $x \equiv z$ which proves that $\equiv$ is a transitive relation on A.

Since the symmetry of the relation $\equiv$ is obvious, we conclude that $\equiv$ is an equivalence relation on A.

(ii) Let $x, y \in A$ be such that $x \equiv y$. This means $x \preccurlyeq y$ and $y \preccurlyeq x$. Then $x * z \preccurlyeq y * z$ and $y * z \preccurlyeq x * z$ according to (11). So, $x * z \equiv y * z$. Therefore, the relation $\equiv$ is right compatible with the operation in $\mathfrak{A}$.

(iii) That the relation $\equiv$ is left compatible with the operation in A can be proved analogously. □

Denote $\mathfrak{A}/\equiv =: \{[x] : x \in A\}$, where $[x] =: \{y \in A : x \equiv y\}$, and define that

$$(\forall x, y \in A)([x] \star [y] =: [x * y]).$$

Since $\equiv$ is a congruence relation on $\mathfrak{A}$, the operation $\star$ is well-defined. The structure $(A/\equiv, \star, [0])$ is a Q-algebra, as shown in [**4**], Theorem 2.7.

We show here the relation between Q-algebras and BCH-algebras. The following lemma can be easily proved.

LEMMA 3.4 ([**16**], Theorem 2.4). *Every BCH-algebra $\mathfrak{A}$ is a Q-algebra. Every Q-algebra $\mathfrak{A}$ satisfying the condition (An) is a BCH-algebra.*

Summarizing the above facts we have the following theorem.

THEOREM 3.8. *Let $\mathfrak{A} =: (A, *, 0, 1)$ be a (bounded) QS-algebra. Then the Q-algebra $\mathfrak{A}/\equiv =: (A/\equiv, \star, [0], [1])$ is a (bounded) BCH-algebra.*

PROOF. To prove the theorem, it remains to check that the formulas (QS), (F) and (An) are valid in $\mathfrak{A}/\equiv$.

Let $x, y, z \in A$ be arbitrary elements. We have:

$([x] \star [y]) \star ([x] \star [z]) = [x * y] \star [x * z] = [(x * y) * (x * z)] = [z * y] = [z] \star [y]$.

$[x] \star [1] = [x * 1] = [0]$.

Let us assume that $[x] \star [y] = [0]$ and $[y] \star [x] = [0]$. Applying (QS) to $[y]$, $[0]$ and $[x]$, we have

$$[y] = [y] \star [0] = ([x] \star [0]) \star ([x] \star [y]) = [x] \star [0] = [x].$$

This shows that the (bounded) Q-algebra $\mathfrak{A}/\equiv$ is a (bounded) BCH-algebra. □

4. Final comments

The algebraic structure, known as 'Q-algebra', as a generalization of BCH/BCI/BCK-algebras, introduced in 2001 by J. Neggers, S. S. Ahn and H. S. Kim. This class of logical algebras has been the subject of study by several researchers. The determination of bounded Q-algebras is discussed in [**1, 2**] by H. K. Abdullah et al.

In this paper we also consider properties of bounded Q/QS-algebras. We relate substructures of (incomplete) subalgebras and substructures of ideals in bounded Q-algebras. Furthermore, we consider (left, right) congruence on bounded Q/QS-algebras and relate them to corresponding substructures in such algebras. Finally, it is shown that in a (bounded) QS-algebra $\mathfrak{A}$ one can determine a congruence relation $\equiv$. It is proved that the corresponding quotient-algebra $\mathfrak{A}/\equiv$ is a BCH-algebra.

The author is convinced that this contribution to the consideration of bounded Q/QS-algebras opens at least one door for a broader and deeper study of (bounded) Q/QS-algebras and their substructures. So, for example, among other things, one could consider in more detail the theories of ideals and filters in (bounded) Q/QS-algebras.

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References

[1] H. K. Abdullah and H. K. Jawad. New types of ideals in Q-algebra, *Journal university of Kerbala*, 16(4)(2018), 9–21.
[2] H. K. Abdullah and H. S. Salman. New types of filters in Q-algebra. *International Journal of Engineering and Information Systems*, 3(11)(2019), 15–22.
[3] S. S. Ahn and H. S. Kim. On QS-algebras, *Journal of the Chungcheong Mathematical Society*, 12 (1999), 33-41.
[4] S. S. Ahn, H. S. Kim and H. D. Lee. R-maps and L-maps in Q-algebras. *International Journal of Pure and Applied Mathematics*, 12(4)(2004), 419–425.
[5] S. S. Ahn and S. E. Kang. The role of $T(X)$ in the ideal theory of Q-algebras. *Honam Mathematical Journal*, 32(3)(2010), 515–523.
[6] A. Anantayasethi, K. Sangsura and N. Sarasit. Direct product of Q-algebras. *European Journal of Pure and Applied Mathematics*, 18(4)(2025), Article Number 6792. DOI: https://doi.org/10.29020/nybg.ejpam.v18i4.6792
[7] R. K. Bandaru, M. A. Öztürk and Y. B. Jun. Bordered GE-algebras. *Journal of Algebraic Systems*, 12(1)(2024), 43–58. DOI:10.22044/JAS.2022.11184.1558
[8] R. Borzooei, A. Borumand Saeid, R. Ameri and A. Rezaei. Involutory BE-algebras, *Journal of Mathematics and Applications*, 37 (2014), 13–26.
[9] Q. P. Hu and X. Li. On BCH-Algebras. *Math. Seminar Notes*, 11(2)(1983), 313–320.
[10] A. Iorgulescu. New generalization of BCI, BCK and Hilbert algebras - Part I. *Journal of the Multi-valued Loguc Soft Computing*, 27(4)(2016), 353–406.
[11] K. Iséki. On bounded BCK-algebras, *Mathematics Seminar Notes* 3(1)(1975), 23-36.
[12] K. Iséki. On bounded BCK-algebras, II, *Mathematics Seminar Notes* 3(1)(1975), 34-36.
[13] K. Iséki. On some problems on BCK-algebras, *Mathematics Seminar Notes*, 7(1979), 173–178.
[14] B. L. Meng. CI-algebras. *Scientiae Mathematicae Japonicae Online*, e-2009, 695-701.
[15] S. M. Mostafa, R. A. K. Omar and M. A. Hassan. Derivations on QS-algebras. *Journal of Semigroup Theory and Applications*, Vol. 2015, 2015:3 (pp. 1–17).
[16] J. Neggers, S. S. Ahn and H. S. Kim. On Q-algebras. *International Journal of Mathematics and Mathematical Sciences*, 27(12)(2001), 749–757.
[17] D. A. Romano. Internal architecture of involutive WE-algebras. *Asian-European Journal of Mathematics*, https://doi.org/10.1142/S179355712550127X
[18] D. A. Romano. Bounded and involutive BI-algebras. *International Journal of Advanced Mathematical Sciences*, (In press) Available on Researchgate
[19] D. A. Romano. Bounded and involutive QI-algebras. *Asian-European Journal of Mathematics*, https://doi.org/10.1142/S1793557125501554
[20] D. A. Romano. On bounded BH-algebras. *Asian-European Journal of Mathematics*, https://doi.org/10.1142/S1793557125501505
[21] D. A. Romano. Strong and weak ideals in BI-algebras. Creative Mathematics and Informatics, 35(1)(2026), (In press) Available on Researchgate.
[22] D. A. Romano. Weak ideals in QI-algebras. Journal of Advanced Mathematical Studie, 19(1)(2026), (In press) Available on Researchgate
[23] N. Th. Varopoulus. Some remarks on Q-algebras. *Annales de L'institut Fourier*, 22(4)(1972), 1–11.
[24] A. Walendziak, On involutive weak exchange algebras, *Bulletin of the Section of Logic*, 54(3)(2025), 383–406. DOI: https://doi.org/10.18778/0138-0680.2025.13

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