Characterization of closed balls via metric projections

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Abstract. Consider the following property (P) for a bounded closed convex set C in a Banach space X. (P): For every $x \in X$, a positive-scalar multiple of x gives a nearest point in C to x. Then it is clear that a closed ball with its center at the origin has this property. The converse of this assertion is the subject of this paper, and it is proved that a bounded closed convex set $C \subset X$ with $0 \in \operatorname{Int} C$ possessing property (P) is a closed ball with center 0, provided $\dim X > 1$. The proof is achieved by reducing the general case to that of 2-dimensional spaces.

1 Introduction Let X denote a real Banach space with norm $\|\cdot\|$ and let A be a subset of X. Then, $x_0 \in A$ is called a nearest point in A to $x \in X$ if $\|x-x_0\| = \min\{\, \|x-y\| \mid y \in A \,\}$ holds. In this sense, for a closed ball C in X with center 0 (the origin) and radius r, it is clear that a nearest point Px in C to $x \in X$ is given by the following formula:

$$Px = \begin{cases} \dfrac{r}{\|x\|} \cdot x & (x \notin C), \\ x & (x \in C). \end{cases}$$

In other words, Px is a positive-scalar multiple of x for every $x \in X$.

The authors happened to wonder if the converse of this fact holds or not. That is to say, suppose that $C \subset X$ is a bounded closed convex set with 0 in its interior, and also suppose that for every $x \in X$ a positive-scalar multiple of x gives a nearest point in C to x, then should C be a closed ball centered at the origin or not?

It is clear that this question is answered in the negative for the extreme case of 1-dimensional spaces: Consider $C = [-1, 2]$ in $\mathbb{R}$ with the usual norm. However, the present authors could firstly answer affirmatively for the case of Hilbert spaces with dimension greater than 1. Their proof depends heavily on the neat geometric property of Hilbert spaces. Namely, for every non-empty closed convex set C in a Hilbert space X, there exists so-called metric projection $P_C \colon X \to C$ that maps $x \in X$ to the unique nearest point in C to x, and the proof utilized the characterization of P_C by inner products, and also the contractivity of P_C.

As an extension of the case of Hilbert spaces, the authors found that the question is affirmatively answered for Lebesgue's L^p spaces with $p \in [2, 4)$. This result is based on the following theorem in Li–Wang–Yang [5] saying that in case of p-uniformly convex and simultaneously q-uniformly smooth Banach spaces, metric projections onto a closed convex set is locally Hölder continuous of order q/p.

Moreover, no example of a Banach space with dimension greater than 1 was found which answers the question in the negative. So, the authors formulated the question into the following conjecture and began to investigate its validity in earnest:

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Conjecture: For every real Banach space X with $\dim X > 1$ the following assertion (A) holds:

(A) If $C \subset X$ is a bounded closed convex set with 0 in its interior, and also suppose that for every $x \in X$ a positive-scalar multiple of x gives a nearest point in C to x, then C should be a closed ball with center 0.

Note that the uniqueness of nearest points is not assumed in the conjecture.

As a result of investigation, the authors have managed to prove that the conjecture is right (Theorem 15). However, in the real process of investigation, they first found that the conjecture is right when it is restricted to the category of *smooth* Banach spaces (Theorem 4).

In Section 2, we give a proof of Theorem 4 since this proof contains the essence of that for general case (Theorem 15) and easier to understand. Moreover, it seems interesting that Theorem 4 is proved by making the best use of undergraduate calculus. In Section 3, the proof of the full conjecture is given, with a detailed description of facts necessary for the line of the proof in Section 2 to work.

Throughout this paper, conventional notations concerning general topology are freely used. For example, $\mathrm{Int}\, A$ denotes the interior of A, ∂A the boundary of A and $\overline{A}$ the closure of A.

2 Preliminaries and the result for smooth Banach spaces

The following Proposition shows that 2-dimensional case is essential for our problem.

Proposition 1. *Let X be a real Banach space with $\dim X > 1$. Then the following assertions hold.*

(1) *If $C \subset X$ is a bounded closed convex set with $0 \in \mathrm{Int}\, C$, then for every 2-dimensional subspace E of X, $C \cap E$ is a bounded closed convex set with $0 \in \mathrm{Int}\,(C \cap E)$ in E. Moreover, provided that a nearest point in C to $x \in X$ is always given by a positive-scalar multiple of x, a nearest point in $C \cap E$ to $x \in E$ is given by a positive-scalar multiple of x.*

(2) *Assertion (A) holds for X if (A) holds for every 2-dimensional subspace of X.*

Proof. Since assertion (1) is trivial, we only give a proof of (2). So, suppose that $C \subset X$ satisfies the assumption in assertion (A). Then, for every 2-dimensional subspace E of X, $C \cap E$ satisfies the assumption in (A) as a subset of E (*cf.* assertion (1) in the present proposition). Therefore if (A) holds for every 2-dimensional subspace of X, $C \cap E$ is a ball in E with center 0. So, take a fixed element $e_0 \in \partial C$, and choose an arbitrary $e_1 \in \partial C$ that is linearly independent of e_0. Then, for the 2-dimensional subspace E generated by $\{e_0, e_1\}$, $e_0, e_1 \in \partial(C \cap E)$ holds and hence $\|e_1\| = \|e_0\|$. Moreover, if $e_1 \in \partial C$ is linearly dependent of e_0, by taking another $e_2 \in \partial C$ that is linearly independent of e_0 and setting E the linear span of $\{e_0, e_2\}$, it is shown that $\|e_0\| = \|e_1\| = \|e_2\|$ since $e_0, e_1, e_2 \in \partial(C \cap E)$. Thus the norm of every element of ∂C is equal, which shows that C is a ball centered at the origin. $\square$

Here let us record the following immediate

Corollary 2. *The conjecture above holds if its restriction to 2-dimensional Banach spaces holds.*

Next we recall the definition of the smoothness of Banach spaces (see e.g. [1]). A Banach space $(X, \|\cdot\|)$ is said to be smooth if there exists a unique bounded linear functional $f_x \in X^*$ such that

$$f_x(x) = \|x\| \text{ and } \|f_x\| = 1$$

for each $x \in X \setminus \{0\}$. It is known that X is smooth if and only if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1}$$

exists for every $x \in X \setminus \{0\}$ and $y \in X$. Moreover it is also known that

$$f_x(y) = \lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2}$$

holds for every $x \in X \setminus \{0\}$ and $y \in X$, and the mapping $x \longmapsto f_x$ from $X \setminus \{0\}$ to X^* is norm to weak* continuous. For these facts, see [1, pp. 20–23].

Although the following lemma is an easy consequence of the definition of smoothness itself, characterization by the existence of the limit in (1) makes the proof completely trivial.

Lemma 3. *Let X denote a real Banach space with* $\dim X > 1$. *Then X is smooth if and only if every 2-dimensional subspace E of X is smooth.*

Now we prove the following

Theorem 4. *For every smooth Banach space X with* $\dim X > 1$, *assertion (A) in our Conjecture holds.*

Proof. **Step 1.** (Preliminaries)

Because of Proposition 1 and Lemma 3, it suffices to show that assertion (A) holds for 2-dimensional smooth Banach spaces. So, hereafter in this proof we assume that $(X, \|\cdot\|)$ is a 2-dimensional smooth Banach space and $C \subset X$ is a bounded closed convex set containing the origin in its interior, and a positive-scalar multiple of $x \in X$ gives a nearest point in C to x.

Now let $e_1, e_2 \in X$ be linearly independent vectors and define

$$p(s, t) = \|se_1 + te_2\|$$

for $(s, t) \in \mathbb{R}^2$. Then p is a norm on $\mathbb{R}^2$. Since the mapping ι from $(\mathbb{R}^2, p)$ to $(X, \|\cdot\|)$ defined by $\iota(s, t) := se_1 + te_2$ is an isometric isomorphism, we may identify $(\mathbb{R}^2, p)$ with $(X, \|\cdot\|)$. Hence we prove the theorem in $(\mathbb{R}^2, p)$ instead of $(X, \|\cdot\|)$. By equation (2), we obtain the following equality for the gradient ∇p:

$$\nabla p(s, t) = (f_{se_1+te_2}(e_1), f_{se_1+te_2}(e_2)),$$

which implies that $p(s, t)$ is of class C^1 on $\mathbb{R}^2 \setminus \{(0, 0)\}$. Moreover,

$$\begin{aligned} \nabla p(\lambda s, \lambda t) &= \nabla p(s, t) \quad (\lambda > 0), \\ (\nabla p)(-s, -t) &= -\nabla p(s, t) \end{aligned} \tag{3}$$

hold by equation (2).

In addition to the Cartesian coordinates (s, t), we also use polar coordinates (r, θ) defined through $s = r \cos \theta$, $t = r \sin \theta$. Then, the boundary of the unit disk with respect to the norm p is described by a polar equation $r = g(\theta)$. Note that $g(-\theta) = g(\theta)$ holds for any θ.

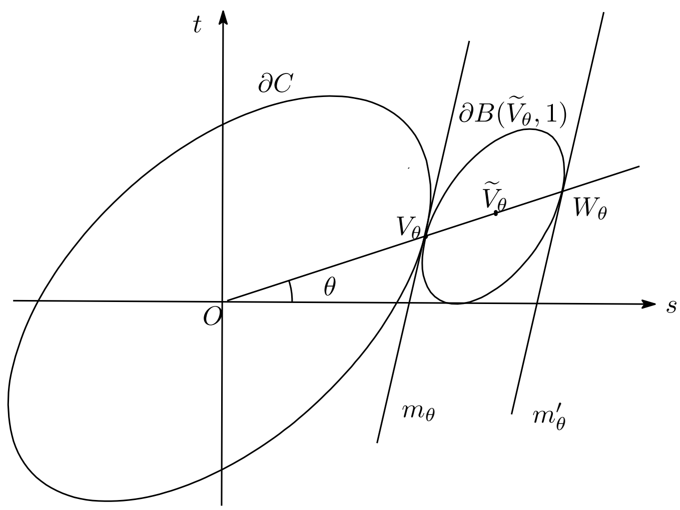

Figure 1: For specification of a supporting line

Note also that the function g is of class C^1 by our assumption of smoothness. Moreover, the boundary ∂C of the closed convex set C is also described by another polar equation $r = l(\theta)$. Lastly, the open disk with center P and radius ρ with respect to the norm p will be denoted by $B(P, \rho)$.

Step 2. (Specification of a tangent-like line at a point of ∂C)

By the well-known Hahn–Banach separation theorem, it is proved that for every point $U \in \partial C$, there exists a line m that supports C and passes U, i.e., $U \in m$ and $\operatorname{Int} C$ is contained in one of the half-spaces separated by m. Since the smoothness of ∂C is not assumed, m might not be uniquely defined. However, an appropriate m could be determined under the assumption that a positive-scalar multiple of x gives a nearest point in C to x. Namely, let $V_\theta \in \partial C$ be the point that is specified as $(l(\theta), \theta)$ in polar coordinates. Then, our assumption about nearest points implies that V_θ becomes a nearest point in C to the following point $\widetilde{V}_\theta$ (see Figure 1):

$$\widetilde{V}_\theta := \left(1 + \frac{g(\theta)}{l(\theta)}\right) \cdot V_\theta.$$

It is easy to see that the distance of $\widetilde{V}_\theta$ and V_θ with respect to p is equal to 1. Hence, $C \cap B(\widetilde{V}_\theta, 1) = \emptyset$ and $V_\theta \in C \cap \overline{B(\widetilde{V}_\theta, 1)}$. Therefore the Hahn–Banach separation theorem yields a line m_θ that passes V_θ and separates C and $B(\widetilde{V}_\theta, 1)$. This time m_θ is uniquely determined since the boundary of $B(\widetilde{V}_\theta, 1)$ is smooth. To obtain the precise description of m_θ, note that because of the symmetry $-B(O, 1) = B(O, 1)$ it is parallel to the tangent m'_θ to $\partial B(\widetilde{V}_\theta, 1)$ at W_θ in Figure 1 (analytically speaking, this is proved by (3)). And m'_θ is further parallel to the tangent m''_θ to $\partial B(O, 1)$ (O denotes the origin) at the point U_θ with angular coordinate θ (see Figure 2). Since m''_θ is not radial, there exists a point T on m''_θ for which the angular coordinate is greater than that of U_θ. Then an angle $\phi(\theta)$ is introduced by the following formula:

$$\angle OU_\theta T = \frac{\pi}{2} + \phi(\theta) \quad \left(-\frac{\pi}{2} < \phi(\theta) < \frac{\pi}{2}\right).$$

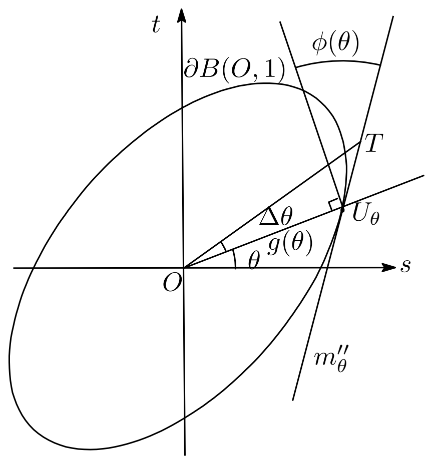

Figure 2: tangent to $\partial B(0,1)$

As to $\phi(\theta)$, an elementary argument in calculus gives

$$\frac{g'(\theta)}{g(\theta)} = \tan\phi(\theta), \tag{4}$$

where g' means the derivative of g. Therefore, the continuity of g' implies the existence of a constant $M > 0$ such that

$$|\tan\phi(\theta)| \le M \quad (\theta \in [0, 2\pi]). \tag{5}$$

Step 3. (Estimate of $l(\theta)$)

Now, we have seen that a supporting line m_θ for C at V_θ is parallel to m''_θ. Hence the angle $\phi(\theta) \in (-\pi/2, \pi/2)$ in Figure 3 is the same as in Figure 2 and so determined by (4). Note also that because of (5), there exists an $\varepsilon > 0$ such that the radial half-line with angular coordinate $\theta + \Delta\theta$ intersect with m_θ for any $\theta \in [0, 2\pi]$ and $\Delta\theta \in (0, \varepsilon)$. So, let $0 < \Delta\theta < \varepsilon$ and let T' be the intersection point of radial half-line with angular coordinate $\theta + \Delta\theta$ and m_θ. Further let $\hat{l}$ be the radial coordinate of T' as in Figure 3, while we have already denoted the radial coordinate of V_θ by $l(\theta)$. Then we obtain

$$\frac{l(\theta)}{\sin\left(\frac{\pi}{2} - \phi(\theta) - \Delta\theta\right)} = \frac{\hat{l}}{\sin\left(\frac{\pi}{2} + \phi(\theta)\right)}$$

by applying the sine rule to $\triangle OV_\theta T'$. Hence

$$l(\theta + \Delta\theta) \le \hat{l} = l(\theta) \cdot \frac{\cos\phi(\theta)}{\cos(\phi(\theta) + \Delta\theta)}$$

and so

$$\frac{l(\theta)}{l(\theta + \Delta\theta)} \ge \frac{\cos(\phi(\theta) + \Delta\theta)}{\cos\phi(\theta)}. \tag{6}$$

Now take $\theta_1, \theta_2 \in [0, 2\pi]$ with $\theta_1 < \theta_2$. Then $\Delta_n\theta := (\theta_2 - \theta_1)/n < \varepsilon$ for sufficiently large $n \in \mathbb{N}$ ($n > 2\pi/\varepsilon$ will do). For such n, inequality (6) yields

$$
\begin{aligned}
\frac{l(\theta_1)}{l(\theta_2)} &= \frac{l(\theta_1)}{l(\theta_1+\Delta_n\theta)} \cdot \frac{l(\theta_1+\Delta_n\theta)}{l(\theta_1+2\Delta_n\theta)} \cdot \cdots \cdot \frac{l\left(\theta_1+(n-1)\Delta_n\theta\right)}{l(\theta_2)} \\
&\geq \prod_{k=0}^{n-1} \frac{\cos(\phi(\theta_1+k\Delta_n\theta)+\Delta_n\theta)}{\cos\phi(\theta_1+k\Delta_n\theta)} \\
&= \cos^n(\Delta_n\theta) \prod_{k=0}^{n-1} \{1-\tan(\phi(\theta_1+k\Delta_n\theta))\tan(\Delta_n\theta)\}\,. \qquad (7)
\end{aligned}
$$

Noting the estimates (5) and $0 < \Delta_n\theta \leq 2\pi/n$, we see that for sufficiently large $n \in \mathbb{N}$,

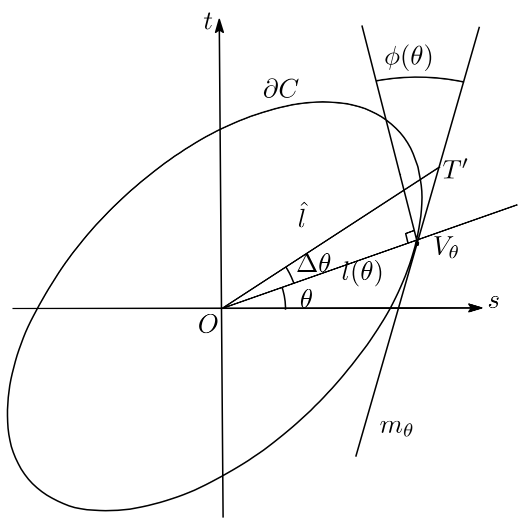

Figure 3: m_θ and ∂C

$$
\begin{aligned}
&\log\{1-\tan(\phi(\theta_1+k\Delta_n\theta))\tan(\Delta_n\theta)\} \\
&= -\tan(\phi(\theta_1+k\Delta_n\theta))\tan(\Delta_n\theta) + \frac{\{\tan(\phi(\theta_1+k\Delta_n\theta))\tan(\Delta_n\theta)\}^2}{2\{1-\eta\tan(\phi(\theta_1+k\Delta_n\theta))\tan(\Delta_n\theta)\}^2}
\end{aligned}
$$

and

$$
\tan(\Delta_n\theta) = \Delta_n\theta + \frac{\sin(\eta'\Delta_n\theta)}{\cos^3(\eta'\Delta_n\theta)} \cdot (\Delta_n\theta)^2
$$

hold for some $\eta, \eta' \in (0,1)$ by applying Taylor's theorem to $\log(1-x)$ and $\tan x$. Hence

$$
\log\{1-\tan\phi(\theta_1+k\Delta_n\theta)\tan(\Delta_n\theta)\} = -\tan\phi(\theta_1+k\Delta_n\theta)\Delta_n\theta + O\left(\frac{1}{n^2}\right) \qquad (8)
$$

holds by (5), where $O\left(1/n^2\right)$ is Landau's big O notation. Precisely speaking, the absolute value of this remainder term is estimated by K/n^2 from above where K is independent of

θ_1, θ_2 and sufficiently large n. Therefore we obtain

$$\begin{aligned}
&\sum_{k=0}^{n-1} \log\{1 - \tan\phi(\theta_1 + k\Delta_n\theta)\tan(\Delta_n\theta)\} \\
&= -\sum_{k=0}^{n-1} \frac{g'(\theta_1 + k\Delta_n\theta)}{g(\theta_1 + k\Delta_n\theta)} \cdot \Delta_n\theta + O\left(\frac{1}{n}\right) \\
&\to -\int_{\theta_1}^{\theta_2} \frac{g'(\theta)}{g(\theta)} d\theta = \log\frac{g(\theta_1)}{g(\theta_2)} \quad (n \longrightarrow \infty)
\end{aligned}$$

by (4) and (8). Hence by taking the logarithm of both sides in (7) and letting $n \longrightarrow \infty$, we obtain

$$\log\frac{l(\theta_1)}{l(\theta_2)} \geq \log\frac{g(\theta_1)}{g(\theta_2)},$$

since $\lim_{n\to\infty} \cos^n(\Delta_n\theta) = 1$.

Therefore $l(\theta)/g(\theta)$ is a decreasing function of θ. However $l(0)/g(0) = l(2\pi)/g(2\pi)$ and so $l(\theta)/g(\theta)$ is a constant, which implies $\partial C = \partial B(O, r)$ for some $r > 0$. Thus C is a closed ball with center 0. □

3 Result without the assumption of smoothness

3.1 Preliminaries This section is devoted to a proof that the conclusion in Theorem 4 holds without the assumption of smoothness. For this purpose, it suffices to prove that assertion (A) (in the *Conjecture* in Section 1) holds for general 2-dimensional real Banach space X, by virtue of Corollary 2.

So, hereafter let X denote a 2-dimensional Banach space and X is identified with $\mathbb{R}^2$ as in Step 1 of the proof of Theorem 4 via a basis of X. We use freely the standard Cartesian coordinate (s, t) and the polar coordinate (r, θ) of $\mathbb{R}^2$ that are defined there. We also adopt the notation $B(P, \rho)$ to denote the open disk with center P and radius ρ with respect to the norm on $\mathbb{R}^2$ that is induced from that of X through the identification.

The boundary $\partial B(O, 1)$ is described by a polar equation $r = g(\theta)$. Firstly, just to be sure, we record the fact that g is continuous without the assumption of smoothness of the norm.

Lemma 5. *$\partial B(O, 1)$ is described by a polar equation $r = g(\theta)$ with a continuous periodic function g.*

This continuity of g could be proved quite easily, e.g., by using the fact that equalities $1 = p(g(\theta)\cos(\theta), g(\theta)\sin(\theta)) = g(\theta)p(\cos(\theta), \sin(\theta))$ hold and p is continuous.

Next we recall well-known fundamental properties of convex functions, which will be crucial to treat the present non-smooth case. For a proof, see e.g., Tiel [4, Chapter 1] or Godement [2, Chapitre V, Théorème 15].

Proposition 6. *Let I be an open interval of $\mathbb{R}$ and $f\colon I \to \mathbb{R}$ a convex function. Then f enjoys the following properties.*

- (i) *For every $t \in I$, the left [resp. right] derivative $f'_-(t)$ [resp. $f'_+(t)$] exists and the inequality $f'_-(t) \leq f'_+(t)$ holds.*
- (ii) *f'_- and f'_+ are increasing functions and are continuous except for points of an at most countable set. Here, the term "increasing" is used in its wider sense, i.e., $t \leq s$ implies $f'_-(t) \leq f'_-(s)$ [resp. $f'_+(t) \leq f'_+(s)$].*

(iii) *For every subinterval* $[a,b]$ *of* I, $|f'_-(t)|$ *and* $|f'_+(t)|$ *are bounded by* $\max\{\,|f'_-(a)|, |f'_+(a)|, |f'_-(b)|, |f'_+(b)|\,\}$. *Hence they are locally bounded on* $(-\rho,\rho)$.

(iv) *Except for at most countably many points,* $f'_-(t) = f'_+(t)$ *holds and hence* f *is differentiable there. Moreover,* f'_- *and* f'_+ *are continuous at points where* f *is differentiable.*

For later use we record the following fact.

Lemma 7. *Let* $\rho > 0$ *and* $h\colon (-\rho,\rho) \to \mathbb{R}$ *be a negative valued convex function. Then the following inequality holds where* $h'_\pm$ *denotes either of the one-sided derivatives* h'_- *and* h'_+*:*

$$uh'_\pm(u) - h(u) > 0 \quad (\forall u \in (-\rho,\rho)) \tag{9}$$

Proof. $uh'_\pm(u) - h(u) > 0$ clearly holds for $u = 0$. Suppose now $0 < u < \rho$. Then,

$$\frac{h(u)}{u} < \frac{h(u)-h(0)}{u} \le h'_-(u) \le h'_+(u)$$

holds, and consequently we obtain $uh'_\pm(u) - h(u) > 0$.

In the case of $-\rho < u < 0$,

$$\frac{h(u)}{u} > \frac{h(u)-h(0)}{u} = \frac{h(0)-h(u)}{0-u} \ge h'_+(u) \ge h'_-(u)$$

imply the desired inequality. $\square$

3.2 Introduction of a collection of coordinate systems A key to our proof is to show that the function g above has the same level of differentiability property as that of convex functions. To do so, it is necessary to introduce a collection of coordinate systems that is fitted to make well use of the convexity of $B(O,1)$. The need for such a collection would be understood by the fact that g itself is convex if and only if it is constant, hence the polar coordinate does not immediately lead to useful knowledge that compensates for the lack of smoothness.

Set $\theta = \theta_1$ in Figure 2 and let the coordinate system (frame) $\mathcal{F}$ for the Cartesian coordinates (s,t) be rotated around the origin by the angle $\frac{\pi}{2} + \theta_1$ to form a new coordinate system $\mathcal{F}_{\theta_1}$ (see Figure 4). In the sequel, $(u,v)_{\mathcal{F}_{\theta_1}}$ denotes the geometric point for which the coordinate with respect to $\mathcal{F}_{\theta_1}$ is (u,v).

The merit of introducing the coordinate system $\mathcal{F}_{\theta_1}$ is well explained by the following

Lemma 8. *There exists a positive constant* ρ_0 *such that for every angle* θ_1 *there exists a unique* convex *function* $h_{\theta_1}(u) < 0$ *of* $u \in (-\rho_0,\rho_0)$ *for which* $(u, h_{\theta_1}(u))_{\mathcal{F}_{\theta_1}} \in \partial B(O,1)$.

Proof. Since the origin is an interior point of $B(O,1)$, there exists a $\rho_0 > 0$ such that $s^2 + t^2 < {\rho_0}^2$ implies $(s,t) \in B(O,1)$.

Now, take an arbitrary angle θ_1. Then, $(u,v)_{\mathcal{F}_{\theta_1}} \in B(O,1)$ holds provided $u^2 + v^2 < {\rho_0}^2$. This means that $(u,0)_{\mathcal{F}_{\theta_1}} \in B(O,1)$ holds for each u with $|u| < \rho_0$. For such u, it is clear that the half-line $\{\,(u,v)_{\mathcal{F}_{\theta_1}} \mid v \le 0\,\}$ intersects $\partial B(O,1)$ at a single point $(u,v_u)_{\mathcal{F}_{\theta_1}}$ for some $v_u < 0$. Then $h_{\theta_1}(u) := v_u$ clearly yields the desired convex function. $\square$

3.3 Differentiability of $g(\theta)$

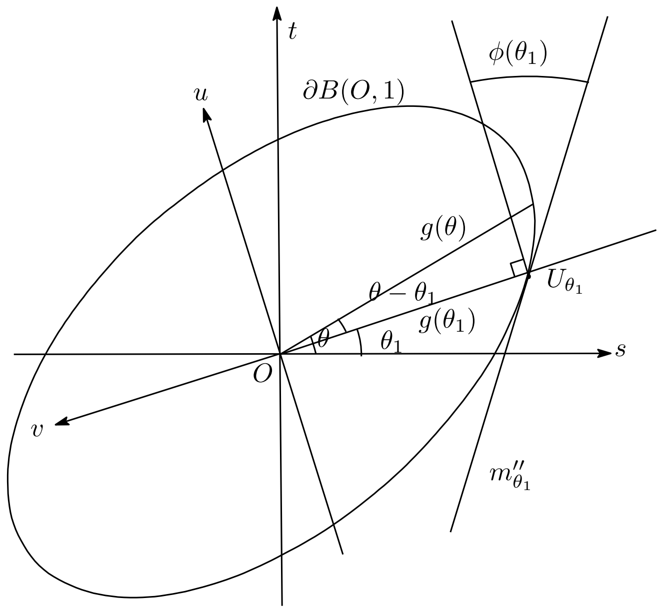

Figure 4: a new coordinate system $(u,v)_{\mathcal{F}_{\theta_1}}$

3.3.1 θ as an independent variable and as a function Lemma 8 implies that $u \in (-\rho_0, \rho_0)$ gives a *local* coordinate for the curve $\partial B(O,1)$ by the correspondence $u \mapsto (u, h_{\theta_1})_{\mathcal{F}_{\theta_1}}$. On the other hand, the angular variable θ could also be used as a local coordinate for $\partial B(O,1)$ by the mapping $\theta \mapsto (g(\theta),\theta)_{rad}$, where $(g(\theta),\theta)_{rad}$ denotes the geometric point with the polar coordinates $(g(\theta),\theta)$, i.e., $(g(\theta)\cos\theta, g(\theta)\sin\theta) \in \mathbb{R}^2$.

We can see that θ is determined (mod 2π) as a continuous function of $u \in (-\rho_0, \rho_0)$ by $(u, h_{\theta_1}(u))_{\mathcal{F}_{\theta_1}} = (g(\theta),\theta)_{rad}$, since $\{(u, h_{\theta_1}(u))_{\mathcal{F}_{\theta_1}} \mid u \in (-\rho_0,\rho_0)\}$ is contained in the half-space $\{(u,v)_{\mathcal{F}_{\theta_1}} \mid u \in \mathbb{R}, v \le 0\}$. Therefore, once a coordinate system $\mathcal{F}_{\theta_1}$ is designated, local coordinate θ for $\partial B(O,1)$ might be considered as a continuous function of $u \in (-\rho_0,\rho_0)$, where u denotes the first component of the coordinate with respect to $\mathcal{F}_{\theta_1}$. In the sequel, θ considered as a function in this way will be denoted simply by $\theta(u)$, avoiding more accurate but rather awkward expression such as $\theta_{\theta_1}(u)$. In addition, intuitively speaking, it is clear that the function $\theta(u)$ defined above is a strictly increasing continuous function of $u \in (-\rho_0,\rho_0)$. To prove these facts analytically, we give an explicit expression of $\theta(u)$.

Lemma 9. *Let θ_1 be arbitrarily fixed and let $h_{\theta_1}(u)$ be the convex function described in Lemma 8. Then, the relation $(u, h_{\theta_1})_{\mathcal{F}_{\theta_1}} = (r,\theta)_{rad}$ $(-\rho_0 < u < \rho_0)$ is satisfied if and only if*

$$\theta = \arcsin\left(\frac{u}{\sqrt{u^2 + h_{\theta_1}(u)^2}}\right) + \theta_1. \tag{10}$$

Therefore, by taking the principal branch of arcsin*, a continuous function $\theta = \theta(u)$ of $u \in (-\rho_0,\rho_0)$ is obtained. Then, the left derivative $\theta'_-(u)$ and the right derivative $\theta'_+(u)$ everywhere exist and are positive, and hence $\theta(u)$ is a strictly increasing function of u. Moreover, the set $\mathcal{N}$ of points where $\theta(u)$ is not differentiable is at most countable and θ'_- and θ'_+ are continuous at points in $(-\rho_0,\rho_0)\setminus\mathcal{N}$.*

Proof. $(u, h_{\theta_1})_{\mathcal{F}_{\theta_1}} = (r,\theta)_{rad}$ is nothing but

$$\begin{aligned} u &= g(\theta)\sin(\theta-\theta_1), \\ h_{\theta_1}(u) &= -g(\theta)\cos(\theta-\theta_1). \end{aligned}$$

Hence

$$g(\theta)^2 = u^2 + h_{\theta_1}(u)^2, \tag{11}$$

$$u = \sqrt{u^2 + h_{\theta_1}(u)^2}\ \sin(\theta-\theta_1), \tag{12}$$

whence follows (10). Since Lemma 8 and Proposition 6 yield one-sided differentiability of h_{θ_1}, (10) shows that $\theta = \theta(u)$ is one-sided differentiable at every $u \in (-\rho_0, \rho_0)$ and

$$\theta'_{\pm}(u) = -\frac{h_{\theta_1}(u) - u(h_{\theta_1})'_{\pm}(u)}{u^2 + h_{\theta_1}(u)^2} \tag{13}$$

holds. Hence we obtain $\theta'_{\pm}(u) > 0$ by virtue of Lemma 7.

The assertions on the differentiability of $\theta(u)$ and the continuity of $\theta'_{\pm}$ follow immediately from (13) and Lemma 8 applied for h_{θ_1}. □

3.3.2 Proof of the differentiability of $g(\theta)$ As a consequence of the results proved so far, the following assertion concerning the differentiability of $g(\theta)$ is obtained.

Lemma 10. *Let θ_1 be arbitrarily fixed and let $\theta = \theta(u)$ is defined by (10) as a function of $u \in (-\rho_0, \rho_0)$ (*arcsin *is construed to mean its principal branch). Then, by virtue of Lemma 9, the range of $\theta(u)$ is an open interval I_{θ_1} containing θ_1, and one-sided derivatives of $g(\theta)$ exist at every $\theta \in I_{\theta_1}$. Moreover, there exists an at most countable set $\mathcal{N}$ for which $g'_{\pm}(\theta)$ is continuous and $g(\theta)$ is differentiable at every point in $I_{\theta_1} \setminus \mathcal{N}$. In addition, $g'_{\pm}(\theta)$ is bounded in a neighbourhood of θ_1.*

Proof. Let $|u| < \rho_0$ and $k \neq 0$ is sufficiently close to 0, then the following algebraic transformation is valid (note that $\theta(u)$ is 1 to 1 by Lemma 9):

$$\begin{aligned} & \frac{g(\theta(u+k)) - g(\theta(u))}{\theta(u+k)-\theta(u)} \cdot \frac{\theta(u+k)-\theta(u)}{k} \\ = {} & \frac{g(\theta(u+k)) - g(\theta(u))}{k} \\ = {} & \frac{\sqrt{(u+k)^2 + h_{\theta_1}(u+k)^2} - \sqrt{u^2 + h_{\theta_1}(u)^2}}{k}. \end{aligned} \tag{14}$$

Here, $k \to +0$ [resp. $k \to -0$] implies $\theta(u+k) \to \theta(u)+0$ [resp. $\theta(u+k) \to \theta(u) - 0$ by Lemma 9. Therefore, by letting $k \to \pm 0$ in (14), the one-sided differentiability of h_{θ_1} and $\theta(u)$ (Lemma 9) with $\theta'_{\pm}(u) > 0$ yield the one-sided differentiability of $g(\theta)$. Explicitly speaking, calculation of the limit of (14) as $k \to \pm 0$ yields

$$g'_{\pm}(\theta)\big|_{\theta=\theta(u)} = \frac{u + h_{\theta_1}(u)(h_{\theta_1}(u))'_{\pm}}{\sqrt{u^2 + h_{\theta_1}(u)^2}} \cdot \frac{1}{\theta'_{\pm}(u)}. \tag{15}$$

The existence of an at most countable set $\mathcal{N}$ as stated in the Lemma follows from this expression, Proposition 6, Lemma 8, Lemma 9 and the fact that $\theta(u)$ is an order preserving homeomorphism from $(-\rho_0, \rho_0)$ to I_{θ_1}.

The boundedness of $g'_{\pm}$ in a neighbourhood of θ_1 is clear from (15) and (13) since $(h_{\theta_1})'_{\pm}$ is locally bounded on $(-\rho_0, \rho_0)$ by Proposition 6 and Lemma 8.

□

Remark 11. *By a detailed analysis, it can be shown that* $g'_\pm$ *is locally bounded on* I_{θ_1}.

Remark 12. *For later use, note that* (15), (13) *and equality* $h_{\theta_1}(0) = -g(\theta_1)$ *imply*

$$(h_{\theta_1})'_\pm(0) = -\frac{g'_\pm(\theta_1)}{g(\theta_1)} \tag{16}$$

for every θ_1.

Arbitrariness of θ_1 in the previous Lemma and the compactness of $[0, 2\pi]$ readily lead to the following

Lemma 13. *One-sided derivatives* $g'_\pm$ *exist everywhere and are bounded on* $[0, 2\pi]$. *In addition there exists an at most countable set* $\mathcal{N} \subset [0, 2\pi]$ *for which* g *is differentiable and* $g'_\pm$ *are continuous at each point of* $[0, 2\pi] \setminus \mathcal{N}$.

Although the next lemma might be well known, we state it with a proof since it is crucial to our purpose.

Lemma 14. *The function* $\log g(\theta)$ *is uniformly Lipschitz continuous on* $[0, 2\pi]$.

Proof. Set $\psi(\theta) := \log g(\theta)$. Then it is clear from Lemma 13 that ψ has left and right derivative at every point and that $\psi'_\pm = g'_\pm/g$ is bounded on $[0, 2\pi]$. So let us take a constant K such that $K \geq |\psi'_\pm(\theta)|$ for every $\theta \in [0, 2\pi]$.

Suppose that $\eta, \tilde\eta \in [0, 2\pi]$, $\eta < \tilde\eta$ and set

$$F(\theta) := \psi(\theta) - \psi(\eta) - \frac{\psi(\tilde\eta) - \psi(\eta)}{\tilde\eta - \eta}(\theta - \eta)$$

for $\theta \in [\eta, \tilde\eta]$. Then

$$F'_\pm(\theta) = \psi'_\pm(\theta) - \frac{\psi(\tilde\eta) - \psi(\eta)}{\tilde\eta - \eta}.$$

Since $F(\theta)$ is continuous, $F(\theta)$ attains its maximum value and minimum value on $[\eta, \tilde\eta]$. Since $F(\eta) = F(\tilde\eta) = 0$, $F(\theta)$ attains at least either of the maximum value or the minimum value at some $\xi \in (\eta, \tilde\eta)$. If $F(\xi)$ is the maximum value, then

$$\begin{aligned} 0 \leq F'_-(\xi) &= \psi'_-(\xi) - \frac{\psi(\tilde\eta) - \psi(\eta)}{\tilde\eta - \eta} \\ 0 \geq F'_+(\xi) &= \psi'_+(\xi) - \frac{\psi(\tilde\eta) - \psi(\eta)}{\tilde\eta - \eta}, \end{aligned}$$

and so

$$\psi'_+(\xi) \leq \frac{\psi(\tilde\eta) - \psi(\eta)}{\tilde\eta - \eta} \leq \psi'_-(\xi).$$

Similarly, if $F(\xi)$ is the minimum value,

$$\psi'_-(\xi) \leq \frac{\psi(\tilde\eta) - \psi(\eta)}{\tilde\eta - \eta} \leq \psi'_+(\xi).$$

Thus, in either case we obtain

$$|\psi(\tilde\eta) - \psi(\eta)| \leq K|\tilde\eta - \eta|$$

and hence ψ is uniformly Lipschitz continuous. □

3.4 Completion of the proof In this last subsection we give a proof of the following theorem by completing the proof for the special case of dim $X = 2$.

Theorem 15. *For every Banach space X with* dim $X > 1$, *assertion (A) in our Conjecture holds.*

So, as stated at the beginning of this Section 3, let X denote a Banach space with dim $X = 2$ identified with $\mathbb{R}^2$ and let the unit "sphere" $\partial B(O, 1)$ be described by a polar equation $r = g(\theta)$. Note that we have shown Lemmas 10 to 14 concerning analytic properties of g.

Now, suppose that $C \subset X$ is a bounded closed convex set with 0 in its interior, and also suppose that for every $x \in X$ a positive-scalar multiple of x gives a nearest point in C to x. Our task is to show that on this supposition C is indeed a closed ball. The proof proceeds along the line of that in Section 2 (the case of smooth Banach spaces).

Firstly, note that for every angle θ our supposition yields the existence of a line m_θ enjoying the following properties: m_θ supports C at V_θ in Fig. 1 and is parallel to a supporting line m''_θ of $B(O, 1)$ at $(g(\theta), \theta)_{rad}$ (see Fig. 2). This is a consequence of the Hahn–Banach separation theorem that does not require smoothness. Although m_θ is not uniquely determined in general, consider that one of such line is assigned for every θ and named m_θ. Then, one can see that the angle $\phi(\theta)$ in Fig. 2 satisfies the following estimate by locally considering the curve $\partial B(O, 1)$ as the graph $\{ (u, h_\theta(u)) \mid |u| < \rho_0 \}$ in the coordinate system $\mathcal{F}_\theta$:

$$(h_\theta)'_-(0) \le -\tan\phi(\theta) \le (h_\theta)'_+(0).$$

Hence (16) yields

$$\frac{g'_+(\theta)}{g(\theta)} \le \tan\phi(\theta) \le \frac{g'_-(\theta)}{g(\theta)} \tag{17}$$

for every θ, and Lemma 10 shows that $|\tan\phi(\theta)|$ is bounded on $[0, 2\pi]$.

Now, prior to going into the heart of the proof, note that the boundedness of various quantities can be readily obtained from Lemma 13: There exists a constant $K > 0$ such that

$$|g'_\pm(\theta)|,\ \frac{|g'_\pm(\theta)|}{g(\theta)},\ |\tan\phi(\theta)| \le K \quad (\forall\theta \in [0, 2\pi]). \tag{18}$$

As in the previous section, let the curve ∂C be described by a polar equation $r = \ell(\theta)$ and let us return to Fig. 3. Because of estimate (18), there exists an $\varepsilon > 0$ for which the ray with angle $\theta + \Delta\theta$ intersects the line m_θ, which separates C and a translation of the unit ball (see Fig. 1), provided $0 < \Delta\theta < \varepsilon$. Now take $\theta_1, \theta_2 \in [0, 2\pi]$ with $\theta_1 < \theta_2$ and $n \in \mathbb{N}$ with $n > 2\pi/\varepsilon$ and set $\Delta_n\theta := (\theta_2 - \theta_1)/n$. Then the argument leading to (7) is also valid in the present case and we obtain

$$\frac{l(\theta_1)}{l(\theta_2)} \ge \cos^n(\Delta_n\theta) \prod_{k=0}^{n-1} \left\{1 - \tan(\phi(\theta_1 + k\Delta_n\theta)) \tan(\Delta_n\theta)\right\}. \tag{19}$$

From (17) and (18), we obtain the following asymptotic formula as $n \to \infty$:

$$
\begin{aligned}
&\sum_{k=0}^{n-1} \log\left(1 - \{\tan\phi(\theta + k\Delta_n\theta)\}\tan(\Delta_n\theta)\right) \\
= \quad & -\sum_{k=0}^{n-1} \tan\phi(\theta_1 + k\Delta_n\theta)\Delta_n\theta + O\left(\frac{1}{n}\right) \\
\geq \quad & -\sum_{k=0}^{n-1} \frac{g'_-(\theta + k\Delta_n\theta)}{g(\theta + k\Delta_n\theta)}\Delta_n\theta + O\left(\frac{1}{n}\right). \qquad (20)
\end{aligned}
$$

By Lemma 13, $g'_-(\theta)/g(\theta)$ is Riemann integrable on $[\theta_1, \theta_2]$, and so

$$
\begin{aligned}
& -\sum_{k=0}^{n-1} \frac{g'_-(\theta + k\Delta_n\theta)}{g(\theta + k\Delta_n\theta)}\Delta_n\theta + O\left(\frac{1}{n}\right) \\
\longrightarrow \quad & -\int_{\theta_1}^{\theta_2} \frac{g'_-(\theta)}{g(\theta)}\, d\theta \quad (n \longrightarrow \infty).
\end{aligned}
$$

Taking the logarithm of (19) and using the asymptotic formula above, we obtain

$$
\log\frac{l(\theta_1)}{l(\theta_2)} \geq -\int_{\theta_1}^{\theta_2} \frac{g'_-(\theta)}{g(\theta)}\, d\theta \qquad (21)
$$

since $\cos^n(\Delta_n\theta) \to 1$ as $n \to \infty$. Moreover, Lemma 14 shows that $\log(g(\theta))$ is absolutely continuous on $[\theta_1, \theta_2]$. Hence the Fundamental Theorem of Calculus for Lebesgue Integral ([3, p. 148]) implies

$$
-(L)\int_{\theta_1}^{\theta_2} \frac{1}{g(\theta)} \cdot \frac{dg}{d\theta}(\theta)\, d\theta = \log\frac{g(\theta_1)}{g(\theta_2)}, \qquad (22)
$$

where $(L)\int$ means the Lebesgue integral. In addition, Lemma 13 shows that

$$
\frac{1}{g(\theta)} \cdot \frac{dg}{d\theta}(\theta) = \frac{g'_-(\theta)}{g(\theta)}
$$

holds except for θ in some at most countable set. Hence

$$
(L)\int_{\theta_1}^{\theta_2} \frac{1}{g(\theta)} \cdot \frac{dg}{d\theta}(\theta)\, d\theta = (L)\int_{\theta_1}^{\theta_2} \frac{g'_-(\theta)}{g(\theta)}\, d\theta \qquad (23)
$$

holds. Since Lebesgue integral and Riemann integral coincide for Riemann integrable functions, (21), (23) and (22) yield

$$
\begin{aligned}
\log\frac{l(\theta_1)}{l(\theta_2)} &\geq -(L)\int_{\theta_1}^{\theta_2} \frac{g'_-(\theta)}{g(\theta)}\, d\theta \\
&= -(L)\int_{\theta_1}^{\theta_2} \frac{1}{g(\theta)} \cdot \frac{dg}{d\theta}(\theta)\, d\theta \\
&= \log\frac{g(\theta_1)}{g(\theta_2)}.
\end{aligned}
$$

Therefore

$$\log \frac{\ell(\theta_1)}{g(\theta_1)} \geq \log \frac{\ell(\theta_2)}{g(\theta_2)}$$

holds and hence $\ell(\theta)/g(\theta)$ is a decreasing function (in the wider sense) of $\theta \in [0, 2\pi]$. This in turn implies that $\ell(\theta)/g(\theta)$ is a constant function since the values at 0 and 2π coincide, and the proof is thus completed.

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