A behavior of the structure tensor on real hypersurfaces in a nonflat complex space form

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Abstract. In the theory of real hypersurfaces in a nonflat complex space form, the behavior of the structure tensor ϕ is significant. In this paper, we investigate generalizations of the parallelism of the structure tensor ϕ.

1 Introduction Contact Riemannian geometry is one of the active field in Riemannian geometry. In particular, it is known that cosymplectic manifolds, Sasakian manifolds and Kenmotsu manifolds are characterized by using a behavior of the structure tensor ϕ on contact Riemannian manifolds (see [1]).

In a nonflat complex space form $\widetilde{M}_n(c)$ (namely, a complex projective space $\mathbb{C}P^n(c)$ of constant holomorphic sectional curvature $c > 0$ or a complex hyperbolic space $\mathbb{C}H^n(c)$ of constant holomorphic sectional curvature $c < 0$), real hypersurfaces admit the almost contact metric structure (ϕ, ξ, η, g) induced from the ambient space. The structure tensor ϕ plays an important role not only contact Riemannian geometry but also the theory of real hypersurfaces in $\widetilde{M}_n(c)$. In this paper, we focus on the parallelism of the structure tensor ϕ of real hypersurfaces in $\widetilde{M}_n(c)$. It is known that *there exists no real hypersurface in $\widetilde{M}_n(c)$ whose the structure tensor ϕ is parallel* (see [4]).

The purpose of this paper is to generalize this fact and to investigate such real hypersurfaces. We first study the following three conditions:

$$\nabla_\xi \phi = 0 \quad (\xi\text{-parallelism}), \tag{1.1}$$

$$\nabla_X \phi = 0 \quad \text{for} \quad \forall X \in T^0M \quad (T^0M\text{-parallelism}), \tag{1.2}$$

$$(\nabla_X \phi)Y - (\nabla_Y \phi)X = 0 \quad \text{for} \quad \forall X, Y \in TM \quad (\text{the Codazzi tensor}), \tag{1.3}$$

where TM is the tangent bundle of M^{2n-1} and T^0M is the holomorphic distribution, that is, $T^0M = \{X \in TM : X \perp \xi\}$. These conditions are simple generalizations of the parallelism of the structure tensor ϕ. In particular, Conditions (1.1) and (1.2) give characterizations of *Hopf hypersurfaces* and *ruled real hypersurfaces* in $\widetilde{M}_n(c)$, respectively. These classes of real hypersurfaces are significant examples. On the other hand, there exists no real hypersurface satisfying Condition (1.3). So, it is natural to consider generalizations of Condition (1.3).

Secondly, we study the following condition which is a certain generalization of (1.3):

$$\operatorname{div} \phi = 0. \tag{1.4}$$

This condition is inspired by Sharma's work (see [6]). By Condition (1.4), we obtain a characterization of Hopf hypersurfaces with constant mean curvature given by $\alpha/(2n-1)$, where $\alpha = g(A\xi, \xi)$ (Theorem 1).

Finally, we give applications of the discussion of Theorem 1. To do this, we focus on the following two classes of real hypersurfaces in $\widetilde{M}_n(c)$:

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(1) The class of minimal Hopf hypersurfaces in $\widetilde{M}_n(c)$;

(2) The class of real hypersurfaces in $\mathbb{C}H^n(c)$ which satisfies the following the condition:

$$\overline{Ric} = \frac{1}{4}(\text{Trace } A)^2 + \frac{c}{2}(n-1),$$

where $\overline{Ric}$ is the maximal Ricci curvature of real hypersurfaces in $\mathbb{C}H^n(c)$. The latter class was investigated by B. Y. Chen (see [2]). He showed that every real hypersurface in $\mathbb{C}H^n(c)$ satisfies the following inequality:

$$\overline{Ric} \leqq \frac{1}{4}(\text{Trace } A)^2 + \frac{c}{2}(n-1).$$

Moreover he also investigated the equality case of the above inequality.

In the latter of this paper, we characterize the above classes of real hypersurfaces by using the modification of Condition (1.4).

2 Preliminaries Let M^{2n-1} be a real hypersurface with a unit local vector field $\mathcal{N}$ of a complex n-dimensional nonflat complex space form $\widetilde{M}_n$. The Riemannian connections $\widetilde{\nabla}$ of $\widetilde{M}_n(c)$ and ∇ of M^{2n-1} are related by

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX,\, Y)\mathcal{N} \quad \text{and} \quad \widetilde{\nabla}_X \mathcal{N} = -AX$$

for vector fields X and Y tangent to M^{2n-1}, where g denotes the induced metric from the standard Riemannian metric of $\widetilde{M}_n(c)$ and A is the shape operator of M^{2n-1} in $\widetilde{M}_n(c)$. The former is called *Gauss's formula*, and the latter is called *Weingarten's formula*. Eigenvalues and eigenvectors of the shape operator A are called *principal curvatures* and *principal vectors* of M^{2n-1} in $\widetilde{M}_n(c)$, respectively.

It is known that M^{2n-1} admits an *almost contact metric structure* (ϕ, ξ, η, g) induced from the Kähler structure J of $\widetilde{M}_n(c)$. *The characteristic vector field* ξ of M^{2n-1} is defined as $\xi = -J\mathcal{N}$ and this structure satisfies

$$\begin{aligned} &\phi^2 = -I + \eta \otimes \xi,\ \eta(X) = g(X, \xi),\ \eta(\xi) = 1,\ \phi\xi = 0,\ \eta(\phi X) = 0, \\ &g(\phi X, Y) = -g(X, \phi Y) \quad \text{and} \quad g(\phi X,\, \phi Y) = g(X\,, Y) - \eta(X)\eta(Y), \end{aligned} \tag{2.1}$$

where I denotes the identity map of the tangent bundle TM of M^{2n-1}. We call ϕ and η *the structure tensor* and *the contact form* of M^{2n-1}, respectively.

The following equation is a fundamental tool in the theory of real hypersurfaces in $\widetilde{M}_n(c)$:

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi. \tag{2.2}$$

We usually call M^{2n-1} a *Hopf hypersurface* if the characteristic vector ξ is a principal curvature vector at each point of M^{2n-1}. It is known that every tube of sufficiently small constant radius around each Kähler submanifold of $\widetilde{M}_n(c)$ is a Hopf hypersurface. This fact tells us that the notion of Hopf hypersurface is natural in the theory of real hypersurfaces in $\widetilde{M}_n(c)$ (see [5]).

The following lemma clarifies a fundamental property which is a useful tool in the theory of Hopf hypersurfaces in $\widetilde{M}_n(c)$.

Lemma 1 ([5])**.** *For a Hopf hypersurface M^{2n-1} with the principal curvature α corresponding to the characteristic vector field ξ in $\widetilde{M}_n(c)$, we have the following:*

(1) *α is locally constant on M^{2n-1};*

(2) *If X is a tangent vector of M^{2n-1} perpendicular to ξ with $AX = \lambda X$, then $(2\lambda - \alpha)A\phi X = (\alpha\lambda + (c/2))\phi X$.*

In $\mathbb{C}P^n(c)$ $(n \geqq 2)$, a Hopf hypersurface all of whose principal curvatures are constant is locally congruent to one of the following:

(A_1) A geodesic sphere $G(r)$ of radius r, where $0 < r < \pi/\sqrt{c}$;

(A_2) A tube of radius r around a totally geodesic $\mathbb{C}P^\ell(c)$ $(1 \leqq \ell \leqq n-2)$, where $0 < r < \pi/\sqrt{c}$;

(B) A tube of radius r around a complex hyper quadric $\mathbb{C}Q^{n-1}$, where $0 < r < \pi/(2\sqrt{c}\,)$;

(C) A tube of radius r around a $\mathbb{C}P^1(c) \times \mathbb{C}P^{(n-1)/2}(c)$, where $0 < r < \pi/(2\sqrt{c}\,)$ and $n(\geqq 5)$ is odd;

(D) A tube of radius r around a complex Grassmann $\mathbb{C}G_{2,5}$, where $0 < r < \pi/(2\sqrt{c}\,)$ and $n = 9$;

(E) A tube of radius r around a Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \pi/(2\sqrt{c}\,)$ and $n = 15$.

These real hypersurfaces are said to be of types (A_1), (A_2), (B), (C), (D) and (E). The principal curvatures of these real hypersurfaces in $\mathbb{C}P^n(c)$ are given as follows (cf. [5]):

	(A_1)	(A_2)	(B)	(C), (D), (E)
λ_1	$\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r\right)$	$\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r\right)$	$\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r - \frac{\pi}{4}\right)$	$\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r - \frac{\pi}{4}\right)$
λ_2	—	$-\frac{\sqrt{c}}{2}\tan\left(\frac{\sqrt{c}}{2}r\right)$	$\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r + \frac{\pi}{4}\right)$	$\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r + \frac{\pi}{4}\right)$
λ_3	—	—	—	$\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r\right)$
λ_4	—	—	—	$-\frac{\sqrt{c}}{2}\tan\left(\frac{\sqrt{c}}{2}r\right)$
α	$\sqrt{c}\cot(\sqrt{c}r)$	$\sqrt{c}\cot(\sqrt{c}r)$	$\sqrt{c}\cot(\sqrt{c}r)$	$\sqrt{c}\cot(\sqrt{c}r)$

The multiplicities of these principal curvatures are given as follows (cf. [5]):

	(A_1)	(A_2)	(B)	(C)	(D)	(E)
$m(\lambda_1)$	$2n-2$	$2n-2\ell-2$	$n-1$	2	4	6
$m(\lambda_2)$	—	2ℓ	$n-1$	2	4	6
$m(\lambda_3)$	—	—	—	$n-3$	4	8
$m(\lambda_4)$	—	—	—	$n-3$	4	8
$m(\alpha)$	1	1	1	1	1	1

In $\mathbb{C}H^n(c)$ $(n \geqq 2)$, a Hopf hypersurface all of whose principal curvatures are constant is locally congruent to one of the following:

(A_0) A horosphere in $\mathbb{C}H^n(c)$;

($\mathrm{A}_{1,0}$) A geodesic sphere $G(r)$ of radius r, where $0 < r < \infty$;

($\mathrm{A}_{1,1}$) A tube of radius r around a totally geodesic $\mathbb{C}H^{n-1}(c)$, where $0 < r < \infty$;

(A_2) A tube of radius r around a totally geodesic $\mathbb{C}H^\ell(c)\,(1 \leqq \ell \leqq n-2)$, where $0 < r < \infty$;

(B) A tube of radius r around a totally real totally geodesic $\mathbb{R}H^n(c/4)$, where $0 < r < \infty$.

These real hypersurfaces are said to be of types (A_0), ($\mathrm{A}_{1,0}$), ($\mathrm{A}_{1,1}$), (A_2) and (B). Summing up, real hypersurfaces of types ($\mathrm{A}_{1,0}$) and ($\mathrm{A}_{1,1}$), we call them real hypersurfaces of type (A_1). The principal curvatures of these real hypersurfaces in $\mathbb{C}H^n(c)$ are given as follows (cf. [5]):

	(A_0)	($\mathrm{A}_{1,0}$)	($\mathrm{A}_{1,1}$)	(A_2)	(B)
λ_1	$\frac{\sqrt{\lvert c\rvert}}{2}$	$\frac{\sqrt{\lvert c\rvert}}{2}\coth(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\coth(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\coth(\frac{\sqrt{\lvert c\rvert}}{2}r)$
λ_2	—	—	—	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh(\frac{\sqrt{\lvert c\rvert}}{2}r)$
α	$\sqrt{\lvert c\rvert}$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}r)$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}r)$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}r)$	$\sqrt{\lvert c\rvert}\tanh(\sqrt{\lvert c\rvert}r)$

Finally, we define ruled real hypersurfaces in a nonflat complex space form $\widetilde{M}_n(c)$. A real hypersurface M^{2n-1} in $\widetilde{M}_n(c)$ is called *a ruled real hypersurface* if the holomorphic distribution $T^0M = \{X \in TM : X \perp \xi\}$ is integrable and each of its leaves (the maximal integrable manifolds) is a totally geodesic submanifold $\widetilde{M}_{n-1}(c)$ in $\widetilde{M}_n(c)$. The following lemma is known as the characterization of ruled real hypersurfaces from the viewpoint of the shape operator A (cf. [5]).

Lemma 2 ([5])**.** *Let M^{2n-1} be a real hypersurface M^{2n-1} in a nonflat complex space form $\widetilde{M}_n(c)$ $(n \geqq 2)$. Then the following three conditions are mutually equivalent*:

1. *M^{2n-1} is a ruled real hypersurface*;
2. *The shape operator A of M^{2n-1} satisfies the following equalities on the open dense subset $M_1 = \{x \in M^{2n-1} | \beta(x) \neq 0\}$ with a unit vector field U orthogonal to ξ :*

$$A\xi = \alpha\xi + \beta U, \ AU = \beta\xi, \ AX = 0$$

for an arbitrary tangent vector X orthogonal to ξ and U, where α, β are differentiable functions on M_1 by $\alpha = g(A\xi, \xi)$ and $\beta = \|A\xi - \alpha\xi\|$;

3. *The shape operator A of M^{2n-1} satisfies $g(AX, Y) = 0$ for arbitrary tangent vectors $X, Y \in T^0M$.*

3 The parallelism of the structure tensor ϕ and its generalizations In the theory of real hypersurfaces in a nonflat complex space form $\widetilde{M}_n(c)$, it is well-known that *there exists no real hypersurface whose structure tensor ϕ is parallel in $\widetilde{M}_n(c)$* (see [4]). This implies that there exists no *cosymplectic* real hypersurfaces in $\widetilde{M}_n(c)$ from the viewpoint of almost contact metric geometry (for detail, see [1]). In this section, we consider simple generalizations of the above fact.

Proposition 1. *Let M^{2n-1} be a real hypersurfaces in a nonflat complex space form $\widetilde{M}_n(c)$ $(n \geqq 2)$. Then the following statements* (1), (2) *and* (3) *hold*:

(1) *M^{2n-1} satisfies the condition $\nabla_\xi\phi = 0$ if and only if M^{2n-1} is locally congruent to a Hopf hypersurfaces in $\widetilde{M}_n(c)$;*

(2) *M^{2n-1} satisfies the condition $\nabla_X\phi = 0$ for any $X \in T^0M$ if and only if M^{2n-1} is locally congruent to a ruled real hypersurface in $\widetilde{M}_n(c)$;*

(3) *There does not exist a real hypersurface M^{2n-1} in $\widetilde{M}_n(c)$ satisfying the condition $(\nabla_X\phi)Y = (\nabla_Y\phi)X$ for any vectors X and Y on M^{2n-1}.*

Proof. (1) Suppose that M^{2n-1} has condition $\nabla_\xi\phi = 0$. By (2.2), we have

$$(\nabla_\xi\phi)X = \eta(X)A\xi - g(A\xi, X)\xi = 0. \tag{3.1}$$

for any vector $X \in TM$. Putting $X = \xi$ in (3.1), then we get $A\xi = g(A\xi, \xi)\xi$. Hence M^{2n-1} is a Hopf hypersurface in $\widetilde{M}_n(c)$. Obviously, the converse holds.

(2) Suppose that M^{2n-1} has the condition $(\nabla_X\phi)Y = 0$ for any vector $X \in T^0M$ and $Y \in TM$. By (2.2), we get $g(AX, Y) = 0$ for any vectors $X, Y \in T^0M$. From Lemma 2, M^{2n-1} is a ruled real hypersurface in $\widetilde{M}_n(c)$.

(3) Suppose that M^{2n-1} satisfies the condition $(\nabla_X\phi)Y = (\nabla_Y\phi)X$ for any vectors $X, Y \in TM$. By (2.2), we have

$$\eta(Y)AX = \eta(X)AY \tag{3.2}$$

for any vectors $X, Y \in TM$.

Now we suppose that M^{2n-1} is a non-Hopf hypersurface in $\widetilde{M}_n(c)$. Then the shape operator A forms $A\xi = \alpha\xi + \beta U$, where the function $\beta \neq 0$ and a unit vector U is orthogonal to the characteristic vector field ξ. We put $X \perp \xi$ and $Y = \xi$ in (3.2). Then we have $AX = 0$ for any vector $X \in T^0M$. Hence we can see

$$0 = g(AU, \xi) = g(U, A\xi) = \beta,$$

which is a contradiction.

Next we suppose that M^{2n-1} is a Hopf hypersurface (with $A\xi = \alpha\xi$) in $\widetilde{M}_n(c)$. We take a unit tangent vector field $V(\perp \xi)$ such that $AV = \lambda V$. By using the equation (3.2), we have $\eta(Y)AV = 0$. Putting $Y = \xi$ in this equation, we can see that $AV = \lambda V = 0$. This implies that

$$\lambda = 0. \tag{3.3}$$

Setting $X = \phi V$ and $Y = \xi$ in (3.2), we get $A\phi V = 0$. From this equation, (3.3) and Lemma 1, we obtain

$$0 = (2\lambda - \alpha)A\phi V = (\alpha\lambda + (c/2))\phi V = (c/2)\phi V \neq 0,$$

which is a contradiction. □

Remark 1. *J. T. Cho studied the condition of transversally Killing of ϕ namely, $(\nabla_X\phi)Y + (\nabla_Y\phi)X = 0$ for any $X, Y \in T^0M$ (for details, see [3]). This condition give the characterization of ruled real hypersurfaces in $\widetilde{M}_n(c)$.*

4 Statements of results Motivated by (3) of the above proposition, we investigate *the divergence of the structure tensor ϕ*. If the structure tensor ϕ is a Codazzi tensor, then we have

$$\begin{aligned}(\mathrm{div}\,\phi)X &= \sum_{i=1}^{2n-1} g((\nabla_{e_i}\phi)X, e_i) = \sum_{i=1}^{2n-1} g((\nabla_X\phi)e_i, e_i)\\ &= \mathrm{Trace}\,(\nabla_X\phi) = \nabla_X(\mathrm{Trace}\,\phi)\end{aligned}$$

for any tangent vector field $X \in TM$. Note that $\mathrm{Trace}\,\phi = 0$, we obtain the condition $\mathrm{div}\,\phi = 0$. Namely, this condition is a generalization of the condition (1.3).

Next we investigate real hypersurfaces in $\widetilde{M}_n(c)$ satisfying $\mathrm{div}\,\phi = 0$.

Theorem 1. *Let M^{2n-1} be a real hypersurface in a nonflat complex space form $\widetilde{M}_n(c)\,(n \geqq 2)$. Then M^{2n-1} satisfies the condition $\mathrm{div}\,\phi = 0$ if and only if M^{2n-1} is locally congruent to a Hopf hypersurface with constant mean curvature given by $\alpha/(2n-1)$, where $\alpha = g(A\xi, \xi)$. If M^{2n-1} has constant principal curvatures then M^{2n-1} is locally congruent to one of the following*:

(i) *A tube of radius r around a totally geodesic $\mathbb{C}P^{\ell}(c)\,(1 \leqq \ell \leqq n-2)$ in $\mathbb{C}P^n(c)$, where $0 < r < \pi/\sqrt{c}$ and $\cot(\sqrt{c}\,r/2) = \sqrt{\ell/(n-\ell-1)}$*;

(ii) *A tube of radius r around a $\mathbb{C}P^1(c) \times \mathbb{C}P^{(n-1)/2}(c)$ in $\mathbb{C}P^n(c)$, where $0 < r < \pi/(2\sqrt{c}\,), n(\geqq 5)$ is odd and $\cot(\sqrt{c}\,r/2) = (\sqrt{n-1} + \sqrt{2}\,)/\sqrt{n-3}$*;

(iii) *A tube of radius r around a complex Grassmann $\mathbb{C}G_{2,5}$ in $\mathbb{C}P^n(c)$, where $0 < r < \pi/(2\sqrt{c}\,), n = 9$ and $\cot(\sqrt{c}\,r/2) = 1 + \sqrt{2}$*;

(iv) *A tube of radius r around a Hermitian symmetric space $SO(10)/U(5)$ in $\mathbb{C}P^n(c)$, where $0 < r < \pi/(2\sqrt{c}\,), n = 15$ and $\cot(\sqrt{c}\,r/2) = \sqrt{5 + \sqrt{21}}/\sqrt{2}$.*

Proof. Suppose M^{2n-1} satisfies $\mathrm{div}\,\phi = 0$. Then we have

$$
\begin{aligned}
(\mathrm{div}\,\phi)X &= \sum_{i=1}^{2n-1} g((\nabla_{e_i}\phi)X, e_i) \\
&= \sum_{i=1}^{2n-1} g(\eta(X)Ae_i - g(Ae_i, X)\xi, e_i) \quad (\text{from } (2.2)) \\
&= \eta(X)(\mathrm{Trace}\,A) - \eta(AX) = 0
\end{aligned} \tag{4.1}
$$

for any vector $X \in TM$. Hence $g(A\xi, X) = 0$ for any vector $X \in T^0M$. This implies that M^{2n-1} is a Hopf hypersurface in $\widetilde{M}_n(c)$. Putting $X = \xi$ in (4.1), then we have

$$\mathrm{Trace}\,A = \alpha. \tag{4.2}$$

This, together with Lemma 1, yields Trace $A = \alpha =$ constant.

Conversely, we suppose that M^{2n-1} is a Hopf hypersurface with Trace $A = g(A\xi, \xi) = \alpha$. Then we can easily check that M^{2n-1} satisfies $\mathrm{div}\,\phi = 0$ (see the relation (4.1)).

Next we suppose that M^{2n-1} has constant principal curvatures. Namely, M^{2n-1} is a Hopf hypersurface with constant principal curvatures. Hence we shall check that M^{2n-1} satisfies the condition (4.2) one by one. Obviously real hypersurfaces of type (A_1) in $\mathbb{C}P^n(c)$, types (A) and (B) in $\mathbb{C}H^n(c)$ do not fulfill the condition (4.2) (see the tables of Section 2).

Let M^{2n-1} be a real hypersurface of type (A_2) in $\mathbb{C}P^n(c)$. We put $x = \cot(\sqrt{c}r/2)$, $0 < r < \pi/\sqrt{c}$. From (4.2), we have $(2n - 2\ell - 2)x - 2\ell(1/x) = 0$. This implies

$$x^2 = \frac{\ell}{n-\ell-1}.$$

Since $x > 0$, we obtain

$$x = \sqrt{\frac{\ell}{n-\ell-1}}.$$

Hence we have the case (i) of our theorem.

Let M^{2n-1} be a real hypersurface of type (B) in $\mathbb{C}P^n(c)$. We put $x = \cot(\sqrt{c}r/2)$, $0 < r < \pi/(2\sqrt{c})$. From (4.2), we have

$$\frac{1+x}{1-x} - \frac{1-x}{1+x} = 0.$$

This means $x = 0$. However, since $x > 1$, M^{2n-1} does not satisfy (4.2).

Let M^{2n-1} be a real hypersurface of type (C) in $\mathbb{C}P^n(c)$. We put $x = \cot(\sqrt{c}r/2)$, $0 < r < \pi/(2\sqrt{c})$. From (4.2), we have

$$\frac{2(1+x)}{1-x} - \frac{2(1-x)}{1+x} + (n-3)x - (n-3)\frac{1}{x} = 0.$$

This implies that $(n-3)x^4 - 2(n+1)x^2 + n - 3 = 0$. Hence we obtain

$$x^2 = \frac{n+1 \pm 2\sqrt{2n-2}}{n-3}.$$

Since $x > 1$, we have

$$x = \frac{\sqrt{n-1} + \sqrt{2}}{\sqrt{n-3}}.$$

Hence we have the case (ii) of our theorem. Similarly, we also obtain the cases (iii) and (iv) of our theorem. $\square$

Next we consider the case of 3-dimensional real hypersurfaces in $\widetilde{M}_2(c)$.

Theorem 2. *There does not exist a real hypersurface M^3 in $\widetilde{M}_2(c)$ satisfying the condition* div $\phi = 0$ *in $\widetilde{M}_2(c)$.*

Proof. We suppose that M^3 satisfies the condition div$\phi = 0$. By Theorem 1, M^3 is locally congruent to a Hopf hypersurface (with $A\xi = \alpha\xi$) in $\widetilde{M}_2(c)$ and M^3 fulfills Trace $A = \alpha$. We take a unit tangent vector field $V(\perp \xi)$ such that $AV = \lambda V$. When $(2\lambda - \alpha)(p) \neq 0$ at some point $p \in M^{2n-1}$, there exists a neighborhood $\mathcal{U}$ of p such that $2\lambda - \alpha \neq 0$ on $\mathcal{U}$. By using Lemma 1, we have

$$\lambda + \frac{\alpha\lambda + (c/2)}{2\lambda - \alpha} = 0.$$

This equation implies that λ is locally constant.

Next we consider the case $2\lambda - \alpha = 0$ at $q \in M^3$. Then there exists a neighborhood $\mathcal{V}$ of the point q such that $2\lambda - \alpha = 0$ on $\mathcal{V}$. Indeed, we suppose that there exists no neighborhood $\mathcal{V}$ of q such that $2\lambda - \alpha = 0$ on $\mathcal{V}$. Then there exists a sequence $\{q_n\}$ on M^3 such that

$$\lim_{n\to\infty} q_n = q \quad \text{and} \quad (2\lambda - \alpha)(q_n) \neq 0 \text{ for each } n.$$

The above discussion in the case $(2\lambda - \alpha)(p) \neq 0$ implies that the continuous function $2\lambda - \alpha$ is constant on some small neighborhood $\mathcal{V}_{q_n}$ of q_n for each n. Then we have $(2\lambda - \alpha)(q) \neq 0$, which is a contradiction. Hence there exists a neighborhood $\mathcal{V}$ of the point q such that $2\lambda - \alpha = 0$ on $\mathcal{V}$. Thus the function λ is locally constant. Therefore M^3 is locally congruent to a Hopf hypersurface with constant principal curvatures. We know that M^3 is one of the real hypersurfaces of types (A_1), (A_2) or (B) in $\widetilde{M}_2(c)$. However these real hypersurfaces do not satisfy the condition Trace $A = \alpha$ (see the table in Section 2). Therefore we obtain the non-existence of real hypersurfaces M^3 satisfying the condition div $\phi = 0$. $\square$

5 Applications of the discussion in Theorem 1 As a immediate consequence of Theorem 1, if both of the divergence of the structure tensor ϕ and the principal curvature α corresponding to the principal vector ξ vanish identically, then M^{2n-1} is a minimal Hopf hypersurface in $\widetilde{M}_n(c)$. However the converse does not hold. Indeed, a minimal real hypersurface of type (A_1) in $\mathbb{C}P^n(c)$ does not satisfy two conditions div $\phi = 0$ and $\alpha = 0$.

In this section, we first characterize the class of minimal Hopf hypersurfaces in $\widetilde{M}_n(c)$ by the modification of the condition (1.4). This is an application of the discussion in Theorem 1.

Proposition 2. *Let M^{2n-1} be a real hypersurface in $\widetilde{M}_n(c)$ $(n \geqq 2)$. Then M^{2n-1} is locally congruent to a minimal Hopf hypersurface in $\widetilde{M}_n(c)$ if and only if M^{2n-1} satisfies the condition*

$$(\mathrm{div}\,\phi)X = -\alpha\eta(X) \tag{5.1}$$

for any vector $X \in TM$.

Proof. Suppose that M^{2n-1} satisfies the condition (5.1). By the calculation (4.1), we have

$$\eta(X)(\mathrm{Trace}\,A) - \eta(AX) = -\alpha\eta(X) \tag{5.2}$$

for any $X \in TM$. This means that $g(A\xi, X) = 0$ for any $X \in T^0M$. Hence M^{2n-1} is a Hopf hypersurface in $\widetilde{M}_n(c)$. Putting $X = \xi$ in (5.2) we get Trace $A = 0$. So we can see that M^{2n-1} is a minimal Hopf hypersurface in $\widetilde{M}_n(c)$. Clearly, the converse holds by the calculation (4.1). □

Remark 2. *By a direct calculation, real hypersurfaces of types* (A_1), (A_2), (B), (C), (D) *and* (E) *in $\mathbb{C}P^n(c)$ whose radius r satisfies the following table are known as minimal Hopf hypersurfaces with constant principal curvatures in $\widetilde{M}_n(c)$.*

	(A_1)	(A_2)	(B)	(C)	(D)	(E)
$\cot\frac{\sqrt{c}r}{2}$	$\frac{1}{\sqrt{2n-1}}$	$\sqrt{\frac{(2\ell+1)}{(2n-2\ell-1)}}$	$\sqrt{n}+\sqrt{n-1}$	$\frac{\sqrt{n}+\sqrt{2}}{\sqrt{n-2}}$	$\sqrt{5}$	$\frac{\sqrt{15}+\sqrt{6}}{3}$

B. Y. Chen studied the maximal Ricci curvature of real hypersurfaces M^{2n-1} in $\mathbb{C}H^n(c)$ (see [2]). Now we denote by $\overline{Ric}$ the maximal Ricci curvature function on M^{2n-1}, namely

$$\overline{Ric}(p) = \mathrm{Max}\{S(X,X) : X \in T_pM^{2n-1}, \|X\| = 1\}, \quad p \in M^{2n-1},$$

where S is the Ricci tensor of M^{2n-1}. In [2], he showed that every real hypersurface in $\mathbb{C}H^n(c)$ satisfies the following inequality:

$$\overline{Ric} \leqq \frac{1}{4}(\mathrm{Trace}\,A)^2 + \frac{c}{2}(n-1). \tag{5.3}$$

In particular, we can characterize real hypersurfaces which satisfy the equality case of (5.3).

Proposition 3. *Let M^{2n-1} be a real hypersurface in $\mathbb{C}H^n(c)$ $(n \geqq 2)$. Then the following three conditions are mutually equivalent*:

(1) *M^{2n-1} satisfies the condition $(\mathrm{div}\,\phi)X = \eta(AX)$ for any tangent vector field X on M^{2n-1};*

(2) *M^{2n-1} satisfies the condition $\overline{Ric} = (1/4)(\mathrm{Trace}\,A)^2 + (c/2)(n-1)$;*

(3) *M^{2n-1} is locally congruent to a Hopf hypersurface with constant mean curvature is given $2\alpha/(2n-1)$.*

Proof. (2) ⇔ (3). See [2].
(1) ⇔ (3). We can prove it by using the same discussion of Theorem 1. □

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References

[1] D. E. Blair, *Riemannian geometry of contact and symplectic manifolds*, Second edition, Progress in Mathematics **203**, (Birkhäuser, Boston, Basel, Berlin, 2010).

[2] B. Y. Chen, *Ricci curvature of real hypersurfaces in complex hyperbolic space*, Arch. Math. (Brno) **38** (2002), 73–80.

[3] J. T. Cho, *Notes on real hypersurfaces in a complex space form*, Bull. Korean Math. Soc. **52** (2015), No. 1, 335–344.

[4] T. Itoh and S. Maeda, *Characterization of totally η-umbilic real hypersurfaces in nonflat complex space forms by some inequality*, Proc. Japan Acad. Ser. A Math. Sci. **80**, No. 5 (2004), 61–64.

[5] R. Niebergall and P. J. Ryan, *Real hypersurfaces in complex space forms*, in: Tight and Taut submanifolds, T.E. Cecil and S.S. Chern (eds.), Cambridge Univ. Press, 1998, 233–305.

[6] R. Sharma, *Notes on contact metric manifolds*, Ulam Quarterly **3** No. 1, (1995), 27–33.