Some topological structures and related groups on digital plane *

Arafa A. Nasef, H. Maki and Abd El Fattah A. El Atik

Received July 28, 2020 ; revised
October 28,2020. January 15,2021. May 2,2021. June 9,2021

Abstract. The aim of the present paper is devoted to discussing some more properties of β-irresolute mappings, contra β-irresolute mappings and two weak homeomorphisms such as βc-homeomorphisms and contra βc-homeomorphisms. Further, we investigate some new groups related to the mappings above and some examples of them on the digital plane and we construct the concept of $\beta_{(2)}$-open sets of the digital plane.

1 Introduction and preliminaries Abd El Monsef el al. [1] and Andrijević [3] introduced independently the concept of β-open sets [1] and semi-preopen sets [3], respectively. Let (X, τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $Cl(A)$ and $Int(A)$, respectively.

Definition 1.1 A subset A of a topological space (X, τ) is called a *β-open set* [1] (or *semi-preopen set* [3]) if $A \subseteq Cl(Int(Cl(A)))$ holds in (X, τ). The complement of a β-open (or semi-preopen) set is called *β-closed* (or *semi-preclosed*).

Throughout the present paper, we use the terminology due to [1] for the naming of the above set, that is, β-open sets, β-closed sets. The β-closure of a subset E of a topological space (X, τ) is defined by $\beta Cl(E) := \bigcap\{F : E \subseteq F, F \text{ is } \beta\text{-closed in } (X, \tau)\}$ and it is the smallest β-closed set containing E. And $\beta Cl(A) = A$ holds if and only if A is β-closed in (X, τ). We recall some importance properties of β-open sets in Section 4 (Theorem 4.1).

In the present paper, we use the following notation and other notation (cf. Notation 3.3, Notation 5.5, Definition 5.12, Remark 5.13, Proposition 5.16(i), Propositon 5.18(i)).

Definition 1.2 Let (X, τ) be a topological space.
$\beta O(X, \tau) = SPO(X, \tau) := \{B : B \text{ is } \beta\text{-open in } (X, \tau)\}$ (cf. [1, Defintion 1.1], [3]),
$\beta C(X, \tau) = SPC(X, \tau) := \{F : F \text{ is } \beta\text{-closed in } (X, \tau)\text{, i.e. } Int(Cl(Int(F))) \subseteq F\}$ [1], [3],
$SO(X, \tau) := \{G : G \text{ is semi-open in } (X, \tau)\text{, i.e. } G \subseteq Cl(Int(G))\}$ [26],
$SC(X, \tau) := \{F : F \text{ is semi-closed in } (X, \tau)\text{, i.e. } Int(Cl(F)) \subseteq F\}$ [8].

One of the purposes of this paper is to investigate some group structures of the new families of mappings, i.e., $\mathcal{G}(X, X \setminus H; \tau) :=$ *con-*$\beta ch(X, X \setminus H; \tau) \cup \beta ch(X, X \setminus H; \tau)$ and $\mathcal{G}_0(X, X \setminus H; \tau)$:=*con-*$\beta ch_0(X, X \setminus H; \tau) \cup \beta ch_0(X, X \setminus H; \tau)$, where $H \subset X$ with $H \neq \emptyset$ (cf. Notation 3.3, Theorems 3.5,3.6 and Theorem 4.7). If we assume $X = H$ (resp. *con-*$\beta ch(X, X \setminus H; \tau) = \emptyset$) in Theorem 3.5(i), then we have the property [4, Theorem 4.4(i)] due to S.C. Arora et al. (resp. [40, Theorem 2.2] due to Sanjay Tahiliani). And, if *con-*$\beta ch(X, X \setminus H; \tau) = \emptyset =$ *con-*$\beta ch_0(X, X \setminus H; \tau)$ are assumed in Theorem 4.7(i), then the properties [40, Theorem 2.7(ii)] etc are obtained.

---

*2010 Math. Subject Classification — : 54C08, 64C40, 54J05.
Key words and phrases — :β-open sets, β-irresolute mappings, contra-β-irresolute mappings, βr-homeomorphisms, contra-βr-homeomorphisms, digital plane, $\beta_{(2)}$-open sets.

In the last Section 5, a characterization (cf. Corollary 5.8) of β-open sets of the digital plane $(\mathbb{Z}^2, \kappa^2)$ will be studied and we prove that $(*)$ $con\text{-}\beta ch(\mathbb{Z}^2;\kappa^2) = \emptyset$ (cf. Corollary 5.11(ii)$'$) and so $\mathcal{G}(\mathbb{Z}^2;\kappa^2) = \beta ch(\mathbb{Z}^2;\kappa^2)$ (cf. Corollary 5.11(iii)$'$). Therefore, we define and construct the new concept, say $\beta_{(2)}$-open sets in a set H, where $H \subseteq \mathbb{Z}^2$ with $|H| \geq 2$ (cf. Definition 5.15). And, using such $\beta_{(2)}$-open sets, we construct new groups, say $\beta_{(2)}ch(H;\kappa^2|H) \cup con\text{-}\beta_{(2)}ch(H;\kappa^2|H)$ and $p.\beta_{(2)}ch(H;\kappa^2|H) \cup con\text{-}p.\beta_{(2)}ch(H;\kappa^2|H)$ etc (cf. Definition 5.21, Theorem 5.25). As examples, it is obtained that some motions, say ρ_{45} and $(\rho_{45})^{-1}$, are elements of $con\text{-}p.\beta_{(2)}ch(U;\kappa^2|U)$ and so $con\text{-}p.\beta_{(2)}ch(U;\kappa^2|U) \neq \emptyset$, where U is the smallest open set containing the origin $(0,0)$ in $(\mathbb{Z}^2, \kappa^2)$ (cf. $(*)$ above and Notation 5.26, Example 5.27; Definition 5.20, Definition 5.21).

2 Contra-β-irresolute mappings and β-irresolute mappings.

Let (X,τ), (Y,σ) and (Z,η) be topological spaces.

Definition 2.1 A mapping $f : (X,\tau) \to (Y,\sigma)$ is said to be

(i) *β-continuous* [1] if $f^{-1}(V)$ is a β-closed set of (X,τ) for each closed set V of (Y,σ),

(ii) *perfectly continuous* [37] if $f^{-1}(V)$ is clopen in (X,τ) for each open set V of (Y,σ),

(iii) *contra-continuous* [13] if $f^{-1}(V)$ is closed in (X,τ) for each open set V of (Y,σ),

(iv) *contra-β-continuous* ([7], [18]) if $f^{-1}(V) \in \beta C(X,\tau)$ for each open set V of (Y,σ),

(v) *strongly contra-β-continuous* if f is a contra-β-continuous mapping such that the inverse image of such nonempty open set of (Y,σ) has an interior point,

(vi) B-*continuous* [43] if $f^{-1}(V)$ is a B-set of (X,τ) for each nonempty open set V of (Y,σ), where the B-set is the intersection of an open set and a semi-closed set of (X,τ) (this is defined by [43]),

(vii) B*-*continuous* [12] (cf. (vi)) if $f^{-1}(V)$ contains a nonempty B-set of (X,τ) for each nonempty open set V of (Y,σ),

(viii) *strongly β-closed* [17] if $f(G)$ is β-closed in (Y,σ) for each β-closed set G of (X,τ).

Definition 2.2 A mapping $f : (X,\tau) \to (Y,\sigma)$ is said to be

(i) *irresolute* [8, Definition 1.1] if $f^{-1}(U) \in SO(X,\tau)$ for every set $U \in SO(Y,\sigma)$,

(ii) *β-irresolute* [36] if $f^{-1}(U) \in \beta O(X,\tau)$ for every set $U \in \beta O(Y,\sigma)$,

(iii) *contra-β-irresolute* [5] if $f^{-1}(U) \in \beta C(X,\tau)$ for every set $U \in \beta O(Y,\sigma)$ (cf. Remark 2.9(ii)),

(iv) *perfectly contra-β-irresolute* if $f^{-1}(V)$ is β-clopen in (X,τ) for each set $V \in \beta O(Y,\sigma)$,

(v) *contra-irresolute* [5] if $f^{-1}(U) \in SC(X,\tau)$ for every set $U \in SO(Y,\sigma)$,

(vi) *perfectly contra-irresolute* [5] if $f^{-1}(U)$ is semi-open and semi-closed in (X,τ) for each set $U \in SO(Y,\sigma)$.

Theorem 2.3 *A mapping $f : (X,\tau) \to (Y,\sigma)$ is* B**-continuous if one of the following conditions* (1) *and* (2) *is satisfied,*

(1) *f is a strongly contra-β-continuous mapping* (cf. Definition 2.1(v)).

(2) *f is an onto and* B*-continuous mapping* (cf. Definition 2.1(vi)).

Proof. Let V be a nonempty open set of (Y,σ). Under the case of the assumption (1), we have that $f^{-1}(V) \in \beta C(X,\tau)$ and $Int(f^{-1}(V)) \neq \emptyset$, and so $\emptyset \neq Int(f^{-1}(V)) = X \cap Int(f^{-1}(V)) \subseteq f^{-1}(V)$. Hence $f^{-1}(V)$ contains a nonempty B-open set U, say $U := X \cap Int(f^{-1}(V))$. Indeed, $X \in SC(X,\tau)$, $Int(f^{-1}(V)) \in \tau$ and $U \neq \emptyset$. Thus, f is B*-continuous, under the assumption (1). Under the case of the assumption (2), we have that $\emptyset \neq f^{-1}(V)$ and $f^{-1}(V)$ is a B-set. And so $f^{-1}(V)$ contains a nonempty B-set $f^{-1}(V)$. Thus, f is B*-continuous, under the assumption (2). $\square$

Remark 2.4 The following diagram shows implications among several mappings defined above, where $p \to q$ (resp. $p \not\leftrightarrow q$) means that p implies q (resp. p and q are independent). The implications are not reversible and the independence is explained (cf. Remark 2.5 below).

$$\begin{array}{ccccc}
 & & \textbf{contra-continuity} & & \\
 & \swarrow & & \searrow & \\
\textbf{contra-}\boldsymbol{\beta}\textbf{-continuity} & \not\leftrightarrow & \textbf{B-continuity} & & \textbf{onto and B-continuity} \\
\uparrow & & & & \downarrow \\
\textbf{strongly contra-}\boldsymbol{\beta}\textbf{-continuity} & & \longrightarrow & & \textbf{B}^{*}\textbf{-continuity}
\end{array}$$

Remark 2.5 (i) Let $(\mathbb{R}, E)$ be the real line with the Euclidean topology E. The mappings $f, 1_{\mathbb{R}} : (\mathbb{R}, E) \to (\mathbb{R}, E)$ of (i-1) below are seen in [14].
(i-1) Let $f : (\mathbb{R}, E) \to (\mathbb{R}, E)$ be a mapping defined by $f(x) = [x]$, where $[x]$ is the Gaussian symbol. Then f is contra-β-continuous (cf. Definition 2.1(iv)). However, f is not contra-continuous, because for an open interval $(\frac{1}{2}, \frac{3}{2})$, $f^{-1}((\frac{1}{2}, \frac{3}{2})) = [1, 2)$ is not closed in $(\mathbb{R}, E)$.
(i-2) The identity mapping $1_{\mathbb{R}} : (\mathbb{R}, E) \to (\mathbb{R}, E)$ is B-continuous (cf. Definition 2.1(vi)) but not contra-β-continuous, since the inverse image of each singleton is not β-open. Moreover, $1_{\mathbb{R}}$ is not contra-continuous.

(ii) The mapping $f : (X, \tau) \to (X, \tau)$ is contra-β-continuous, but f is not B-continuous. Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a, b\}\}$. Then we have $\beta C(X, \tau) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{b, c\},$-$\{a, c\}, X\}$ and $SC(X, \tau) = \{\emptyset, \{c\}, X\}$. We define the mapping $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = a$, $f(b) = c$ and $f(c) = b$.

(iii) The converse of Theorem 2.3 under the assumption (1) is not reversible. Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. Let $f : (X, \tau) \to (X, \tau)$ be a mapping defined by $f(a) = b$, $f(b) = c$ and $f(c) = a$. Then since $\beta C(X, \tau) = SC(X, \tau) = P(X) \backslash \{\{a, b\}\}$, we show f is B-continuous and onto. By Theorem 2.3 under the assumption (2), it is obtained that f is B*-continuous. This mapping f is contra-β-continuous, but $Int(f^{-1}(\{a\})) = Int(\{c\}) = \phi$ hold. And so f is not strongly contra-β-continuous.

(iv) The converse of Theorem 2.3 under the assumption (2) is not reversible. The mapping $f : (X, \tau) \to (X, \tau)$ defined in (ii) above is not B-continuous (cf. (ii)). But f is B*-continuous, because $\{c\}$ and X are the nonempty B-sets.

(v) The contra-β-continuous mapping $f : (X, \tau) \to (X, \tau)$ of (ii) above is not strongly contra-β-continuous (cf. Definition 2.1(v), because $Int(f^{-1}(\{a, b\})) = \emptyset$.

Remark 2.6 (i) Let $X = Y = \{a, b\}$ and $\tau = \{X, \emptyset, \{a\}\}$ and $\sigma = \{Y, \emptyset, \{b\}\}$. Then the identity mapping $1_X : (X, \tau) \to (Y, \sigma)$ is a contra-β-continuous mapping but not β-continuous.

(ii) The identity mapping $1_{\mathbb{R}} : (\mathbb{R}, E) \to (\mathbb{R}, E)$ of Remark 2.5(i)(i-2) is β-continuous but not contra-β-continuous.

Remark 2.7 The following properties are well known.

(i) If $f : (X, \tau) \to (Y, \sigma)$ is contra-β-irresolute and $g : (Y, \sigma) \to (Z, \eta)$ is β-continuous, then the composition $g \circ f : (X, \tau) \to (Z, \eta)$ is contra-β-continuous (cf. [7, Theorem 2.18]).

(ii) ([4, Theorem 2.3(iv)]) Every homeomorphism is β-irresolute.

Remark 2.8 (i) By the following examples (i-1) and (i-2), it is shown that the contra-β-irresoluteness and β-irresoluteness are independent. Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset,$ -$\{a\}, \{a, b\}\}$. Then

(i-1) The identity mapping on (X, τ) above is β-irresolute but not contra-β-irresolute.

(i-2) Let $f : (X, \tau) \to (X, \tau)$ be a mapping defined by: $f(a) = b = f(b)$, $f(c) = a$. Then f is contra-β-irresolute but not β-irresolute.

(ii) In general, for any topological space (X, τ), the identity mapping $1_X : (X, \tau) \to (X, \tau)$ is contra-β-irresolute if and only if $\beta O(X, \tau) = \beta C(X, \tau)$ holds. And, 1_X on any topological space (X, τ) is β-irresolute.

Remark 2.9 (i) Every contra-β-irresolute mapping is contra-β-continuous, but it is shown that its converse is not true, by the following example. Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. Let $f : (X, \tau) \to (X, \tau)$ be a mapping defined by $f(a) = c$, $f(b) = a$, $f(c) = b$. One can deduce that f is contra-β-continuous, but it is not contra-β-irresolute.

(ii) For a mapping $f : (X, \tau) \to (Y, \sigma)$, f is contra-β-irresolute if and only if the inverse image $f^{-1}(F)$ of each β-closed set F of (Y, σ) is β-open in (X, τ).

(iii) For a mapping $f : (X, \tau) \to (Y, \sigma)$, f is β-irresolute if and only if the inverse image $f^{-1}(F)$ of each β-closed set F of (Y, σ) is β-closed in (X, τ).

Remark 2.10 (i) The following diagram of implications is well known.

Contra-irresolute $\leftarrow$ **Perfectly contra-irresolute** $\rightarrow$ **Irresolute**.

We have also the following diagram of implications.

Contra-β-irresolute $\leftarrow$ **Perfectly contra-β-irresolute** $\rightarrow$ **β-irresolute**;

and they are not reversible (cf. Remark 2.8(i) above and Remark 2.11 below).

(ii) In the implications above, the irresoluteness (resp. contra-irresoluteness, perfectly contra-irresoluteness) and the β-irresoluteness (resp. contra-β-irresoluteness, perfectly contra-β-irresoluteness) are independent (cf. (a), (b), (c) below).

Let $X = \{a, b, c\}$. We consider the topologies on X: $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$, $\tau_1 = \{X, \emptyset, \{a\}, \{a, b\}\}$, $\tau_2 = \{X, \emptyset, \{c\}, \{a, b\}\}$ and $\tau_3 = \{X, \emptyset\}$. We have the following: $SO(X, \tau) = \beta O(X, \tau) = P(X) \setminus \{\{c\}\}$, $SO(X, \tau_1) = \beta O(X, \tau_1) = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$, $SO(X, \tau_2) = \tau_2$, $\beta O(X, \tau_2) = P(X)$, $SO(X, \tau_3) = \{\emptyset, X\}$, $\beta O(X, \tau_3) = P(X)$.

(a)(a-1) Define a mapping $f : (X, \tau) \to (X, \tau_2)$ as follows: $f(a) = a$, $f(b) = c$ and $f(c) = b$. Then f is irresolute; f is not β-irresolute.

(a-2) Let $f : (X, \tau_3) \to (X, \tau)$ be the identity mapping. Then f is β-irresolute; f is not irresolute.

(b)(b-1) Let $f : (X, \tau_2) \to (X, \tau_1)$ be the identity mapping. Then f is contra-β-irresolute, f is not contra-irresolute.

(b-2) Define a mapping $f : (X, \tau_1) \to (X, \tau_2)$ as follows: $f(a) = a$, $f(b) = a$, $f(c) = b$. Then f is contra-irresolute, f is not contra-β-irresolute.

(c)(c-1) Let $f : (X, \tau_3) \to (X, \tau_2)$ be the identity mapping. Then f is perfectly contra β-irresolute, f is not perfectly contra-irresolute.

(c-2) Define a mapping $f : (X, \tau) \to (X, \tau_2)$ as follows: $f(a) = c$, $f(b) = a$, $f(c) = b$. Then f is perfectly contra-irresolute, f is not perfectly contra-β-irresolute.

Remark 2.11 We have a decomposition of perfectly contra-β-irresolute mappings. The following conditions (1) and (2) are equivalent: (1) $f : (X, \tau) \to (Y, \sigma)$ is perfectly contra-β-irresolute; (2) $f : (X, \tau) \to (Y, \sigma)$ is contra-β-irresolute and β-irresolute.

3 Groups $\mathcal{G}(X, X \setminus H; \tau)$ and $\mathcal{G}_0(X, X \setminus H; \tau)$.

Main purposes of the present Section 3 are to prove Theorems 3.5 and Theorem 3.6 (cf. Notation 3.3).

Definition 3.1 Let (X, τ) and (Y, σ) be topological spaces.

(i) A mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be

(i-1) a *βc-homeomorphism* ([4, Definition 3.1(ii)]) if f is a β-irresolute bijection and f^{-1} is β-irresolute,

(i-2) a *contra-βc-homeomorphism* if f is a contra-β-irresolute ([6], [4, Definition 4.1]) bijection and f^{-1} is contra-β-irresolute (cf. Definition 2.2(iii)).

(ii) (ii-1) $\beta ch(X; \tau) := \{f |\ f : (X, \tau) \to (X; \tau)$ is a βc-homeomorphism$\}$ ([4, notation (3) after Definiton 3.1],

(ii-2) *con-$\beta ch(X; \tau)$*$:=\{f |\ f : (X, \tau) \to (X, \tau)$ is a contra-βc-homeomorphism$\}$ ([4, Definition 4.3(1)]).

(iii) $h(X; \tau) := \{f |\ f : (X, \tau) \to (X, \tau)$ is a homeomorphism$\}$ (e.g., [4, notation (3) after Definiton 3.1]).

(iv) ([40, Definition 2.1]) Let H be a nonempty subset of X.
(iv-1) $\beta ch(X, X \setminus H;\tau) := \{a|\ a \in \beta ch(X;\tau)$ and $a(X \setminus H) = X \setminus H\}$.
(iv-1)$'$ con-$\beta ch(X, X \setminus H;\tau) := \{a|\ a \in con$-$\beta ch(X;\tau)$ and $a(X \setminus H) = X \setminus H\}$ (cf. (ii)(ii-2)).
(iv-2) $\beta ch_0(X, X \setminus H;\tau) := \{a|\ a \in \beta ch(X, X \setminus H;\tau)$ and $a(x) = x$ for every point $x \in X \setminus H\}$.
(iv-2)$'$ con-$\beta ch_0(X, X \setminus H;\tau) := \{a|\ a \in con$-$\beta ch(X, X \setminus H;\tau)$ and $a(x) = x$ for every point $x \in X \setminus H\}$, where $H \neq X$ (cf. (ii)(ii-2) above).

Remark 3.2 (i) In 2010, N. Arora et al. [4, Theorem 4.4(i)] proved that $\beta ch(X;\tau) \cup con$-$\beta ch(X;\tau)$ forms a group under the composition of mappings.
(ii) In 2019, Sanjay Tahiliani [40, Theorem 2.2] proved that $\beta ch(X, X \setminus H;\tau)$ and $\beta ch_0(X, X \setminus H;\tau)$ form groups under the composition of mappings, where H is a subset of X, and Sanjay Tahiliani proved important propertes [40, Theorem 2.7].

Notation 3.3 Let (X,τ) be a topological space and $H \subseteq X$ with $H \neq \emptyset$.
(i) $\mathcal{G}(X;\tau) := \beta ch(X;\tau) \cup con$-$\beta ch(X;\tau)$.
(ii) $\mathcal{G}(X, X \setminus H;\tau) := con$-$\beta ch(X, X \setminus H;\tau) \cup \beta ch(X, X \setminus H;\tau)$.
(ii)$'$ $\mathcal{G}_0(X, X \setminus H;\tau) := con$-$\beta ch_0(X, X \setminus H;\tau) \cup \beta ch_0(X, X \setminus H;\tau)$.

Remark 3.4 Let us consider especially the case where that $H = X$ in Notation 3.3(ii) above. Then, we have that $\mathcal{G}(X, X \setminus X;\tau) = \mathcal{G}(X;\tau)$ holds. (cf. Definition 3.1(ii),(iv)).

Theorem 3.5 *Let H be a nonempty subset of (X,τ) and $\mathcal{G}(X, X \setminus H;\tau)$, $\mathcal{G}_0(X, X \setminus H;\tau)$ and $\mathcal{G}(X;\tau)$ be the families defined in* Notation 3.3 *above, respectively. Then,*
(i) *$\mathcal{G}(X, X \setminus H;\tau)$ forms a group under the composition of mappings.*
(i)$'$*$\mathcal{G}_0(X, X \setminus H;\tau)$ forms a subgroup of $\mathcal{G}(X, X \setminus H;\tau)$, where $H \neq X$.*
(ii) *The group $\beta ch_0(X, X \setminus H;\tau)$ forms a subgroup of $\mathcal{G}_0(X, X \setminus H;\tau)$, where $H \neq X$.*
(iii) *The groups $\mathcal{G}(X, X \setminus H;\tau)$ and $\mathcal{G}_0(X, X \setminus H;\tau)$ (where $H \neq X$) are subgroups of $\mathcal{G}(X;\tau)$* (cf. Notation 3.3, [4, Theorem 4.4(i)]).

Proof. Throghout the present proofs of (i),(i)$'$ and (ii), let us denote: $\mathcal{G} := \mathcal{G}(X, X \setminus H;\tau)$ and $\mathcal{G}_0 := \mathcal{G}_0(X, X \setminus H;\tau)$. **(i)** A binary operation $w : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is well defined by $w(a, b) := b \circ a$, where $b \circ a$ is the composite function of the functions a and b. Indeed, it is shown by the following four cases.
Case 1 (resp. Case 1$'$) a (resp. b) $\in \beta ch(X, X \setminus H;\tau)$ and b (resp. a) $\in con$-$\beta ch(X, X \setminus H;\tau)$**.** For the present case, $b \circ a : (X,\tau) \to (X,\tau)$ is a *contra*-β-irresolute bijection such that $(b \circ a)^{-1}$ is also *contra*-β-irresolute and $(b \circ a)(X \setminus H) = X \setminus H$ (cf. [4, Lemma 4.2(i-2)]). And so, $w(a, b) \in con$-$\beta ch(X, X \setminus H) \subseteq \mathcal{G}$.
Case 2 (resp. Case 3) $a, b \in con$-$\beta ch(X, X \setminus H;\tau)$ (resp. $\beta ch(X, X \setminus H;\tau)$)**.** For the present case, $b \circ a : (X,\tau) \to (X,\tau)$ is a β-irresolute bijection such that $(b \circ a)^{-1}$ is also β-irresolute and $(b \circ a)(X \setminus H) = X \setminus H$ (cf. [4, Lemma 4.2(i-1)]). And so, $w(a, b) \in \beta ch(X, X \setminus H) \subseteq \mathcal{G}$.

Thus, the binary operation $w : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is well defined. For all elements $a, b, c \in \mathcal{G}$, $w(w(a, b), c) = w(a, w(b, c))$ holds. The identity element $e \in \mathcal{G}$ is well defined by the identity mapping $1_X : (X,\tau) \to (X,\tau)$, i.e., $e := 1_X \in \mathcal{G}$; and so $w(e, a) = a = w(a, e)$ hold for all element $a \in \mathcal{G}$. The inverse element of an element $a \in \mathcal{G}$ is well defined by the inverse mapping a^{-1} of $a : (X,\tau) \to (X,\tau)$ and so $w(a, a^{-1}) = e = w(a^{-1}, a)$ hold for all element $a \in \mathcal{G}$. And hence $(\mathcal{G}, w)$ forms a group under the composition of mappings, i.e., $\mathcal{G}$ is a group. **(i)$'$** Since $1_X \in \beta ch_0(X, H;\tau)$, we have the following: $\mathcal{G}_0 \neq \emptyset$. For any element $a, b \in \mathcal{G}_0$ and the binary operation $w : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, it is seen that $w(a, b^{-1}) = b^{-1} \circ a \in \mathcal{G}$ and $(b^{-1} \circ a)(x) = b^{-1}(x) = x$ for every point $x \in X \setminus H$; and so $w(a, b^{-1}) \in \mathcal{G}_0$. **(ii)** By (i)$'$, the subgroup $\mathcal{G}_0$ has the binary operation $w|\mathcal{G}_0$. Let $a, b \in \beta ch_0(X, X \setminus H;\tau)$. Then, $1_X \in \beta ch_0(X, X \setminus H;\tau) \neq \emptyset$ and $(w|\mathcal{G}_0)(a, b^{-1}) = b^{-1} \circ a \in \beta ch_0(X, X \setminus H;\tau) \subseteq \mathcal{G}_0$; and so

$\beta ch_0(X, X \setminus H; \tau)$ is a subgroup of $\mathcal{G}_0$. **(iii)** The group $\mathcal{G}(X;\tau)$ (cf. Notation 3.3(i)) forms a group under the compositon of mappings ([4, Theorem 4.4(i)]). And, $\mathcal{G}$ forms a group by the composition of mappings (cf. (i)) such that $\mathcal{G} \subseteq \mathcal{G}(X;\tau)$ (cf. Notation 3.3). Thus, $\mathcal{G}$ is a subgroup of $\mathcal{G}(X,\tau)$. And using (i)$'$, $\mathcal{G}_0$ is a subgroup of $\mathcal{G}(X,\tau)$. □

Theorem 3.6 (i) *If $f : (X,\tau) \to (Y,\sigma)$ is a contra-βc-homeomorphism* (cf. Definition 3.1(i)(i-2))*, then f induces also an isomorphism $f_* : \mathcal{G}(X, X \setminus H;\tau) \to \mathcal{G}(Y, Y \setminus f(H);\sigma)$, where f_* is defined by $f_*(a) := f \circ a \circ f^{-1}$.*

(ii) *If $f : (X,\tau) \to (Y,\sigma)$ is a βc-homeomorphism* (cf. Definition 3.1(i))*, then f induces an isomorphism $f_* : \mathcal{G}(X, X \setminus H;\tau) \to \mathcal{G}(Y, Y \setminus f(H);\sigma)$, where f_* is defined in* (i) *above.*

(iii) *Suppose one of the following properties* (a), (b) *below on mappings $f : (X,\tau) \to (Y,\sigma)$ and $g : (Y,\sigma) \to (Z,\eta)$:* (a) *$f$ and g are contra-βc-homeomorphisms,* (b) *f and g are βc-homeomorphisms.*
Then, $(g \circ f)_* = g_* \circ f_* : \mathcal{G}(X, X \setminus H;\tau) \to \mathcal{G}(Z, Z \setminus g(f(H);\eta))$ holds and $(1_X)_* = 1 : \mathcal{G}(X, X \setminus H;\tau) \to \mathcal{G}(X, X \setminus H;\tau)$ is the identity isomorphism, where $1_X : (X,\tau) \to (X,\tau)$ on the identity and 1 is the identity on $\mathcal{G}(X, X \setminus H)$.

Proof. **(i)** Under the assumption that f is a contra-βc-homeomorphism, it is proved that f_* is an isomorphism between the groups. Indeed, we have the following properties: **(1)** mapping $f_* : \mathcal{G}(X, X \setminus H;\tau) \to \mathcal{G}(Y, Y \setminus f(H);\sigma)$ is well defined; **(2)** f_* is a homomorphism; **(3)** f_* is a bijection. **Proof of (1)** Let $a \in \mathcal{G}(X, X \setminus H;\tau)$. We first have the following: $f_*(a)(Y \setminus f(H)) = Y \setminus f(H)$ holds. And, we consider the following two cases.
Case 1 $a \in \beta ch(X, X \setminus H;\tau)$ **(resp. Case 2** $a \in$ *con-βch*$(X, X \setminus H;\tau)$**).** Let $B \in \beta O(Y,\sigma)$. Then, for the Case 1 (resp. Case 2), we have the following: $(f_*(a))^{-1}(B) = f(a^{-1}(f^{-1}(B))) \in \beta O(Y;\sigma)$ (resp. $\beta C(Y,\sigma)$) and so $f_*(a) : (Y,\sigma) \to (Y,\sigma)$ is β-irresolute (resp. contra-β-irresolute) bijection. Moreover, we have the following: $f_*(a)(B) \in \beta O(Y,\sigma)$ (resp. $\beta C(Y,\sigma)$). Then, $(f_*(a))^{-1} : (Y,\sigma) \to (Y,\sigma)$ is β-irresolute (resp. contra-β-irresolute). Thus, for the Case 1 (resp. Case 2), we prove that $f_*(a) : (Y,\sigma) \to (Y,\sigma)$ is a βc-homeomorhism (resp. *contra*-βc-homeomorphism) such that $f_*(a)(Y \setminus f(H)) = Y \setminus f(H)$. Namely, using Notation 3.3, we have that $f_*(a) \in \mathcal{G}(Y, Y \setminus f(H);\sigma)$.
Proof of (2) Let a and b be elements of $\mathcal{G}(X, X \setminus H;\tau)$. Then, we have the following: $f_*(w_X(a,b)) = (f \circ b \circ f^{-1}) \circ (f \circ a \circ f^{-1}) = w_Y(f_*(a), f_*(b))$.
Proof of (3) Let $a, b \in \mathcal{G}(X, X \setminus H;\tau)$ such that $f_*(a) = f_*(b)$. Then, $f \circ a \circ f^{-1} = f \circ b \circ f^{-1}$ and so $a = b$. Let $d \in \mathcal{G}(Y, Y \setminus f(H);\sigma)$. Then, it is proved that $f^{-1} \circ d \circ f \in \mathcal{G}(X, X \setminus H;\tau)$ and $f_*(f^{-1} \circ d \circ f) = d$. **(ii)** Under the assumption that f is a βc-homeomorphism, it is proved similarly that of (i) that f_* is isomorphism between the groups. **(iii)** By definitions and (i) (resp. (ii)), the present properties are shown. □

Corollary 3.7 (i) (resp. (ii)) *If $f : (X,\tau) \to (Y,\sigma)$ is a contra-βc-homeomorphism* (resp. *βc-homeomorphism*)*, then f induces an isomorphism $f_* : \mathcal{G}(X;\tau) \to \mathcal{G}(Y;\sigma)$, where f_* is defined by $f_*(a) := f \circ a \circ f^{-1}$ for any $a \in \mathcal{G}(X;\tau)$.*

(iii) *Suppose one of the following properties* (a), (b) *below on mappings $f : (X,\tau) \to (Y,\sigma)$ and $g : (Y,\sigma) \to (Z,\eta)$,* (a) *$f$ and g are contra-βc-homeomorphisms,* (b) *f and g are βc-homeomorphisms.*
Then, we have the following properties (1), (2) *and* (3) *on f_*, g_*.*
(1) *$(g \circ f)_* = g_* \circ f_* : \mathcal{G}(X;\tau) \to \mathcal{G}(Z;\eta)$.* (2) *$(1_X)_* = 1 : \mathcal{G}(X;\tau) \to \mathcal{G}(X;\tau)$ is the identity isomorphism, where $1_X : (X,\tau) \to (X,\tau)$ is the identity.*
(3) (3-1) *f_*(con-$\beta ch(X;\tau)$) = con-$\beta ch(Y;\sigma)$ holds.* (3-2) *$f_*(\beta ch(X;\tau))$=$\beta ch(Y;\sigma)$ holds.*
(3-3) *$f_*(h(X;\tau)) \subseteq \beta c$-$h(Y;\sigma)$ holds* (cf. Definition 3.1(iii)).

(iv) (cf. [4, Theorem 4.5(i)]) *Especially, if $f : (X,\tau) \to (Y,\sigma)$ and $g : (Y,\sigma) \to (Z,\eta)$ are homeomorphisms, then the induced mappings $f_* : \mathcal{G}(X;\tau) \to \mathcal{G}(Y;\sigma)$ and $g_* : \mathcal{G}(Y;\sigma) \to \mathcal{G}(Z;\eta)$ are isomorphisms* (cf. (i))*. Moreover, they have the same property of* (1), (2) *and* (3)(3-1)(3-2) *in* (iii)*. We note that, in* (3)(3-3)*, $f_*(h(X;\tau)) = h(Y;\sigma)$ holds.*

Proof. **(i), (ii), (iii)(1)(2)** are obtained respectively, by setting that $H = X$ in Theorem 3.6 above (cf. Remark 3.4). **(iii)(3) Proof of (3-1) (resp. (3-2))** By setting the case where $H = X$ in the proof of Theorem 3.6(i) (resp. (ii)), it is obtained that $f_*(con\text{-}\beta ch(X;\tau)) \subseteq con\text{-}\beta ch(Y;\sigma)$ (resp. $f_*(\beta ch(X;\tau)) \subseteq \beta ch(Y;\sigma)$) holds, under the assumption (a) (resp. (b)) on f. Conversely, for each element $d \in con\text{-}\beta ch(Y,\sigma)$ (resp. $\beta ch(Y;\sigma)$), we take a mapping $f^{-1} \circ d \circ f : (X,\tau) \to (X,\tau)$. Then, it is shown that $f^{-1} \circ d \circ f \in con\text{-}\beta ch(X;\tau)$ (resp. $\beta ch(X;\tau)$) and $f_*(f^{-1} \circ d \circ f) = d$ and so $d \in f_*(con\text{-}\beta ch(X;\tau))$ (resp. $\beta ch(X;\tau)$). **Proof of (3-3)** By [4, Theorems 3.2(iii), 3.3(vi)], it is well known that $h(X;\tau) \subseteq \beta ch(X,\tau)$; and so $f_*(h(X;\tau)) \subseteq f_*(\beta ch(X;\tau)) = \beta ch(Y;\sigma)$ (cf. (3)(3-1) above).

(iv) Since any homeomorphism is a βc-homeomorhism ([4, Theorems 3.2(iii), 3.3(vi)]), then by (ii) it is shown that f_* and g_* are isomorphisms. By (1),(2) of (iii), the same properties (1), (2) and (3)(3-1)(3-2) are obtained;the present property (3-3) is well known. □

Corollary 3.8 (cf. Notation 3.3, Corollary 3.7(i)(ii)) (i) *If $\mathcal{G}(X;\tau) \not\cong \mathcal{G}(Y;\sigma)$, then*

(i-1) *there does not exist any contra-βc-homeomorphism between two topological spaces (X,τ) and (Y,σ), and* (i-2) *there is not any βc-homeomorphism between (X,τ) and (Y,σ), and hence* (i-3) $(X,\tau) \not\cong (Y,\sigma)$ (i.e., (X,τ) *is not homeomorphic to* (Y,σ)).

(ii) *If $\beta ch(X;\tau) \not\cong \beta ch(Y;\sigma)$, then there does not exist any βc-homeomorphism between (X,τ) and (Y,σ).* □

Example 3.9 Let $(X,\tau),(Y,\sigma),(Y(1),\sigma_1)$ and $(Y(2),\sigma_2)$ be four topological spaces, where $X = Y = Y(1) = Y(2) := \{a,b,c\}$, $\tau := \{\emptyset,\{a\},\{b,c\},X\}$, $\sigma := \{\emptyset,\{a\},\{b\},\{a,b\},Y\}$, $\sigma_1 := \{\emptyset,\{a\},\{b\},\{a,b\},\{a,c\},Y(1)\}$ and $\sigma_2 := \{\emptyset,\{a\},Y(2)\}$. And, let $h_x : X \to X$ be a bijection such that $h_x(x) = x$ and $h_x \neq 1_X$ for a given point $x \in X$. Then we have the following properties.

(i) $\mathcal{G}(X;\tau) \not\cong \mathcal{G}(Y;\sigma)$ (cf. Corollary 3.8(i) above). Indeed, it is shown that $\beta O(X,\tau) = P(X) = \beta C(X,\tau)$ hold and so $\beta ch(X;\tau) = con\text{-}\beta ch(X;\tau) \cong S_3$(=the symmetric group of degree 3) and hence $\mathcal{G}(X;\tau) \cong S_3$. And, it is shown that $\beta O(Y,\sigma) = P(Y) \setminus \{\{c\}\}$ and $\beta C(Y,\sigma) = P(Y) \setminus \{\{a,b\}\}$ hold; and so $con\text{-}\beta ch(Y;\sigma) = \emptyset$ and $\mathcal{G}(Y;\sigma) = \beta ch(Y;\sigma) = \{1_Y, h_c\}$.

(ii) $\mathcal{G}(X;\tau) \not\cong \mathcal{G}(Y(1);\sigma_1)$. Indeed, it is shown that $\beta O(Y(1),\sigma_1) = \sigma_1$ and $\beta ch(Y(1);\sigma_1)$ $=\{1_{Y(1)}\}$ and $con\text{-}\beta ch(Y(1);\sigma_1) = \{h_b\}$; and so $\mathcal{G}(Y(1);\sigma_1) = \{1_{Y(1)}, h_b\} \not\cong S_3$ (cf. (i) above).

(iii) $\mathcal{G}(Y(1);\sigma_1) \cong \mathcal{G}(Y(2);\sigma_2)$ and $\beta ch(Y(1);\sigma_1) \not\cong \beta ch(Y(2);\sigma_2)$. Indeed, it is shown that $\beta O(Y(2),\sigma_2) = \{\emptyset,\{a\}, \{a,b\},\{a,c\},Y(2)\}$ and $\beta ch(Y(2);\sigma_2) = \{1_{Y(2)}, h_a\}$, $con\text{-}\beta ch(Y(2);\sigma_2) = \emptyset$.

4 Groups $\mathcal{G}(X,X \setminus H;\tau)/Ker((r_H)_*))$, $\mathcal{G}_0(X,X \setminus H;\tau)/Ker((r_H)_*)$ and $\mathcal{G}(H;\tau|H)$.

The purpose of the present section is to prove Theorem 4.7. We first recall the concept of α-open sets et al. due to [35], i.e., $(*)$ a subset H of a topological space (X,τ) is said to be *α-open* in (X,τ) if $H \subseteq Int(Cl(Int(H)))$ holds in (X,τ) and the compliment of an α-open set is called *α-closed*. The family of all α-open sets (resp. α-closed sets) of (X,τ) is denoted by $\alpha O(X,\tau)$ (resp. $\alpha C(X,\tau)$).
And we recall some importante properties on β-open sets as follows.

Theorem 4.1 (i)([1], e.g., [32, Lemma 3.3(b)],[19, Lemma 4.1(2)]) *Let $A \subseteq H \subseteq X$. If $A \in \beta O(H,\tau|H)$ and $H \in \beta O(X,\tau)$, then $A \in \beta O(X,\tau)$.*

(ii)([1, Lemma 2.5 and its Proof], e.g. [32, Lemma 3.2(b)], [19, Lemma 4.1(1)]) *Let $H \subseteq X$ and $A_1 \subseteq X$. If $H \in \alpha O(X,\tau)$ and $A_1 \in \beta O(X,\tau)$, then $A_1 \cap H \in \beta O(H,\tau|H)$.*

(ii)$'$ (cf. (ii),(i) above, [2, Corollary 2.14(a)]) *Let $H \subseteq X$ and $A_1 \subseteq X$. If $H \in \alpha O(X,\tau)$ and $A_1 \in \beta O(X,\tau)$, then $A_1 \cap H \in \beta O(X,\tau)$.*

(iii)([1, Remark 1.1]) *Arbitrary union of β-open sets of (X,τ) is β-open in (X,τ).*

(iv)([19, Lemma 4.3(2)]) *If* $A \subseteq H \subseteq X$ *and* $H \in \alpha O(X,\tau)$*, then* $\beta Cl(A) \cap H = \beta Cl_H(A)$*, where* $\beta Cl_H(A)$ *denotes the* β*-closure of* A *in the subspace* $(H, \tau|H)$.

(iv-1) *Let* $F \subseteq H \subseteq X$*. If* $H \in \alpha O(X,\tau)$ *and* $F \in \beta C(X,\tau)$*, then* $F \in \beta C(H,\tau|H)$ (i.e., $\beta_H Cl(F) = F$ *holds*).

(iv-2) *Let* F_1 *and* H *be subsets of* X*. If* $H \in \alpha O(X,\tau)$ *and* $F_1 \in \beta C(X,\tau)$*, then* $F_1 \cap H \in \beta C(H,\tau|H)$ (i.e., $\beta_H Cl(F_1 \cap H) = F_1 \cap H$ *holds*).

(iv-3) *Let* $F \subseteq H \subseteq X$*. If* $H \in \alpha O(X,\tau) \cap \beta C(X,\tau)$ *and* $F \in \beta C(H,\tau|H)$*, then* $F \in \beta C(X,\tau)$.

Proof. **(iv-1) (resp. (iv-2))** By the assumptions and (iv), it is shown that $\beta_H Cl(F) = F$ (resp. $\beta_H Cl(F_1 \cap H) = \beta Cl(F_1 \cap H) \cap H \subseteq \beta Cl(F_1) \cap H = F_1 \cap H$ and so $\beta_H Cl(F_1 \cap H) = F_1 \cap H$). Therefore, we have the following: $F \in \beta C(H,\tau|H)$ (resp. $F_1 \cap H \in \beta C(H,\tau|H)$).
(iv-3) By the assumptions, (iv) and (iii), it is shown that $F = \beta_H Cl(F) = H \cap \beta Cl(F)$ and so $H \cap \beta Cl(F) \in \beta C(X,\tau)$. □

Remark 4.2 It follows from the following example that one of assumptions of Theorem 4.1(ii) above (i.e., $H \in \alpha O(X,\tau)$) is not removed. Let $X := \{a,b,c\}$ and $\tau := \{\emptyset, \{a\}, \{b\}, \{a,b\}, X\}$. Then, a subset $H := \{b,c\} \notin \alpha O(X,\tau)$ and $H \in \beta O(X,\tau)$. And, for a set $A_1 := \{a,c\} \in \beta O(X,\tau), A_1 \cap H = \{c\} \notin \beta O(H,\tau|H)$. Indeed, we have that $\beta O(X,\tau) = P(X)\backslash\{\{c\}\}, \beta O(H,\tau|H) = \{\emptyset, \{b\}, H\}$ and $\alpha O(X,\tau) = \tau$ hold and $Cl_H(Int_H(Cl_H(A_1 \cap H))) = Cl_H(Int_H(\{c\})) = \emptyset \not\supseteq A_1 \cap H$.

Remark 4.3 (i) Let H and K be subsets of X and Y, respectively. For a mapping $f : X \to Y$ satisfying $K = f(H)$, we define the map $r_{H,K}(f) : H \to K$ by $(r_{H,K}(f))(x) := f(x)$ for every $x \in H$. Then, we have the following: $j_K \circ (r_{H,K}(f)) = f|H : H \to Y$, where $j_K : K \to Y$ is the inclusion defined by $j_K(y) := (1_Y|K)(y) = y$ for every $y \in K$ and $f|H : H \to Y$ is the restriction of f to H defined by $(f|H)(x) := f(x)$ for every $x \in H$. Especially, if $X = Y = H = K$, then $r_{H,H}(f) = f$ holds.

(ii) Especially, we suppose that $X = Y, H = K \subseteq X$ and $a(H) = H, b(H) = H$ for mappings $a, b : X \to X$. Then, $r_{H,H}(b \circ a) = (r_{H,H}(b)) \circ (r_{H,H}(a))$ holds.
Moreover, if $a : X \to X$ is a bijection such that $a(H) = H$, then $r_{H,H}(a) : H \to H$ is bijective and $r_{H,H}(a^{-1}) = (r_{H,H}(a))^{-1}$.

Theorem 4.4 (cf. [40, Lemma 2.8]) (i) *Let* $H \in \alpha O(X,\tau)$*. If* $f : (X,\tau) \to (Y,\sigma)$ *is contra-*β*-irresolute* (resp.β*-irresolute*)*, then the restriction of* f *to* H*, say* $f|H : (H,\tau|H) \to (Y,\sigma)$*, is contra-*$\beta$*-irresolute* (resp.$\beta$*-irresolute*).

(ii) *Let* $k : (X,\tau) \to (K,\sigma|K)$ *be a mapping and* $j_K : (K,\sigma|K) \to (Y,\sigma)$ *be the inclusion, where* $K \subseteq Y$*. If* K *is* α*-open in* (Y,σ)*, then the following properties* (1), (2) *are equivalent:*

(1) $k : (X,\tau) \to (K,\sigma|K)$ *is contra-*β*-irresolute* (resp. β*-irresolute*)*;*

(2) $j_K \circ k : (X,\tau) \to (Y,\sigma)$ *is contra-*β*-irresolute* (resp. β*-irresolute*).

(iii) *Suppose that* $f : (X,\tau) \to (Y,\sigma)$ *is contra-*β*-irresolute* (resp. β*-irresolute*)*,* $H \in \alpha O(X,\tau)$ *and* $f(H) \in \alpha O(Y,\sigma)$*. Then,* $r_{H,f(H)}(f) : (H,\tau|H) \to (f(H),\sigma|f(H))$ *is contra-*β*-irresolute* (resp.β*-irresolute*).

Proof. **(i)** Let $A \in \beta O(Y,\sigma)$ (resp. $\beta C(Y,\sigma)$). Then, we have the following: $f^{-1}(A) \in \beta C(X,\tau)$. By Theorem 4.1(iv-2), it is shown that $(f|H)^{-1}(A) = f^{-1}(A) \cap H \in \beta C(H,\tau|H)$ and so $f|H : (H,\tau|H) \to (Y,\sigma)$ is contra-β-irresolute (resp. β-irresolute).
(ii) (1)⇒(2) Let $A_1 \in \beta O(Y, \sigma)$. We have the followng: $(*1)$ $j_K^{-1}(A_1) = K \cap A_1 \in \beta O(K,\sigma|K)$ (cf. Theorem 4.1(ii)). Then, using $(*1)$ above and assumption (1), we have that $(j_K \circ k)^{-1}(A_1) = k^{-1}(j_K^{-1}(A_1)) \in \beta C(X,\tau)$ (resp. $\beta O(X,\tau)$). **(2)⇒(1)** Let $B \in \beta O(K,\sigma|K)$. Then, we have that $B \in \beta O(Y,\sigma)$ (cf. Theorem 4.1(i)). Then, using (2), we show that $k^{-1}(B) = (j_K \circ k)^{-1}(B) \in \beta C(X,\tau)$ (resp. $\beta O(X,\tau)$). **(iii)** By assumption and (i), it is obtained that $f|H : (H,\tau|H) \to (Y,\sigma)$ is contra-β-irresolute (resp. β-irresolute). Since $f|H = j_{f(H)} \circ (r_{H,f(H)}(f))$ (cf. Remark 4.3(i)), by (ii) it is shown that the mapping $r_{H,f(H)}(f)$ is contra-β-irresolute (resp. β-irresolute). □

Theorem 4.5 (cf. [40, Definition 2.5, Theorem 2.7(i)]) *Suppose that* $H \in \alpha O(X,\tau)$.

(i) *If* $f \in$ *con-*$\beta ch(X, X\backslash H;\tau)$ (resp. $\beta ch(X, X\backslash H;\tau)$), *then* $r_{H,H}(f) \in$ *con-*$\beta ch(H;\tau|H)$ (resp. $\beta ch(H;\tau|H)$).

(ii) *The following mapping* $(r_H)_* : \mathcal{G}(X, X\setminus H;\tau) \to \mathcal{G}(H;\tau|H)$ *is well defined by* $(r_H)_*(f) := r_{H,H}(f)$ *for every* $f \in \mathcal{G}(X, X\setminus H;\tau)$.

(iii) *The following mapping* $(r_H)_{*,0} : \mathcal{G}_0(X, X\setminus H;\tau) \to \mathcal{G}(H;\tau|H)$ *is well defined by* $(r_H)_{*,0}(f) := r_{H,H}(f)$ *for every* $f \in \mathcal{G}_0(X, X\setminus H;\tau)$.

(iv) (cf. [4, Theorem 4.4(i)]), Notation 3.3, Theorem 3.5(i),(i)$'$)
(iv-1) $(r_H)_* : \mathcal{G}(X, X\setminus H;\tau) \to \mathcal{G}(H;\tau|H)$ *is a homomorphism of group.*
(iv-2) $(r_H)_{*,0} : \mathcal{G}_0(X, X\setminus H;\tau) \to \mathcal{G}(H;\tau|H)$ *is a homomorphism of group.*
(iv-3) $(r_H)_*|\mathcal{G}_0(X, X\setminus H;\tau) = (r_H)_{*,0}$.

Proof. **(i)** In Theorem 4.4(iii), let consider the case where $Y = X$ and $\tau = \sigma$. Since $f \in$ *con-*$\beta ch(X, X\setminus H;\tau)$ (resp. $\beta ch(X, X\setminus H;\tau)$), we have the following property that both f and f^{-1} are contra-β-irresolute (resp. β-irresolute) bijections from (X,τ) onto itself such that $f(X\setminus H) = X\setminus H = f^{-1}(X\setminus H)$) (cf. Definition 3.1), and so $f(H) = H = f^{-1}(H)$, $f(H)$ and $f^{-1}(H)$ are α-open in (X,τ). Then, by Theorem 4.4(iii), it is shown that $r_{H,H}(f)$ and $(r_{H,H}(f))^{-1} = r_{H,H}(f^{-1}) : (H,\tau|H) \to (H,\tau|H)$ are contra-β-irresolure (resp. β-irresolute) bijections (cf. Remark 4.3(ii)). Namely, we have the followng: $r_{H,H}(f) \in$ *con-*$\beta ch(H;\tau|H)$ (resp. $\beta ch(H;\tau|H)$). **(ii)** Let $a \in \mathcal{G}(X, X\setminus H)$. For the case where that $a \in$ *con-*$\beta ch(X, X\setminus H;\tau)$ (resp. $\beta ch(X, X\setminus H;\tau)$), by using (i) it is shown that $r_{H,H}(a) \in$ *con-*$\beta ch(H;\tau|H)$ (resp. $\beta ch(H;\tau|H)$) and so $r_{H,H}(a) \in \mathcal{G}(H;\tau|H)$. Therefore, $(r_H)_*(a) := r_{H,H}(a) \in \mathcal{G}(H;\tau|H)$ holds for any element $a \in \mathcal{G}(X, X\setminus H;\tau)$ and so $(r_H)_*$ is well defined.

(iii) We recall that $\mathcal{G}_0(X, X\setminus H;\tau) \subseteq \mathcal{G}(X, X\setminus H;\tau)$. Then, by the definition of $(r_H)_{*,0}$ and (ii), it is obtaned that $(r_H)_{*,0}(a) := r_{H,H}(a) \in \mathcal{G}(H;\tau|H)$ for every $a \in \mathcal{G}_0(X, X\setminus H)$.

(iv) We denote $\mathcal{G} := \mathcal{G}(X, X\setminus H;\tau)$ and $\mathcal{G}_0 := \mathcal{G}_0(X, X\setminus H;\tau)$, throughout the present proof of (iv). **(iv-1)** Let $a, b \in \mathcal{G}$ and $w : \mathcal{G}\times\mathcal{G} \to \mathcal{G}$ be the binary operation of the group $\mathcal{G}$ (cf. Proof of Theorem 3.5). Then, by definition, $w(a,b) := b\circ a$ for $a, b \in \mathcal{G}$ and $(r_H)_*(w(a,b)) = r_{H,H}(b\circ a) = (r_{H,H}(b))\circ(r_{H,H}(a))$ hold (cf. (ii) above, Remark 4.3(ii)). Here, we recall that the group $\mathcal{G}(H;\tau|H) :=$ *con-*$\beta ch(H;\tau|H) \cup \beta ch(H;\tau|H)$ has the binary operation $w_H : \mathcal{G}(H;\tau|H)\times\mathcal{G}(H;\tau|H) \to \mathcal{G}(H;\tau|H)$ defined by the composite mapping: $w_H(f,g) := g\circ f$, where $f, g \in \mathcal{G}(H;\tau|H)$ (cf. [4, Theorem 4.4(i)]). Thus, we have the following: $(r_H)_*(w(a,b)) = (r_{H,H}(b))\circ(r_{H,H}(a)) = w_H(r_{H,H}(a), r_{H,H}(b)) = w_H((r_H)_*(a), (r_H)_*)(b))$ and hence $(r_H)_* : \mathcal{G} \to \mathcal{G}(H;\tau|H)$ is a homomorphism of group. **(iv-2)** Since $\mathcal{G}_0$ is a subgroup of $\mathcal{G}$ (cf. Theorem 3.5(i)$'$), by an argument similar to that of (iv-1) it is shown that $(r_H)_{*,0} : \mathcal{G}_0 \to \mathcal{G}(H;\tau|H)$ is a homomorphism of group. **(iv-3)** For an element $a \in \mathcal{G}_0$, we have the following: $((r_H)_*|\mathcal{G}_0)(a) = (r_H)_*(a) = r_{H,H}(a)$, on the other hand, $(r_H)_{*,0}(a) = r_{H,H}(a)$ and hence $(r_H)_*|\mathcal{G}_0 = (r_H)_{*,0}$. □

Lemma 4.6 ([40, Lemma 2.6] *for the case where* β*-irresoluteness*) *Let* (X,τ) *and* (Y,σ) *be topological spaces such that* $X = U_1\cup U_2$ *with* $U_j \neq \emptyset$ $(j \in \{1,2\})$. *Let* $f_1 : (U_1,\tau|U_1) \to (Y,\sigma)$ *and* $f_2 : (U_2,\tau|U_2) \to (Y,\sigma)$ *be the two contra-*β*-irresolute* (resp. β*-irresolute*) *mappings with* $f_1(x) = f_2(x)$ *for every point* $x \in U_1\cap U_2$. *If* U_1 *and* U_2 *are* α*-open sets of* (X,τ), *then its combination* $f_1\nabla f_2 : (X,\tau) \to (Y,\sigma)$ *is contra-*β*-irresolute* (resp. β*-irresolute*), *where* $(f_1\nabla f_2)(z) = f_j(z)$ *for every* $z \in U_j$ $(j \in \{1,2\})$.

Proof. By using Theorem 4.1(i)(iii) and above definitions, this lemma is proved. □

Theorem 4.7 (cf. [40, Theorem 2.7 (ii),(iii)]) (i) *Suppose that* $H \in \alpha O(X,\tau)$. *Then, we have the following isomorphisms of groups* (cf. Theorem 4.5(ii),(iii),(iv)).
(i-1) $\mathcal{G}(X, X\setminus H;\tau)/Ker((r_H)_*) \cong Im((r_H)_*)$.
(i-2) $\mathcal{G}_0(X, X\setminus H;\tau) \cong Im((r_H)_{*,0})$, *where* $Ker((r_H)_*) := \{a \in \mathcal{G}(X, X\setminus H;\tau) | (r_H)_*(a) = 1_H\}$ *is a normal subgroup of* $\mathcal{G}(X, X\setminus H;\tau)$, *and* $Im((r_H)_*) := \{(r_H)_*(a) | a \in \mathcal{G}(X, X\setminus H;\tau)\}$ *and* $Im((r_H)_{*,0}) := \{(r_H)_{*,0}(b) | b \in \mathcal{G}_0(X, X\setminus H;\tau)\}$ *are subgroups of* $\mathcal{G}(H;\tau|H)$.

(ii) *Suppose that* $H \in \alpha O(X,\tau) \cap \alpha C(X,\tau)$ (cf. Lemma 4.6, the top of the present Section 4). *Then, under the assumption above, we have the following properties on the homomorphisms* $(r_H)_*$ *and* $(r_H)_{*,0}$ (cf. Theorem 4.5).
(ii-1) *If con-*$\beta ch(X, X \setminus H;\tau) \neq \emptyset$*, then* $(r_H)_* : \mathcal{G}(X, X \setminus H;\tau) \to \mathcal{G}(H;\tau|H)$ *is onto.*
(ii-2) *If con-*$\beta ch_0(X, X \setminus H;\tau) \neq \emptyset$*, then* $(r_H)_{*,0} : \mathcal{G}_0(X, X \setminus H;\tau) \to \mathcal{G}(H;\tau|H)$ *is onto.*

(iii) *Suppose that* $H \in \alpha O(X,\tau) \cap \alpha C(X,\tau)$. *Then, we have the following isomorphisms of groups.*
(iii-1) *If con-*$\beta ch(X, X \setminus H;\tau) \neq \emptyset$*, then* $\mathcal{G}(X, X \setminus H;\tau)/Ker((r_H)_*) \cong \mathcal{G}(H;\tau|H)$.
(iii-2) *If con-*$\beta ch_0(X, X \setminus H;\tau) \neq \emptyset$*, then* $\mathcal{G}_0(X, X \setminus H;\tau) \cong \mathcal{G}(H;\tau|H)$.

Proof. **(i)** Since $H \in \alpha O(X,\tau)$, the mappings $(r_H)_*$ and $(r_H)_{*,0}$ are the well defined homomorphisms of groups (cf. Theorem 4.5, Remark 4.3(i)). Then, by using the first isomorphism theorem of group theory, it is obtained that there are group isomorphisms below, under the α-openness of H in (X,τ): **(i-1)** $\mathcal{G}(X, X \setminus H;\tau)/Ker((r_H)_*) \cong Im((r_H)_*)$ and **(i-2)**$_1$ $\mathcal{G}_0(X, X \setminus H;\tau)/Ker((r_H)_{*,0}) \cong Im((r_H)_{*,0})$. In (i-2)$_1$ above, it is shown that **(i-2)**$_2$ $Ker((r_H)_{*,0}) = \{1_X\}$. Indeed, let $u_0 \in Ker((r_H)_{*,0}) \subseteq \mathcal{G}_0(X, X \setminus H;\tau)$. Then, $(r_H)_{*,0}(u_0) = 1_H$ holds,where 1_H is the identity element of $\mathcal{G}(H;\tau|H)$, by definitions (cf. Theorem 4.5(iii),(ii) and Remark 4.3(i)), it is shown that, for any point $x \in H, 1_H(x) = ((r_H)_{*,0}(u_0))(x) = (r_{H,H}(u_0))(x) = u_0(x)$ and so $u_0(x) = x$ holds for any point $x \in H$. Moreover, for any point $x \in X \setminus H, u_0(x) = x$ holds, because of $u_0 \in \mathcal{G}_0(X, X \setminus H;\tau)$ (cf. Notation 3.3(ii)$'$, Definition 3.1(iv)) and hence we prove (i-2)$_2 Ker((r_H)_{*,0}) = \{1_X\}$. Thus, by using (i-2)$_1$ and (i-2)$_2$ above, the isomorphism (i-2) is proved.

(ii) (ii-1) Let $h \in \mathcal{G}(H;\tau|H)$. We find a mapping, say $h_1 \in \mathcal{G}(X, X \setminus H;\tau)$ such that $(r_H)_*(h_1) = h$. Indeed, we consider the following two cases (because of $\mathcal{G}(H;\tau|H) := \beta ch(H;\tau|H) \cup$ *con-*$\beta ch(H;\tau|H)$).
Case 1 $h \in$ *con-*$\beta ch(H;\tau|H)$**.** For the present case, we select an element g belonging to *con-*$\beta ch(X, X \setminus H;\tau) \neq \emptyset$ by one of assumptions. By Theorem 4.4(i), it is obtained that $g|(X \setminus H) : (X \setminus H, \tau|X \setminus H) \to (X,\tau)$ is a contra-β-irresolute mapping such that $(g|(X \setminus H))(X \setminus H) = X \setminus H$. Then, since $j_H \circ h : (H,\tau|H) \to (X,\tau)$ is a contra-β-irresolute mapping (cf. Theorem 4.4(ii)), by Lemma 4.6, it is shown that the combination, say $h_1 := (j_H \circ h)\nabla(g|(X \setminus H)) : (X,\tau) \to (X,\tau)$, is a contra-$\beta$-irresolute bijection. And, we have the followng:$h_1(X \setminus H) = X \setminus H$ and $h_1^{-1} = (j_H \circ h^{-1})\nabla(g^{-1}|g(X \setminus H))$. Using Theorem 4.4(ii) and (i) above, it is shown that $j_H \circ h^{-1} : (H,\tau|H) \to (X,\tau)$ and $g^{-1}|g(X \setminus H) : (X \setminus H, \tau|(X \setminus H)) \to (X,\tau)$ are contra-β-irresolute mappings; and so the mapping h_1^{-1} is contra-β-irresolute (cf. Lemma 4.6). Thus, we proved that $h_1 \in$ *con-*$\beta ch(X, X \setminus H;\tau)$ and $(r_H)_*(h_1) = r_{H,H}(h_1) = r_{H,H}((j_H \circ h)\nabla(g|(X \setminus H))) = r_{H,H}(j_H \circ h) = h$ (cf. Theorem 4.5(ii), Remark 4.3(i)). Namely, for the present case, there exists an element $h_1 \in$ *con-*$\beta ch(X, X \setminus H;\tau) \subseteq \mathcal{G}(X, X \setminus H;\tau)$ such that $(r_H)_*(h_1) = h$.
Case 2 $h \in \beta ch(H;\tau|H)$**.** For the present case, using [40, Theorem 2.7 (i)(i-2)]. there is an element $h'_1 \in \beta r\text{-}h(X, X \setminus H;\tau) \subseteq \mathcal{G}(X, X \setminus H;\tau)$ such that $(r_H)_*(h'_1) = h$, where $h'_1 := (j_H \circ h)\nabla(1_X|(X \setminus H)) : (X,\tau) \to (X,\tau)$. Therefore, using Case 1 and Case 2, we prove that $(r_H)_*$ is onto.

(ii-2) Let $h \in \mathcal{G}(H;\tau|H)$. We consider the following two cases.
Case 1 $h \in$ *con-*$\beta ch(H;\tau|H)$**.** For the present case, we select an element $g_0 \in$ *con-*$\beta ch_0(X, X \setminus H;\tau) \neq \emptyset$. By an argument similar to that in the proof of (ii)(ii-1) Case 1, it is proved that $h_2 := (j_H \circ h)\nabla(g_0|(X \setminus H)) \in$ *con-*$\beta ch(X, X \setminus H) \subseteq \mathcal{G}_0(X, X \setminus H;\tau)$ and $(r_H)_{*,0}(h_2) = h$.
Case 2 $h \in \beta ch(H;\tau|H)$**.** For the present case, using [40, Theorem 2.7 (i)(i-2)], there exists an element $h'_2 \in \beta ch_0(X, X \setminus H;\tau) \subseteq \mathcal{G}_0(X, X \setminus H;\tau)$ such that $(r_H)_{*,0}(h'_2) = h$, where $h'_2 := (j_H \circ h)\nabla(1_X|(X \setminus H))$. Therefore, $(r_H)_{*,0}$ is onto. **(iii)** By (i) and (ii), the isomorphisms (iii-1) and (iii-2) are obtained. $\square$

5 A characterization of β-open sets of the digital plane $(\mathbb{Z}^2, \kappa^2)$ and some new groups on $(\mathbb{Z}^2, \kappa^2)$. In the present Section 5, we have the following four subsections (I), (II), (III) and (IV).

(I) Introduction of some related notation. We recall the concept of the digital plane.

Definition 5.1 (E.D.Khalimsky, R.Koppermann,P.R.Meyer,T.Y.Kong, cf. [22, p.175,-Definition 4], [20, p.905,p.908], [29, Section 2], [30, Example 4 in Section 2]).

(i) The *digital line* or the *Khalimsky line* $(\mathbb{Z}, \kappa)$ is the set $\mathbb{Z}$ of all integers, equipped with the topology κ having $\{\{2m-1, 2m, 2m+1\} | m \in \mathbb{Z}\}$ as a subbase (e.g., [27, Section 3 (I)], [16], [38, Section 6 in p.6]).

(ii) The *digital plane* or the *Khalimsky plane* is the Cartesian product (=topological product) of 2-copies of the digital line $(\mathbb{Z}, \kappa)$. This topological space is denoted by $(\mathbb{Z}^2, \kappa^2)$, where $\mathbb{Z}^2 := \mathbb{Z} \times \mathbb{Z}$ and $\kappa^2 := \kappa \times \kappa$ (e.g., [16], [9, Section 6], [39, Section 5], [11, Section 7], [10], [33, Section 6], [27, Section 3(II) in p.322]).

($\bullet$) In $(\mathbb{Z}, \kappa)$, for each integer s, $\{2s\}$ is closed and it is not open, and $\{2s+1\}$ is open and it is not closed. And so $Cl(\{2s\}) = \{2s\}$, $Cl(\{2s+1\}) = \{2s, 2s+1, 2s+2\}$, $Int(\{2s\}) = \emptyset$ and $Int(\{2s+1\}) = \{2s+1\}$.

($\bullet$) In $(\mathbb{Z}^2, \kappa^2)$, for each integers s and m, $\{(2s, 2m)\}$ is closed and it is not open, and $\{(2s+1, 2m+1)\}$ is open and it is not closed, and $\{(2s+1, 2m)\}$ and $\{(2s, 2m+1)\}$ are not open and they are not closed. And so we have the following properties:

$\cdot$ $Cl(\{(2s, 2m)\}) = \{(2s, 2m)\}$, $Cl(\{(2s+1, 2m+1)\}) = \{2s, 2s+1, 2s+2\} \times \{2m, 2m+1, 2m+2\}$, $Cl(\{(2s+1, 2m)\}=\{2s, 2s+1, 2s+2\} \times \{2m\}$, $Cl(\{(2s, 2m+1)\} = \{2s\} \times \{2m, 2m+1, 2m+2\}$, and

$\cdot$ $Int(\{(2s, 2m)\}) = \emptyset$, $Int(\{(2s+1, 2m+1)\}) = \{(2s+1, 2m+1)\}$, $Int(\{(2s+1, 2m)\}) = Int(\{(2s, 2m+1)\}) = \emptyset$.

Definition 5.2 (cf. Notation 5.5 below) Let A be a subset of $(\mathbb{Z}^2, \kappa^2)$.

(i) $A_{\kappa^2} := \{x | \ x \in A$ and $\{x\} \in \kappa^2\}$, (ii) $A_{\mathcal{F}^2}$:=$\{x | \ x \in A$ and $\{x\}$ is closed in $(\mathbb{Z}^2, \kappa^2)\}$,

(iii) A_{mix} :=$\{x | \ x \in A, x \notin A_{\kappa^2}$ and $x \notin A_{\mathcal{F}^2}\}$, and

(iv) for the set $A = \emptyset$, $A_{\kappa^2} := \emptyset$, $A_{\mathcal{F}^2} := \emptyset$, $A_{mix} := \emptyset$.

(v) Note that, sometimes, the set A_{κ^2} (resp. $A_{\mathcal{F}^2}, A_{mix}$) above is denoted by $(A)_{\kappa^2}$ (resp. $(A)_{\mathcal{F}^2}$, $(A)_{mix}$ (cf. Notation 5.5).

Definition 5.3 (i) For an open set E and a point $x \in E, E$ is said to be *the smallest open set containing* x, if $E \subseteq G$ holds for every open set G containng x (e.g., [31, Definition 2.5, Remark 2.6 (ii)], [27, Section 3], [25, p.6 of Section 1]).

(ii) The smallest open set containing a point x in $(\mathbb{Z}^2, \kappa^2)$ is denoted by $U(x)$ throughout the present section (cf. Remark 5.4(iv) below).

Remark 5.4 The following properties are well known. Let A be a subset of $(\mathbb{Z}^2, \kappa^2)$ in (i), (ii) and (iii).

(i) $(\mathbb{Z}^2)_{\kappa^2} = \{(2s+1, 2m+1) | \ s, \ m \in \mathbb{Z}\}$, $A_{\kappa^2} = A \cap (\mathbb{Z}^2)_{\kappa^2}$,

(ii) $(\mathbb{Z}^2)_{\mathcal{F}^2} = \{(2s, 2m) | \ s, \ m \in \mathbb{Z}\}$, $A_{\mathcal{F}^2} = A \cap (\mathbb{Z}^2)_{\mathcal{F}^2}$,

(iii) $(\mathbb{Z}^2)_{mix} = \{(2s+1, 2m) | \ s, \ m \in \mathbb{Z}\} \cup \{(2s', 2m'+1) | \ s', m' \in \mathbb{Z}\}$, $A_{mix} = A \cap (\mathbb{Z}^2)_{mix}$.

(iv) Moreover, we have the following properties:

(iv-1) if $x \in (\mathbb{Z}^2)_{\kappa^2}$, then $x := (2s+1, 2m+1)$ for some $s, m \in \mathbb{Z}$ and $U((2s+1, 2m+1))$=$\{(2s+1, 2m+1)\}$ (cf. (i) above and Definition 5.3(ii) for the notation $U(\bullet)$),

(iv-2) if $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$, then $x := (2s, 2m)$ for some $s, m \in \mathbb{Z}$ and $U((2s, 2m))$=$\{2s-1, 2s, 2s+1\} \times \{2m-1, 2m, 2m+1\}$ (cf. (ii) above),

(iv-3) if $x \in (\mathbb{Z}^2)_{mix}$, then $x = (2s+1, 2m)$ or $x = (2s, 2m+1)$ for some $s, m \in \mathbb{Z}$ and $U((2s+1, 2m)) = \{2s+1\} \times \{2m-1, 2m, 2m+1\}$, $U((2s, 2m+1)) = \{2s-1, 2s, 2s+1\} \times \{2m+1\}$ (cf. (iii) above).

(II) A characterization of β-open sets of $(\mathbb{Z}^2,\kappa^2)$. We prepare the following notation which are used in Theorem 5.7 and Corollary 5.8 below. And, we note $(X)_{\kappa^2}$:=$\{y|y\in X$ and $\{y\}\in\kappa^2\}$ for a subset X of $(\mathbb{Z}^2,\kappa^2)$.

Notation 5.5 (cf. Definition 5.2) Let A be a nonempty subset of $(\mathbb{Z}^2,\kappa^2)$.
$V(A_{\mathcal{F}^2}) := \bigcup\{\{x\}\cup(A\cap U(x))_{\kappa^2}|\ x\in A_{\mathcal{F}^2}$ and $(A\cap U(x))_{\kappa^2}\neq\emptyset\}$,
$V(A_{mix}) := \bigcup\{\{y\}\cup(A\cap U(y))_{\kappa^2}|\ y\in A_{mix}$ and $(A\cap U(y))_{\kappa^2}\neq\emptyset\}$. And, for the case where $A_{\mathcal{F}^2}=\emptyset$ (resp. $A_{mix}=\emptyset$), we set that $V(A_{\mathcal{F}^2}):=\emptyset$ (resp. $V(A_{mix}):=\emptyset$).

Example 5.6 Let $A := \{x,p_x,y,y^-,y',z\}\cup\{a\}\subset\mathbb{Z}^2$ and $B := A\setminus\{a\}$, where $x := (0,0), p_x := (1,1), y := (2,1), y^- := (3,1), y' := (3,0)$, $z := (5,1)$ and $a := (-2,0)$. Then, we have the following properties: **(1)** the set A is not β-open and B is β-open,
(2) $V(A_{\mathcal{F}^2})\cup V(A_{mix})\cup A_{\kappa^2}=\{p_x,y^-,z,x,y,y'\}$=$A\setminus\{a\}\neq A$,
(2)$'$ $V(B_{\mathcal{F}^2})\cup V(B_{mix})\cup B_{\kappa^2}$=$\{x,p_x\}\cup\{p_x,y,y^-,y'\}\cup\{p_x,y^-,z\}$=$B$.
Proof of (1) We see that $Cl(Int(Cl(A)))=\{0,1,2,3,4,5,6\}\times\{0,1,2\}\not\ni(-2,0)=a$ and so $Cl(Int(Cl(A)))\not\supseteq A$ holds. For the set B, we see that $Cl(Int(Cl(B)))=\{0,1,2,3,4,5,6\}\times\{0,1,2\}\supset B$. **Proof of (2)** Since $A_{\mathcal{F}^2}=\{a,x\}$, we see that $(A\cap U(a))_{\kappa^2}=\emptyset$ and $(A\cap U(x))_{\kappa^2}\neq\emptyset$, $V(A_{\mathcal{F}^2})=\{x\}\cup(A\cap U(x))_{\kappa^2}=\{x,p_x\}$ hold. Since $A_{mix}=\{y,y'\}$, we see that $V(A_{mix})=[\{y\}\cup(A\cap U(y))_{\kappa^2}]\cup[\{y'\}\cup(A\cap U(y'))_{\kappa^2}]$ =$\{y,p_x,y^-\}\cup\{y',y^-\}$=$\{y,p_x,y^-,y'\}$. Since $A_{\kappa^2}=\{p_x,y^-,z\}$, we prove (2). **Proof of (2)$'$** For this β-open set B, we are able to see that: $B_{\mathcal{F}^2}=\{x\}$, $B_{mix}=A_{mix}$, $B_{\kappa^2}=A_{\kappa^2}$ and so we have the following: $V(B_{\mathcal{F}^2})\cup V(B_{mix})\cup B_{\kappa^2}$=$\{x,p_x\}\cup\{p_x,y,y^-,y'\}\cup\{p_x,y^-,z\}$=$B$.

By investigating Example 5.6 above, we find one of the characterization of β-open sets of $(\mathbb{Z}^2,\kappa^2)$ (cf. Theorem 5.7(i)(i-2),(ii) and Corollary 5.8 below).

Theorem 5.7 (i) (i-1) *If B is a nonempty β-open subset of $(\mathbb{Z}^2,\kappa^2)$, then $(B\cap U(x))_{\kappa^2}\neq\emptyset$ holds for each point $x\in B_{\mathcal{F}^2}\cup B_{mix}$* (cf. Remark 5.4(i)(ii)).
(i-2) *If B is a β-open set of $(\mathbb{Z}^2,\kappa^2)$, then B is expressible as follows:* $B=V(B_{\mathcal{F}^2})\cup V(B_{mix})\cup B_{\kappa^2}$ (cf. Notation 5.5, Remark 5.4(i)).
(ii) *If a subset B is expressible as $B=V(B_{\mathcal{F}^2})\cup V(B_{mix})\cup B_{\kappa^2}$, then B is β-open in $(\mathbb{Z}^2,\kappa^2)$.*

Proof. We note that, in general, for a point $w\in\mathbb{Z}^2$ and a subset B of $(\mathbb{Z}^2,\kappa^2)$,
(1) $(B\cap U(w))_{\kappa^2}$=$B\cap(U(w))_{\kappa^2}$ holds, where $(U(w))_{\kappa^2}$:=$\{z|z\in U(w)$ and $\{z\}\in\kappa^2\}$.
(i)(i-1) Let $x\in B_{\mathcal{F}^2}\cup B_{mix}$. Then, since $B\subseteq Cl(Int(Cl(B)))$, we have the following: $U(x)\cap Int(Cl(B))\neq\emptyset$ holds and so there exists a point $z(x)$ such that $z(x)\in U(x)\cap Int(Cl(B))$. Then **(2)** $U(z(x))\subseteq Cl(B)$ and $\emptyset\neq(U(z(x))\subseteq U(x)$ (cf. Definition 5.3(ii)). Then, using (2), we see that $\emptyset\neq(U(z(x)))_{\kappa^2}\subseteq(U(x))_{\kappa^2}$ and we can take an open singleton $\{p(x)\}$ such that $p(x)\in U(z(x))$ and so $p(x)\in B$. Thus, we have the following:$p(x)\in B\cap(U(x))_{\kappa^2}$, i.e., $(B\cap U(x))_{\kappa^2}\neq\emptyset$ (cf. (1) above). **(i-2)** First we note that the sets $V(B_{\mathcal{F}^2})$ and $V(B_{mix})$ are well defined by (i-1) above, respectively, for a nonempty β-open set B. We prove that: **(3)** $V(B_{\mathcal{F}^2})\cup V(B_{\mathcal{F}^2})\cup B_{\kappa^2}\subseteq B$ holds and **(4)** $B\subseteq V(B_{\mathcal{F}^2})\cup V(B_{mix})\cup B_{\kappa^2}$ holds. **Proof of (3)** We see that $V(B_{\mathcal{F}^2})\subseteq\bigcup\{\{x\}\cup B_{\kappa^2}|x\in B_{\mathcal{F}^2}\}$=$B_{\mathcal{F}^2}\cup B_{\kappa^2}\subseteq B$ and $V(B_{mix})\subseteq\bigcup\{\{y\}\cup B_{mix}|y\in B_{mix}\}$=$B_{mix}\subseteq B$. And so we prove (3).
Proof of (4) Let $z\in B$. And we consider the following two cases.
Case 1 $z\in B_{\mathcal{F}^2}\cup B_{mix}$**.** For the present case, by (i-1), it is shown that $(B\cap U(z))_{\kappa^2}\neq\emptyset$ and $z\in V(B_{\mathcal{F}^2})\cup V(B_{mix})$, and so **(5)** $z\in V(B_{\mathcal{F}^2})\cup V(B_{mix})\cup B_{\kappa^2}$.
Case 2 $z\in B_{\kappa^2}$**.** For the present case, it is seen clearly that (5) above holds. Thus, by Case 1 and Case 2 above, (4) is proved. Therefore, by (3), (4) and Notation 5.5, it is proved that $B=V(B_{\mathcal{F}^2})\cup V(B_{mix})\cup B_{\kappa^2}$ holds if B is β-open in $(\mathbb{Z}^2,\kappa^2)$.
(ii) Let $B\neq\emptyset$. By using the assumption of (ii), the definition of $V(B_{\mathcal{F}^2})$ and $V(B_{mix})$ (cf. Notation 5.5), it is shown that $(B\cap U(x))_{\kappa^2}\neq\emptyset$ holds for each point $x\in B_{\mathcal{F}^2}\cup B_{mix}$. We show firstly that: **(6)** $\{x\}\cup(B\cap U(x))_{\kappa^2}$ is β-open for each point $x\in B_{\mathcal{F}^2}\cup B_{mix}$. Indeed,

since $(B\cap U(x))_{\kappa^2} \neq \emptyset$, there exists a point, say $z(x)$, such that $z(x) \in B_{\kappa^2}\cap(U(x))_{\kappa^2}$. Then, it is shown that $x \in Cl(\{z(x)\})$, because $z(x) \in (U(x))_{\kappa^2} \subset U(x) \subseteq W$ hold for every open set W containing x. And we have the following: $Cl(Int(Cl(\{z(x)\}) \supset Cl(Int(\{z(x)\})) = Cl(\{z(x)\}) \ni x$, and so $Cl(Int(Cl(\{x\} \cup (B \cap U(x))_{\kappa^2}))) \supseteq Cl(Int(Cl(\{z(x)\}))) \supset \{x\}$. Then, $Cl(Int(Cl(\{x\} \cup (B \cap U(x))_{\kappa^2}))) \supseteq \{x\} \cup Cl(Int(Cl((B \cap U(x))_{\kappa^2}))) \supseteq \{x\} \cup (B \cap U(x))_{\kappa^2}$, because $(B \cap U(x))_{\kappa^2} \in \beta O(\mathbb{Z}^2, \kappa^2)$. Thus we prove the property (6), i.e., $\{x\} \cup (B \cap U(x))_{\kappa^2} \in \beta O(\mathbb{Z}^2, \kappa^2)$ for each point $x \in B_{\mathcal{F}^2} \cup B_{mix}$.

Therefore, since any union of β-open sets is β-open (cf. Theorem 4.1(iii)), by using (6), Notation 5.5 and the assumption of (ii), it is proved that the set B is β-open in $(\mathbb{Z}^2, \kappa^2)$. □

Corollary 5.8 *A subset B of $(\mathbb{Z}^2, \kappa^2)$ is β-open if and only if B is expressible as $B = V(B_{\mathcal{F}^2}) \cup V(B_{mix}) \cup B_{\kappa^2}$.* □

(III) A proof of $con\text{-}\beta ch(\mathbb{Z}^2;\kappa^2) = \emptyset$ (cf. Corollary 5.11(ii)$'$ below). We first prepare the following notation: **(III-1)** $U := \{-1,0,1\} \times \{-1,0,1\}$ (i.e., $U := U((0,0))$: the smallest open set of $(\mathbb{Z}^2;\kappa^2)$ containing $(0,0)$): **(III-2)** $O := (0,0), p^{(1)} := (1,-1), p^{(2)} := (-1,-1), p^{(3)} := (-1,1), p^{(4)} := (1,1)$ and $y^{(1)} := (0,-1), y^{(2)} := (-1,0), y^{(3)} := (0,1)$,-$y^{(4)} := (1,0)$, and so **(III-3)** $(\cdot)$ $U = \{O, p^{(1)}, p^{(2)}, p^{(3)}, p^{(4)}, y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}\}$ and $(\cdot)$ $U_{\kappa^2} = \{p^{(1)}, p^{(2)}, p^{(3)}, p^{(4)}\}$, $(\cdot)$ $U_{\mathcal{F}^2} = \{O\}, (\cdot)$ $U_{mix} = \{y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}\}$ and so $(\cdot)$ $U = U_{\kappa^2} \cup U_{mix} \cup U_{\mathcal{F}^2}$ (disjoint union) and $(\cdot)$ the smallest open sets $U(y^{(i)})$ of $(\mathbb{Z}^2;\kappa^2)$ containing the point $y^{(i)}(1 \leq i \leq 4)$ is defined as follows: $U(y^{(i)}) = \{p^{(i+1)}, y^{(i)}, p^{(i)}\}(1 \leq i \leq 4)$, where $p^{(5)} = p^{(1)}$ (cf. $(**)$ in the first part of the subsection (IV) below).

Proposition 5.9 *Let* $U := U((0,0))$ (cf. (III-1) above) *and* $f : (U, \kappa^2|U) \to (U, \kappa^2|U)$ *be a mapping. If f is bijective, then*

(i) f^{-1} *is not contra-β-irresolute* (cf. Definition 2.2) *and*

(ii) f^{-1} *and f are not contra-βc-homeomorphisms* (cf. Definition 3.1).

Proof. **(i)** We select three points, say $p^{(1)} \in U_{\kappa^2}, p^{(2)} \in U_{\kappa^2}$ and $y^{(1)} \in U_{mix}$ with the smallest open set $U(y^{(1)}) = \{p^{(1)}, y^{(1)}, p^{(2)}\}$ (cf. (III-2) above). For the point $y^{(1)}$ there exists an only one point, say $z(y^{(1)})$, such that $z(y^{(1)}) \in U$ and $f(z(y^{(1)})) = y^{(1)}$. Then, $(\bullet)$ we take a set $B := U \setminus \{z(y^{(1)})\}$. Then, we claime that:

(1) the set B is β-open in $(U, \kappa^2|U)$ and

(2) $f(B)$ is not β-closed in $(U, \kappa^2|U)$. **Proof of (1)** We consider the following two cases, because of $z(y^{(1)}) \in U = U_{\kappa^2} \cup (U_{mix} \cup U_{\mathcal{F}^2})$ (cf. (III)-1, (III)-2, (III)-3 above).

Case 1 $z(y^{(1)}) \in U_{\kappa^2}$**.** For the present case, since $\{z(y^{(1)})\}$ is open in $(\mathbb{Z}^2, \kappa^2)$ and $z(y^{(1)}) \in U$, we see that $z(y^{(1)}) = p^{(j_0)}$ for some j_0 with $1 \leq j_0 \leq 4$. And, so we have the followng: $Cl(B) = \bigcup\{Cl(\{p^{(i)}\}|i \neq j_0$ with $1 \leq i \leq 4\}$ and $Int(Cl(B)) \supseteq \bigcup\{Int(Cl(\{p^{(i)}\}))|i \neq j_0$ with $1 \leq i \leq 4\}$=$\bigcup\{\{p^{(i)}\}|i \neq j_0$ with $1 \leq i \leq 4\}$ and so $Cl(Int(Cl(B))) \supseteq \bigcup\{Cl(\{p^{(i)}\})|i \neq j_0$ with $1 \leq i \leq 4\} = Cl(B) \supset B$, i.e., for the present Case 1, B is β-open in $(\mathbb{Z}^2, \kappa^2)$.

Case 2 $z(y^{(1)}) \in U_{mix} \cup U_{\mathcal{F}^2}$**.** For the present case, since $B = U \cap (\mathbb{Z}^2 \setminus \{z(y^{(1)})\})$, we have the following: $Cl(Int(Cl(B))) \supseteq Cl(Int(U \cap Cl(\mathbb{Z}^2 \setminus \{z(y^{(1)})\}))$=$Cl(Int(U \cap \mathbb{Z}^2)) = Cl(Int(U)) = Cl(U) \supset B$ and so $B \in \beta O(\mathbb{Z}^2, \kappa^2)$. Thus, for each case, $B \in \beta O(\mathbb{Z}^2, \kappa^2)$ and so, by using Theorem 4.1(ii), it is shown that $B = B \cap U$ is β-open in $(U, \kappa^2|U)$ (note: $U \in \kappa^2 \subset \alpha O(\mathbb{Z}^2, \kappa^2)$). **Proof of (2)** For the point $z(y^{(1)})$ with $y^{(1)} = f(z(y^{(1)}))$, $B := U \setminus \{z(y^{(1)}\}$ and the bijection $f : U \to U$, we see that $f(B) = U \setminus \{y^{(1)}\}$, where $U := U((0,0))$. Using Theorem 4.1(iv), we have the following: $\beta Cl_U(f(B)) = U \cap \beta Cl(f(B))$= $U \cap [f(B) \cup Int(Cl(Int(f(B))))] = U \cap [(U \setminus \{y^{(1)}\}) \cup U] = U$ and so $\beta Cl_U(f(B)) = U \neq U \setminus \{y^{(1)}\} = f(B)$. Thus, we prove (2). Therefore, by (1), (2) and definitions, it is proved that $f^{-1} : (U, \kappa^2|U) \to (U, \kappa^2|U)$ is not contra-β-irresolute (cf. Definition 2.2(iii)). **(ii)** By (i) above and Definition 3.1(i)(i-2), it is obtained that f^{-1} and f are not contra-βc-homeomorphisms. □

Remark 5.10 Let $g : (\mathbb{Z}^2, \kappa^2) \to (\mathbb{Z}^2, \kappa^2)$ be a bijective function. Then, the inverse $g^{-1} : (\mathbb{Z}^2, \kappa^2) \to (\mathbb{Z}^2, \kappa^2)$ is not necessarily contra-β-irresolute and so g and g^{-1} are not necessarily a contra-βr-homeomorphism (cf. Definitions 2.2, 3.1). Indeed, we take a mixed point, say $y \in (\mathbb{Z}^2)_{mix}$, and a set $B := \mathbb{Z}^2 \setminus \{g^{-1}(y)\}$. We prove that $B \in \beta O(\mathbb{Z}^2, \kappa^2)$ and $g(B) = \mathbb{Z}^2 \setminus \{y\}$ is not β-closed in $(\mathbb{Z}^2, \kappa^2)$ (cf. Proof of Proposition 5.9(i)).

Corollary 5.11 *Let $U := U((0,0))$, i.e., $U := \{-1,0,1\} \times \{-1,0,1\} (\subset \mathbb{Z}^2)$. Then, we have the following properties.*

(i) *Every homeomorphism $f : (U, \kappa^2|U) \to (U, \kappa^2|U)$ is not a contra-βr-homeomorphism.*

(i)′ *Every homeomorphism $g : (\mathbb{Z}^2, \kappa^2) \to (\mathbb{Z}^2, \kappa^2)$ is not a contra-βr-homeomorphism.*

(ii) *$h(U; \kappa^2|U) \not\subseteq$ con-$\beta ch(U; \kappa^2|U)$ and con-$\beta ch(U; \kappa^2|U) = \emptyset$.*

(ii)′ *$h(\mathbb{Z}^2; \kappa^2) \not\subseteq$ con-$\beta ch(\mathbb{Z}^2; \kappa^2)$ and con-$\beta ch(\mathbb{Z}^2; \kappa^2) = \emptyset$.*

(iii) *$\mathcal{G}(U; \kappa^2|U) = \beta ch(U; \kappa^2|U)$* (cf. Notation 3.3, Definition 3.1(ii)).

(iii)′ *$\mathcal{G}(\mathbb{Z}^2; \kappa^2) = \beta ch(\mathbb{Z}^2; \kappa^2)$* (cf. Notation 3.3, Definition 3.1(ii)). □

(IV) New groups $\beta_{(2)}ch(H; \kappa^2|H)$, $\beta_{(2)}ch(H; \kappa^2|H) \cup$ *con*-$\beta_{(2)}ch(H; \kappa^2|H)$, - $p.\beta_{(2)}ch(H; \kappa^2|H)$ and $p.\beta_{(2)}ch(H; \kappa^2|H) \cup$ *con*-$p.\beta_{(2)}ch(H; \kappa^2|H)$.

Our main results of (IV) are Theorems 5.23, 5.25 and Example 5.27 (cf. Definition 5.15). We introduce some new concepts $\beta_{(2)}$-open sets (cf. Definition 5.12, Definiton 5.15). We first recall the following notation **(**$*$**)** et al. and we prepare new definitions (Definition 5.12, Remark 5.13).

($*$**)** Let $x = (x_1, x_2) \in (\mathbb{Z}^2)_{\mathcal{F}^2}$ (i.e., x_1 and x_2 are even). For this point x, we denote the points belonging to the smallest open set $U(x)$ containg the point x as follows:
$U(x) := \{x_1-1, x_1, x_1+1\} \times \{x_2-1, x_2, x_2+1\} = \{x, p_x^{(1)}, p_x^{(2)}, p_x^{(3)}, p_x^{(4)}, y_x^{(1)}, y_x^{(2)}, y_x^{(3)}, y_x^{(4)}\}$, where $p_x^{(1)} := (x_1+1, x_2-1)$, $p_x^{(2)} := (x_1-1, x_2-1)$, $p_x^{(3)} := (x_1-1, x_2+1)$, $p_x^{(4)} := (x_1+1, x_2+1)$, $y_x^{(1)} := (x_1, x_2-1)$, $y_x^{(2)} := (x_1-1, x_2)$, $y_x^{(3)} := (x_1, x_2+1)$, $y_x^{(4)} := (x_1+1, x_2)$.

($**$**)** The following illustration shows the points belonging in $U(x)$, where $\{x\} = \{(x_1, x_2)\}$ is closed in $(\mathbb{Z}^2, \kappa^2)$ (i.e., $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$), $(U(x))_{\mathcal{F}^2} = \{x\}$, $(U(x))_{\kappa^2} = \{p_x^{(i)} | i \in \{1,2,3,4\}\}$ and $(U(x))_{mix} = \{y_x^{(i)} | i \in \{1,2,3,4\}\}$.

$$
\begin{array}{lllllll}
\mathbb{Z} & & & & & & \\
\uparrow & & & & & & \\
\cdots & \circ p_x^{(3)} & & \cdot y_x^{(3)} & & \circ p_x^{(4)} & \cdots \\
\cdots & \cdot y_x^{(2)} & & \bullet\ x & & \cdot y_x^{(4)} & \cdots \\
\cdots & \circ p_x^{(2)} & & \cdot y_x^{(1)} & & \circ p_x^{(1)} & \cdots \\
\bullet\ O & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \to \quad \mathbb{Z}
\end{array}
$$

When $x = (0,0)$ (i.e., $x =$"the origin O" of $\mathbb{Z}^2$), we simply denote $p_x^{(i)}$ and $y_x^{(i)}$ as $p^{(i)}$ and $y^{(i)}$, respectively, and so $U := U((0,0)) = \{(0,0), p^{(1)}, p^{(2)}, p^{(3)}, p^{(4)}, y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}\}$.

Definition 5.12 Let $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$ and $U(x)$ be the smallest open set containing x. We define the following two families, $\beta_{(2)}O(U(x))$ and $\beta_{(2)}C(U(x))$ as follow (cf. Propositions 5.16(i), 5.18(i)):

(i) $\beta_{(2)}O(U(x)) := \{B | B \in \beta O(\mathbb{Z}^2, \kappa^2), B \subset U(x)$ and $|B| = 2\}$ (cf. Remark 5.13(i)),

(ii) $\beta_{(2)}C(U(x)) := \{F | F \in \beta C(\mathbb{Z}^2, \kappa^2), F \subset U(x)$ and $|F| = 2\}$ (cf. the definition of $p.\beta_{(2)}C(U(x)) \subset \beta_{(2)}C(U(x))$ in Remark 5.13(ii) below).

Remark 5.13 For a point $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$ and the smallest open set $U(x)$ containing x, we have the following precise form of families $\beta_{(2)}O(U(x))$ and $\beta_{(2)}C(U(x))$ above, respectively (cf. Propositions 5.16(i), 5.18(i)).

(i) $\beta_{(2)}O(U(x)) = \{\{x, p_x^{(1)}\}, \{x, p_x^{(2)}\}, \{x, p_x^{(3)}\}, \{x, p_x^{(4)}\}, \{y_x^{(1)}, p_x^{(1)}\}, \{y_x^{(2)}, p_x^{(2)}\},$- $\{y_x^{(3)}, p_x^{(3)}\}, \{y_x^{(4)}, p_x^{(4)}\}, \{p_x^{(2)}, y_x^{(1)}\}, \{p_x^{(3)}, y_x^{(2)}\}, \{p_x^{(4)}, y_x^{(3)}\}, \{p_x^{(1)}, y_x^{(4)}\}, \{p_x^{(2)}, p_x^{(1)}\},$- $\{p_x^{(3)}, p_x^{(2)}\}, \{p_x^{(4)}, p_x^{(3)}\}, \{p_x^{(1)}, p_x^{(4)}\}, \{p_x^{(3)}, p_x^{(1)}\}, \{p_x^{(4)}, p_x^{(2)}\}\}$. (We use the following abbreviated notation: $\beta_{(2)}O(U(x)) := \{\{x, p_x^{(i)}\}, \{y_x^{(i)}, p_x^{(i)}\}, \{p_x^{(i+1)}, y_x^{(i)}\}, \{p_x^{(i+1)}, p_x^{(i)}\}, \{p_x^{(3)}, p_x^{(1)}\},$ $\{p_x^{(4)}, p_x^{(2)}\} | i \in \{1,2,3,4\}\}$, where $p_x^{(5)} := p_x^{(1)}$.)

(ii) (ii-1) $\beta_{(2)}C(U(x))$=$\{\{x, y_x^{(i)}\}, \{y_x^{(i+1)}, y_x^{(i)}\}, \{y_x^{(1)}, y_x^{(3)}\}, \{y_x^{(2)}, y_x^{(4)}\}, \{y_x^{(i)}, p_x^{(i)}\},$ - $\{p_x^{(i+1)}, y_x^{(i)}\}, \{p_x^{(3)}, p_x^{(1)}\}, \{p_x^{(4)}, p_x^{(2)}\}, \{x, p_x^{(i)}\}, \{p_x^{(i+2)}, y_x^{(i)}\}, \{p_x^{(i+3)}, y_x^{(i)}\} | i \in \{1,2,3,4\}\}$, and

(ii-2) we introduce the following importante subfamily, say $p.\beta_{(2)}C(U(x))$, of $\beta_{(2)}C(U(x))$ above and this concept is used in Theorem 5.23, 5.25 and Example 5.27, $p.\beta_{(2)}C(U(x))$:=$\{\{x, y_x^{(i)}\}, \{y_x^{(i+1)}, y_x^{(i)}\}, \{y_x^{(1)}, y_x^{(3)}\}\{y_x^{(2)}, y_x^{(4)}\}, \{y_x^{(i)}, p_x^{(i)}\}, \{p_x^{(i+1)}, y_x^{(i)}\}$ - $|i \in \{1,2,3,4\}\})$, where $y_x^{(5)} := y_x^{(1)}$ and $p_x^{(5)} := p_x^{(1)}, p_x^{(6)} := p_x^{(2)}, p_x^{(7)} := p_x^{(3)}$.

Remark 5.14 (cf. Definition 5.12) We have the following inclusions of families.

(i) $\beta_{(2)}O(U(x)) \subset \beta O(\mathbb{Z}^2, \kappa^2)$. (i)′ $\beta_{(2)}O(U(x)) \subset \beta O(U(x), \kappa^2|U(x)) \subset \beta O(\mathbb{Z}^2, \kappa^2)$ (cf. (i) above, Definition 5.12(i), Theorem 4.1(i),(ii)).

(ii) $p.\beta_{(2)}C(U(x)) \subset \beta_{(2)}C(U(x)) \subset \beta C(\mathbb{Z}^2, \kappa^2)$. (ii)′ $\beta_{(2)}C(U(x)) \subset \beta C(U(x), \kappa^2|$-$U(x)) \subset \beta C(\mathbb{Z}^2, \kappa^2)$ (cf. (ii) above, Definition 5.12(ii), Remark 5.13(ii) and, Theorem 4.1(iv-1),(iv-3)).

(Note) By definition, we say that: (1) $\emptyset \notin \beta_{(2)}O(U(x))$ and $\emptyset \notin \beta_{(2)}C(U(x))$ and (2) $\beta_{(2)}O(U(x)) \cap \beta_{(2)}C(U(x))$=$\{\{x, p_x^{(i)}\}, \{y_x^{(i)}, p_x^{(i)}\}, \{p_x^{(i+1)}, y_x^{(i)}\}, \{p_x^{(3)}, p_x^{(1)}\}, \{p_x^{(4)}, p_x^{(2)}\} | i \in \{1,2,3,4\}\}$, where $p_x^{(5)} := p_x^{(1)}$.

Definition 5.15 Let $H \subseteq \mathbb{Z}^2$ with $|H| \geq 2$. A subset B (resp. F, F_1) of $(\mathbb{Z}^2, \kappa^2)$ is said to be *a $\beta_{(2)}$-open* (resp. *$\beta_{(2)}$-closed, $p.\beta_{(2)}$-closed*) *set of* H, if $B \subseteq H$ (resp. $F \subseteq H$, $F_1 \subseteq H$) and there exists a point $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$ such that $B \in \beta_{(2)}O(U(x))$ (resp. $F \in \beta_{(2)}C(U(x))$, $F_1 \in p.\beta_{(2)}C(U(x))$) (cf. Definition 5.12(i) and Remark 5.13(i) (resp. Definition 5.12(ii) and Remark 5.13(ii))). The family of all $\beta_{(2)}$-open (resp. $\beta_{(2)}$-closed, $p.\beta_{(2)}$-closed) sets of H is denoted by $\beta_{(2)}O(H)$ (resp. $\beta_{(2)}C(H)$, $p.\beta_{(2)}C(H)$). (Note: Proposition 5.16 (resp. 5.18(i), 5.18(i)) below.)

Proposition 5.16 *Let $H \subseteq \mathbb{Z}^2$ with $|H| \geq 2$.*

(i) $\beta_{(2)}O(H) \subseteq \beta O(\mathbb{Z}^2, \kappa^2)$ *holds* (cf. Remark 5.17(i) below).

(ii) $\beta_{(2)}O(H) \subseteq \beta O(H, \kappa^2|H)$ *holds* (cf. Remark 5.17(i) below).

(iii) If H is β-open in $(\mathbb{Z}^2, \kappa^2)$, then $\beta O(H, \kappa^2|H) \subseteq \beta O(\mathbb{Z}^2, \kappa^2)$ (cf. Theorem 4.1(i), Remark 5.17(ii) below).

Proof **(i), (ii)** Let $B \in \beta_{(2)}O(H)$. By Definition 5.15 and Remark 5.14(i), it is obtained that $(*)B \in \beta_{(2)}O(U(x))$ and $B \subset H$ for some point $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$.

The proof of (i) By Definition 5.12(i) (or Remark 5.14(i)), it is obtained that $B \in \beta O(\mathbb{Z}^2, \kappa^2)$ and so **(i)**:$\beta_{(2)}O(H) \subset \beta O(\mathbb{Z}^2, \kappa^2)$.

The proof of (ii) is as follows. Let $B \in \beta_{(2)}O(H)$. We show that $B \in \beta O(H, \kappa^2|H)$. Indeed, by $(*)$ above in the top of the present proof, $B \subset H$ and $B \in \beta_{(2)}O(U(x))$ for some point $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$. We investigate the proof with the following cases: Case 1, Case 2 and Case 3 (cf. Remark 5.13(i) etc.).

Case 1 $B \in \{\{x,\ p_x^{(i)}\} | i \in \{1,2,3,4\}\}$ **(resp. Case 2** $B \in \{\{y_x^{(i)}, p_x^{(i)}\}, \{y_x^{(i)}, p_x^{(i+1)}\} | i \in \{1,2,3,4\}\}$, where $p_x^5 := p_x^{(1)}$.) For the present case, we put $B = \{u, p\}$, where $u \in (U(x))_{\mathcal{F}^2}$ (i.e., $u = x$) and $p \in (U(x))_{\kappa^2}$ (resp. $u \in (U(x))_{mix}$ and $p \in (U(u))_{\kappa^2}$). Then, we have the following: **(1)** $Int_H(B) \supseteq \{p\}$ and **(2)** $Cl_H(\{p\}) \supseteq B$. **Proof of (1)** Since $U(p) \cap H = \{p\} \cap H = \{p\} \in \kappa^2|H$, the set $U(p) \cap H = \{p\}$ is the smallest open set of $(H, \kappa^2|H)$ containing p such that $p \in B$ and so $p \in Int_H(B)$. **Proof of (2)** Since $U(u) \cap H$ is the smallest open set of $(H, \kappa^2|H)$ containing u and $(U(u) \cap H) \cap \{p\} \neq \emptyset$, we have the

following: $u \in Cl_H(\{p\})$ and so $Cl_H(\{p\}) \supset \{u, p\} = B$.
Then, using (1) and (2), we see that $Cl_H(Int_H(Cl_H(B))) \supseteq Cl_H(Int_H(B)) \supseteq Cl_H(\{p\}) \supseteq B$ and so B is β-open in $(H, \kappa^2|H)$ for the present Case 1 (resp. Case 2).
Case 3 $B \in \{\{p_x^{(i)},\ p_x^{(i+1)}\}, \{p_x^{(i)},\ p_x^{(i+2)}\} | i \in \{1, 2, 3, 4\}\}$, where $p_x^{(5)} := p_x^{(1)}$ and $p_x^{(6)} := p_x^{(2)}$. For the present case, we put $B = \{p,\ p'\}$ where $p \in (U(x))_{\kappa^2}$ and $p' \in (U(x))_{\kappa^2}$ with $p \neq p'$. Then, we have **(3)** $Int_H(B) = B$. **Proof of (3)** Since $B = \{p, p'\}$, where $p \neq p'$ and $p, p' \in (U(x))_{\kappa^2}$, it is shown that $U(p) \cap H = \{p\} \cap H = \{p\} \subset B$ and $U(p') \cap H \subset B$, and so $U(p) \cap H$ (resp. $U(p') \cap H$) is the smallest open set of $(H, \kappa^2|H)$ containing p (resp. p'). Thus we have the following: $p \in Int_H(B)$ (resp. $p' \in Int_H(B)$) and so $B \subset Int_H(B)$, i.e., $B = Int_H(B)$. Then, the set B is β-open in $(H, \kappa^2|H)$ for the present Case 3. Therefore, by all cases above, it is shown that $\beta_{(2)}O(H) \subseteq \beta O(H, \kappa^2|H)$ (cf. Remark 5.17(i) below).
(iii) Suppose that $B \in \beta O(H, \kappa^2|H)$. Since $B \subseteq H \subseteq \mathbb{Z}^2$ and H is β-open in $(\mathbb{Z}^2, \kappa^2)$ (by assumptions), using Theorem 4.1(i), we have the followng: $B \in \beta O(\mathbb{Z}^2, \kappa^2)$. Thus we prove that $\beta O(H, \kappa^2|H) \subseteq \beta O(\mathbb{Z}^2, \kappa^2)$ if H is β-open in $(\mathbb{Z}^2, \kappa^2)$. □

Remark 5.17 (i) In Proposition 5.16(i)(ii)), $\beta_{(2)}O(H)$ is a proper subfamily of $\beta O(H, \kappa^2|H)$ and $\beta O(\mathbb{Z}^2, \kappa^2)$, respectively. Indeed, let $B := \{p, p'\}$ and $H := B \cup \{x\}$, where $p := (-1, -1)$, $p' := (3, -1)$ and $x := (0, 0)$. Then, $B \in \beta O(H; \kappa^2|H)$ and $B \in \beta O(\mathbb{Z}^2, \kappa^2)$. However, $B \notin \beta_{(2)}O(H)$, because $B \notin \beta_{(2)}O(U(x))$ for any $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$.

(ii) It follows from the following example that the assumption of Proposition 5.16(iii) is not removed. Let $H := \{p, p', y, x, y'\}$ and $B := \{p, p', y, y'\}$, where $p := (-1, -1), p' := (3, -1), y := (-1, 0), x := (0, 0), y' := (1, 0)$. Then, H is not β-open in $(\mathbb{Z}^2, \kappa^2)$, because $Cl(Int(Cl(H))) = Cl(\{p, p'\}) \not\ni y'$ and $y' \in H$. And, B is β-open in $(H, \kappa^2|H)$, because $Cl_H(Int_H(Cl_H(B))) = H \supset B$; however, $B \notin \beta O(\mathbb{Z}^2, \kappa^2)$, because $Cl(Int(Cl(B))) = Cl(\{p, p'\}) \not\ni y'$ and $y' \in B$.

Proposition 5.18 *Let $H \subset \mathbb{Z}^2$ with $|H| \geq 2$.*

(i) *$p.\beta_{(2)}C(H) \subseteq \beta_{(2)}C(H) \subseteq \beta C(\mathbb{Z}^2, \kappa^2)$ hold.*

(ii) *If H is α-open in $(\mathbb{Z}^2, \kappa^2)$, then $\beta_{(2)}C(H) \subseteq \beta C(H; \kappa^2|H)$* (cf. Theorem 4.1(iv-2), Remark 5.19(i) below).

(iii) *If H is α-open and β-closed in $(\mathbb{Z}^2, \kappa^2)$, then $\beta C(H, \kappa^2|H) \subseteq \beta C(\mathbb{Z}^2, \kappa^2)$* (cf. Theorem 4.1(iv-3), Remark 5.19(ii) below).

Proof. The proof is analogous to the case of $\beta_{(2)}O(H)$ (cf. Proposition 5.16 above) and so is omitted. □

Remark 5.19 (i) In Proposition 5.18(i), $\beta_{(2)}C(H)$ is a proper subfamily of $\beta C(\mathbb{Z}^2, \kappa^2)$. Indeed, let $H := U(x) \cup U(x')$, where $x := (0, 0)$ and $x' := (2, 0)$. Then, a β-closed set $F := \{x, x'\}$ of $(\mathbb{Z}^2, \kappa^2)$ is not $\beta_{(2)}$-closed in H, because $F \notin \beta_{(2)}C(U(z))$ for any point $z \in (\mathbb{Z}^2)_{\mathcal{F}^2}$ (cf. Remark 5.13(ii)).

(ii) It follows from the following example that the assumption of Proposition 5.18(iii) is not removed. Let $F := \{p, x, p'\}$ and $H := F \cup \{q\}$, where $x := (0, 0)$, $p := (-1, -1)$, $p' := (1, -1)$, $q := (3, -1)$. Then, $Int(Cl(Int(H))) = \{-1, 0, 1, 2, 3\} \times \{-1\} \not\supset H$ and so H is not α-open in $(\mathbb{Z}^2, \kappa^2)$. Since $Int_H(Cl_H(Int_H(F))) = F$ holds, we see that $F \in \beta C(H, \kappa^2|H)$. However, $F \notin \beta C(\mathbb{Z}^2, \kappa^2)$, because $Int(Cl(Int(F))) = \{-1, 0, 1\} \times \{-1\} \not\subset F$.

Definition 5.20 Let $(H, \kappa^2|H)$ be a subspace of $(\mathbb{Z}^2, \kappa^2)$ with $|H| \geq 2$ and $f : (H, \kappa^2|H) \to (H, \kappa^2|H)$ be a mapping. And, let $\mathcal{A}_H$ and $\mathcal{B}_H$ be collections of subsets of H such that:-
$\mathcal{A}_H, \mathcal{B}_H \in \{\beta_{(2)}O(H),\ \beta_{(2)}C(H), p.\beta_{(2)}C(H)\}$ (cf. Definition 5.15, Remark 5.13).

Then, f is said to be $(\mathcal{A}_H, \mathcal{B}_H)$-*irresolute*, if $f^{-1}(E) \in \mathcal{B}_H$ for every set $E \in \mathcal{A}_H$, where $(\mathcal{A}_H, \mathcal{B}_H)$ denotes the ordered pair of the collections $\mathcal{A}_H$ and $\mathcal{B}_H$.

(Note 1) Especially, if $\mathcal{A}_H = \mathcal{B}_H$, then the concept of the $(\mathcal{A}_H, \mathcal{A}_H)$-irresolute mapping is simply said to be $\mathcal{A}_H$-*irresolute*.

(Note 2) In the present definition, we are able to define the concepts of the following mappings: the *$\beta_{(2)}O(H)$-irresolute mappings, $\beta_{(2)}C(H)$-irresolute mappings, $p.\beta_{(2)}C(H)$-irresolute mappings, $(\beta_{(2)}O(H), \beta_{(2)}C(H))$-irresolute mappings, $(\beta_{(2)}C(H), \beta_{(2)}O(H))$-irresolute mappings, $(\beta_{(2)}O(H), p.\beta_{(2)}C(H))$-irresolute mappings, $(p.\beta_{(2)}C(H),\ \beta_{(2)}O(H))$-irresolute mappings.*

Definition 5.21 For a subspace $(H, \kappa^2|H)$ of $(\mathbb{Z}^2, \kappa^2)$, where $|H| \geq 2$, we define the following collections of mappings as follows (cf. Definitions 5.20, 5.15).

(i) $\beta_{(2)}ch(H;\kappa^2|H)$:=$\{f|\ f : (H, \kappa^2|H) \to (H, \kappa^2|H)$ is a bijection such that f and f^{-1} are both $\beta_{(2)}O(H)$-irresolute and they are $\beta_{(2)}C(H)$-irresolute$\}$.

(i)′ $p.\beta_{(2)}ch(H;\kappa^2|H)$:=$\{f|\ f : (H, \kappa^2|H) \to (H, \kappa^2|H)$ is a bijection such that f and f^{-1} are both $\beta_{(2)}O(H)$-irresolute and they are $p.\beta_{(2)}C(H)$-irresolute$\}$.

(ii) *con*-$\beta_{(2)}ch(H;\kappa^2|H)$:=$\{f|\ f : (H, \kappa^2|H) \to (H, \kappa^2|H)$ is a bijection such that f and f^{-1} are both $(\beta_{(2)}O(H), \beta_{(2)}C(H))$-irresolute and they are $(\beta_{(2)}C(H), \beta_{(2)}O(H))$-irresolute$\}$.

(ii)′ *con*-$p.\beta_{(2)}ch(H;\kappa^2|H)$:=$\{f|\ f : (H, \kappa^2|H) \to (H, \kappa^2|H)$ is a bijection such that f and f^{-1} are both $(\beta_{(2)}O(H), p.\beta_{(2)}C(H))$-irresolute and they are $(p.\beta_{(2)}C(H), \beta_{(2)}O(H))$-irresolute$\}$.

Lemma 5.22 *Let $(H, \kappa^2|H)$ be a subspace of $(\mathbb{Z}^2, \kappa^2)$ such that:*
$(*)$ $H = \bigcup\{U(z)|z \in A_{\mathcal{F}^2}\}$, where $A_{\mathcal{F}^2}$ is a nonempty subset of $\mathbb{Z}^2$. If $f : (H, \kappa^2|H) \to (H, \kappa^2|H)$ is a homeomorphism, then for a point $x \in H_{\mathcal{F}^2}$,

(i) *$f(H_{\kappa^2}) = H_{\kappa^2}$, $f(H_{\mathcal{F}^2}) = H_{\mathcal{F}^2}$ and $f(H_{mix}) = H_{mix}$ hold in $(\mathbb{Z}^2, \kappa^2)$,*

(ii) *$f(U(p)) = U(f(p))$ holds for each point $p \in (U(x))_{\kappa^2}$,*

(iii) *$f(U(y)) = U(f(y))$ holds for each point $y \in (U(x))_{mix}$, and*

(iv) *$f(U(x)) = U(f(x))$.*

Proof. Since f is homeomorphic, we have the property (i) (cf. Definition 5.2, Remark 5.4). Then, by the standard method, the properties (ii), (iii) and (iv) are proved. □

Let us consider the case $H = \mathbb{Z}^2$, i.e., $H = \bigcup\{U(z)|z \in (\mathbb{Z}^2)_{\mathcal{F}^2}\}$. Then, we have the following properties on $(\mathbb{Z}^2, \kappa^2)$.

Theorem 5.23 *Let $h(\mathbb{Z}^2;\kappa^2)$ be the group of all homeomorphisms from $(\mathbb{Z}^2, \kappa^2)$ onto itself* (cf. Definition 3.1(iii))*.*

(i) *If $f : (\mathbb{Z}^2, \kappa^2) \to (\mathbb{Z}^2, \kappa^2)$ is a homeomorphism, then* (cf. Definition 5.20, Note:1)
(1a) *f is $\beta_{(2)}C(\mathbb{Z}^2)$-irresolute,* (1b) *$f$ is $\beta_{(2)}O(\mathbb{Z}^2)$-irresolute, and*
(1c) *f is $p.\beta_{(2)}C(\mathbb{Z}^2)$-irresolute.*

(ii) *$h(\mathbb{Z}^2;\kappa^2) \subseteq \beta_{(2)}ch(\mathbb{Z}^2;\kappa^2)$ holds* (cf. Theorem 5.25(iv)′ below).

(ii)′ *$h(\mathbb{Z}^2;\kappa^2) \subseteq p.\beta_{(2)}ch(\mathbb{Z}^2;\kappa^2)$ holds* (cf. Theorem 5.25(iv)′ below).

Proof. **(i) (Proof of (1a))** Let $F \in \beta_{(2)}C(\mathbb{Z}^2)$ (cf. Definition 5.15). We claim that $f^{-1}(F) \in \beta_{(2)}C(\mathbb{Z}^2)$. Indeed, there exists a point $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$ such that $F \in \beta_{(2)}C(U(x))$ and we note that $f^{-1}(x) \in (\mathbb{Z}^2)_{\mathcal{F}^2}$ (cf. Lemma 5.22(i) above). Since $F \in \beta_{(2)}C(U(x))$, we have that $F \subset U(x), |F| = 2$ and $F \in \beta C(\mathbb{Z}^2, \kappa^2)$ (cf. Definition 5.12(ii), or Remark 5.14(ii)). By definitions., Remark 2.7(ii) and Lemma 5.22(i)(iv), it is shown that $f^{-1}(F) \subset U(f^{-1}(x)) \subset \mathbb{Z}^2, |f^{-1}(F)| = 2$, $f^{-1}(x) \in (\mathbb{Z}^2)_{\mathcal{F}^2}$ and $f^{-1}(F) \in \beta C(\mathbb{Z}^2, \kappa^2)$, and so $f^{-1}(F) \in \beta_{(2)}C(U(f^{-1}(x)))$ (cf. Definition 5.12(ii)). Then, we conclude that $f^{-1}(F) \in \beta_{(2)}C(\mathbb{Z}^2)$ holds (cf. Definition 5.15).
(Proof of (1b)) Let $B \in \beta_{(2)}O(\mathbb{Z}^2)$. We claim that $f^{-1}(B) \in \beta_{(2)}O(\mathbb{Z}^2)$. The proof is analogous to the proof of (1a) above using Definitions 5.15, 5.12(i), Remark 5.13(i) and Lemma 5.22. And so the proof is omitted.
(Proof of (1c)) Let $F_1 \in p.\beta_{(2)}C(\mathbb{Z}^2)$ (cf. Definition 5.15, Remark 5.13(ii)(ii-2)). Then, $F_1 \subset \mathbb{Z}^2$, $|F_1| = 2$ and there exists a point $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$ such that $F_1 \in p.\beta_{(2)}C(U(x))$. Then, we have the following: $F_1 \in \{\{x, y_x^{(i)}\}, \{y_x^{(i)}, y_x^{(i+1)}\}$, $\{y_x^{(1)}, y_x^{(3)}\}$, $\{y_x^{(2)}, y_x^{(4)}\}$, $\{y_x^{(i)}$, -

$p_x^{(i)}\}, \{p_x^{(i+1)}, y_x^{(i)}\} | i \in \{1,2,3,4\}\}$, where $p_x^{(5)} := p_x^{(1)}$, $y_x^{(5)} := y_x^{(1)}$. Using Lemma 5.22, we note that $f^{-1}(x) \in (U(f^{-1}(x)))_{\mathcal{F}^2}$, $f^{-1}(U(p_x^{(i)})) = U(f^{-1}(p_x^{(i)}))$ and $f^{-1}(U(y_x^{(i)})) = U(f^{-1}(y_x^{(i)}))$. Put $z := f^{-1}(x)$. Then, $f^{-1}(p_x^{(i)}) = p_z^{(k(i))}$ and $f^{-1}(y_x^{(i)}) = y_z^{(k'(i))}$ are well defined in $U(z)$ for some integers $k(i), k'(i) \in \{1,2,3,4\}$, where $i \in \{1,2,3,4\}$. We show that $(\bullet\bullet)$ $f^{-1}(F_1) \in p.\beta_{(2)}C(U(z))$. Indeed, we show $(\bullet\bullet)$ above for the following precise cases.

Case 1 $F_1 := \{p_x^{(i)}, y_x^{(i)}\}$**.** For the present case, we see that: $f^{-1}(F_1) = \{p_z^{k(i)}, y_z^{k'(i)}\}$. Since $p_x^{(i)} \in U(y_x^{(i)})$, by Lemma 5.22, it is shown that $f^{-1}(p_x^{(i)}) = p_z^{(k(i))} \in U(y_z^{(k'(i))})$, and $|k(i) - k'(i)| \le 1$ (cf. $(**)$ of the first part of the present (IV)) and so we have the following: $k'(i) = k(i)$ or $k'(i) = k(i) - 1$ because of $k'(i) \le k(i)$. Thus, we show that $f^{-1}(F_1) = \{p_z^{(k(i))}, y_z^{(k(i))}\}$ or $f^{-1}(F_1) = \{p_z^{(k(i))}, y_z^{(k(i)-1)}\}$, where $y_z^{(0)} := y_z^{(4)}$ and so $f^{-1}(F_1) \in pure\beta_{(2)}C(U(z))$.

Case 1′ $F_1 := \{p_x^{(i+1)}, y_x^{(i)}\}$**.** For the present case, we see that: $f^{-1}(F_1) = \{p_z^{(k(i+1))}, y_z^{(k'(i))}\}$. We claim that $f^{-1}(F_1) \in p.\beta_{(2)}C(U(z))$. The proof is analogous to the proof of Case 1 above, using Definition 5.12(ii), Remark 5.13(ii)(ii-2) and Lemma 5.22. And so the proof is omitted.

Case 2 $F_1 := \{x, y_x^{(i)}\}$**.** For the present case, we see that: $f^{-1}(F_1) = \{z, y_z^{(k(i))}\} \in pure\beta_{(2)}C(U(z))$.

Case 3 $F_1 = \{y_x^{(1)}, y_x^{(3)}\}$**.** For the present case, we see that: $f^{-1}(F_1) = \{y_z^{(k'(1))}, y_z^{(k'(3))}\}$, where $\{k'(1), k'(3)\} \subset \{1,2,3,4\}$. Since $U(y_z^{(k'(1))}) \cap U(y_z^{(k'(3))}) = \emptyset$ and $U(y_z^{(k'(i))}) \subset U(z)$ for each $i \in \{1,3\}$, we have the following: $0 < |k'(1) - k'(3)| \le 2$. And, if $|k'(1) - k'(3)| = 1$ then $U(y_z^{(k'(1))}) \cap U(y_z^{(k'(3))}) \ne \emptyset$ and so $|k'(1) - k'(3)| = 2$. Thus, we see that $f^{-1}(F_1) = \{y_z^{(k'(1))}, y_z^{(k'(3))}\} \in p.\beta_{(2)}C(U(z))$.

Case 3′ $F_1 = \{y_x^{(2)}, y_x^{(4)}\}$**.** For the present case, we see that: $f^{-1}(F_1) = \{y_z^{(k'(2))}, y_z^{(k'(4))}\} \in p.\beta_{(2)}C(U(z))$, where $\{k'(2), k'(4)\} \subset \{1,2,3,4\}$. The proof is analogous to the proof of Case 3 above and so the proof is omitted.

Case 4 $F_1 = \{y_x^{(i)}, y_x^{(i+1)}\}$**.** For the present case, we see that:
$f^{-1}(F_1) = \{y_z^{(k'(i))}, y_z^{(k'(i+1))}\}$, where $\{k'(i), k'(i+1)\} \subset \{1,2,3,4\}$ and $i \in \{1,2,3,4\}$. Since $U(y_x^{(i)}) \cap U(y_x^{(i+1)}) = \{p_x^{(i+1)}\}$, we have the following that: $U(y_z^{(k'(i))}) \cap U(y_z^{(k'(i+1))}) = \{p_z^{(k(i+1))}\}$ (cf. Lemma 5.22) and $|k'(i) - k'(i+1)| = 1$, i.e., $k'(i) - k'(i+1) = 1$ (if $k'(i) > k'(i+1)$) or $k'(i+1) - k'(i) = 1$ (if $k'(i) < k'(i+1)$). Thus, we prove that $f^{-1}(F_1) = \{y_z^{(k'(i))}, y_z^{(k'(i+1))}\} = \{y_z^{(k'(i))}, y_z^{(k'(i)-1)}\}$ or $f^{-1}(F_1) = \{y_z^{(k'(i))}, y_z^{(k'(i)+1)}\}$, and so $f^{-1}(F_1) \in p.\beta_{(2)}C(U(z))$ (cf. Remark 5.13(ii)(ii-2), where $y_z^{(5)} := y_z^{(1)}$ and $y_z^{(0)} := y_z^{(4)}$.

Thus, by all cases above, the property $(\bullet\bullet)$ is proved. And, it is shown that, for each set $F_1 \in p.\beta_{(2)}C(\mathbb{Z}^2)$, there exists a point $z \in (\mathbb{Z}^2)_{\mathcal{F}^2}$ such that $f^{-1}(F_1) \in p.\beta_{(2)}C(U(z))$, $f^{-1}(F_1) \subset \mathbb{Z}^2$ with $|f^{-1}(F_1)| = 2$, i.e., $f^{-1}(F_1) \in p.\beta_{(2)}C(\mathbb{Z}^2)$ (cf. Definition 5.15). Therefore, the homeomorphism $f : (\mathbb{Z}^2, \kappa^2) \to (\mathbb{Z}^2, \kappa^2)$ is $p.\beta_{(2)}C(\mathbb{Z}^2)$-irresolute.

(ii) (resp. (ii)′) By Definition 5.21(i) (resp. (i)′) and (i)(1a) (1b) (resp. (i)(1b) (1c)) above, the present (ii) (resp. (ii)′) is proved. □

Remark 5.24 The properties (1a), (1b) and (1c) in Theorem 5.23(i) are not hold, in general. This can be shown in the following example. Let $f : (\mathbb{Z}^2, \kappa^2) \to (\mathbb{Z}^2, \kappa^2)$ be a bijection defined by $f((x_1, x_2) := (x_1 + 1, x_2)$ for each point $(x_1, x_2) \in \mathcal{Z}^2$. Then, $f^{-1}(\{(1,1)\}) = \{(0,1)\} \notin \kappa^2$ for the set $\{(1,1)\} \in \kappa^2$ and so f is not a homeomorphsm. For the set $V := \{(1,1), (1,-1)\} \in \beta_{(2)}O(\mathbb{Z}^2)$, we have the following: $f^{-1}(V) = \{(0,1), (0,-1)\} \notin \beta_{(2)}O(\mathbb{Z}^2)$ and so f is not $\beta_{(2)}O(\mathbb{Z}^2)$-irresolute. And, for a set $F := \{(0,1), (0,-1)\} \in p.\beta_{(2)}C(\mathbb{Z}^2)$, we have the following: $f^{-1}(F) = \{(-1,1), (-1,-1)\} \notin p.\beta_{(2)}C(\mathbb{Z}^2)$ and so f is not p.$\beta_{(2)}C(\mathbb{Z}^2)$-irresolute. Moreover, for the above sets F and $f^{-1}(F)$, since $F \in \beta_{(2)}C(\mathbb{Z}^2)$ and $f^{-1}(F) \notin \beta_{(2)}C(\mathbb{Z}^2)$, the bijection f is not $\beta_{(2)}C(\mathbb{Z}^2)$-irresolute.

Theorem 5.25 *Let* $(H, \kappa^2|H)$ *be a subspace of* $(\mathbb{Z}^2, \kappa^2)$ *where* $|H| \geq 2$.

(i) (resp. (i)′) *The collection* $\beta_{(2)}ch(H; \kappa^2|H)$ (resp. $p.\beta_{(2)}ch(H; \kappa^2|H)$) *forms a group under the composition of mappings* (cf. Definition 5.21(i) (resp. (i)′).

(ii) (resp. (ii)′) *The union of two collections:* $\beta_{(2)}ch(H; \kappa^2|H) \cup con\text{-}\beta_{(2)}ch(H; \kappa^2|H)$ (resp. $p.\beta_{(2)}ch(H; \kappa^2|H) \cup con\text{-}p.\beta_{(2)}ch(H; \kappa^2|H)$) *forms a group under the composition of mappings* (cf. Definition 5.21(i),(ii) (resp. (i)′, (ii)′).

(iii) (resp. (iii)′) *The group* $\beta_{(2)}ch(H; \kappa^2|H)$ (resp. $p.\beta_{(2)}ch(H; \kappa^2|H)$) *is a non-empty subgroup of* $\beta_{(2)}ch(H; \kappa^2|H) \cup con\text{-}\beta_{(2)}ch(H; \kappa^2|H)$ (resp. $p.\beta_{(2)}ch(H; \kappa^2|H) \cup$ - $con\text{-}p.\beta_{(2)}ch(H; \kappa^2|H)$).

(iv) (resp. (iv)′) *The group* $h(\mathbb{Z}^2; \kappa^2)$ *is a subgroup of the group* $\beta_{(2)}ch(\mathbb{Z}^2; \kappa^2)$ (resp. $p.\beta_{(2)}ch(\mathbb{Z}^2; \kappa^2)$) *and so* $h(\mathbb{Z}^2; \kappa^2)$ *is a subgroup of the group* $\beta_{(2)}ch(\mathbb{Z}^2; \kappa^2) \cup con\text{-}\beta_{(2)}ch$ - $(\mathbb{Z}^2; \kappa^2)$ (resp. $p.\beta_{(2)}ch(\mathbb{Z}^2; \kappa^2) \cup con\text{-}p.\beta_{(2)}ch(\mathbb{Z}^2; \kappa^2)$).

Proof. **(i) (resp. (i)′)** A binary operation $\eta_H : \beta_{(2)}ch(H; \kappa^2|H) \times \beta_{(2)}ch(H; \kappa^2|H) \to \beta_{(2)}ch(H; \kappa^2|H)$ (resp. $\eta'_H : p.\beta_{(2)}ch(H; \kappa^2|H) \times p.\beta_{(2)}ch(H; \kappa^2|H) \to p.\beta_{(2)}ch(H;$- $\kappa^2|H)$) is well defined by $\eta_H(g_1, g_2) := g_2 \circ g_1$ (resp. $\eta'_H(g_1, g_2) := g_2 \circ g_1$). Indeed, by using Definitions 5.20(Note:1),5.21(i) (resp. (i)′), it is shown that $g_2 \circ g_1$ and $(g_2 \circ g_1)^{-1}$ are both $\beta_2 O(H)$-irresolute and they are $\beta_2 C(H)$-irresolute (resp. $g_2 \circ g_1$ and $(g_2 \circ g_1)^{-1}$ are both $\beta_{(2)} O(H)$-irresolute and they are $p.\beta_{(2)} C(H)$-irresolute). Thus, we prove that $\eta_H(g_1, g_2) \in \beta_{(2)}ch(H; \kappa^2|H)$ (resp. $\eta'_H(g_1, g_2) \in p.\beta_{(2)}ch(H; \kappa^2|H)$) and the binary operation η_H (resp. η'_H) satisfies the axiom of group. Therefore, the pair $(\beta_{(2)}ch(H; \kappa^2|H), \eta_H)$ (resp. $(p.\beta_{(2)}ch(H; \kappa^2|H), \eta'_H)$) forms a group under compositions of mappings.

(ii) (resp. (ii)′) We first note on the following notation that: let $\mathcal{G}_H := \beta_{(2)}ch(H; \kappa^2|H) \cup con\text{-}\beta_{(2)}ch(H; \kappa^2|H)$ (resp. $p\mathcal{G}_H := p.\beta_{(2)}ch(H; \kappa^2|H) \cup con\text{-}p.\beta_{(2)}ch(H; \kappa^2|H)$) throughout the present proof of (ii) (resp. (ii)′). A binary operation $w_H : \mathcal{G}_H \times \mathcal{G}_H \to \mathcal{G}_H$ (resp. $w'_H : p\mathcal{G}_H \times p\mathcal{G}_H \to p\mathcal{G}_H$) is well defined by $w_H(f, f') := f' \circ f$ (resp. $w'_H(f, f') := f' \circ f$). Indeed, let $(f, f') \in \mathcal{G}_H \times \mathcal{G}_H$.

Case 1 $f \in \beta_{(2)}ch(H; \kappa^2|H)$ and $f' \in con\text{-}\beta_{(2)}ch(H; \kappa^2|H)$ (resp. **Case 1′** $f \in p.\beta_{(2)}ch(H; \kappa^2|H)$ and $f' \in con\text{-}p.\beta_{(2)}ch(H; \kappa^2|H)$)**.** For the present case, it is claimed that $w_H(f, f') \in con\text{-}\beta_{(2)}ch(H; \kappa^2|H) \subseteq \mathcal{G}_H$ (resp. $w'_H(f, f') \in con\text{-}p.\beta_{(2)}ch(H; \kappa^2|H)$), because $f' \circ f$ and $(f' \circ f)^{-1}$ are both $(\beta_{(2)} O(H), \beta_{(2)} C(H))$-irresolute and they are $(\beta_{(2)} C(H), \beta_{(2)} O(H))$-irresolute (resp. $f' \circ f$ and $(f' \circ f)^{-1}$ are both $(\beta_{(2)} O(H), p.\beta_{(2)} C(H))$-irresolute and they are $(p.\beta_{(2)} C(H), \beta_{(2)} O(H))$-irresolute). And so, we have that $w_H(f, f') \in con\text{-}\beta_{(2)}ch(H; \kappa^2|H) \subseteq \mathcal{G}_H$ (resp. $w'_H(f, f') \in con\text{-}p.\beta_{(2)}ch$ - $(H; \kappa^2|H) \subseteq p\mathcal{G}_H$).

Case 2 $f \in con\text{-}\beta_{(2)}ch(H; \kappa^2|H)$ and $f' \in \beta_{(2)}ch(H; \kappa^2|H)$ (resp. **Case 2′** $f \in con\text{-}p.\beta_{(2)}ch(H; \kappa^2|H)$ and $f' \in p.\beta_{(2)}ch(H; \kappa^2|H)$)**.** For the present case, by similar argument of that of Case 1 (resp. Case 1′) above, it is shown that $w_H(f, f') = f' \circ f \in con\text{-}\beta_{(2)}ch(H; \kappa^2|H) \subseteq \mathcal{G}_H$ (resp. $w'_H(f, f') = f' \circ f \in con\text{-}p.\beta_{(2)}ch(H; \kappa^2|H) \subseteq p\mathcal{G}_H$).

Case 3 $f \in con\text{-}\beta_{(2)}ch(H; \kappa^2|H)$ and $f' \in con\text{-}\beta_{(2)}ch(H; \kappa^2|H)$ (resp. **Case 3′** $f \in con\text{-}p.\beta_{(2)}ch(H; \kappa^2|H)$ and $f' \in con\text{-}p.\beta_{(2)}ch(H; \kappa^2|H)$)**.** For the present case, it is shown that $w_H(f, f') \in \beta_{(2)}ch(H; \kappa^2|H) \subseteq \mathcal{G}_H$ (resp. $w'_H(f, f') \in p.\beta_{(2)}ch(H; \kappa^2|H) \subseteq p\mathcal{G}_H$), because $f' \circ f$ and $(f' \circ f)^{-1}$ are both $\beta_{(2)} O(H)$-irresolute and they are $\beta_{(2)} C(H)$-irresolute (resp. $f' \circ f$ and $(f' \circ f)^{-1}$ are both $\beta_{(2)} O(H)$-irresolute and they are $p.\beta_{(2)} C(H)$-irresolute).

Case 4 $f \in \beta_{(2)}ch(H; \kappa^2|H)$ and $f' \in \beta_{(2)}ch(H; \kappa^2|H)$ (resp. **Case 4′** $f \in p.\beta_{(2)}ch(H; \kappa^2|H)$ and $f' \in p.\beta_{(2)}ch(H; \kappa^2|H)$)**.** For the present case, by definitions, it is shown that $w_H(f, f') \in \beta_{(2)}ch(H; \kappa^2|H) \subseteq \mathcal{G}_H$ (resp. $w_H(f, f') \in p.\beta_{(2)}ch(H; \kappa^2|H) \subseteq p\mathcal{G}_H$), because $f' \circ f$ and $(f' \circ f)^{-1}$ are both $\beta_{(2)} O(H)$-irresolute and they are $\beta_{(2)} C(H)$-irresolute (resp. $f' \circ f$ and $(f' \circ f)^{-1}$ are both $\beta_{(2)} O(H)$-irresolute and they are $p.\beta_{(2)} C(H)$-iresolute). Finally, the binary operation $w_H : \mathcal{G}_H \times \mathcal{G}_H \to \mathcal{G}_H$ (resp. $w'_H : p\mathcal{G}_H \times p\mathcal{G}_H \to p\mathcal{G}_H$) satisfies the axiom of group (cf. the proof of (i)) the identity function on $H, 1_H \in \beta_{(2)}ch(H; \kappa^2|H) \subseteq \mathcal{G}_H$ (resp. $1_H \in p.\beta_{(2)}ch(H; \kappa^2|H) \subseteq p\mathcal{G}_H$); and so the pair $(\mathcal{G}_H, w_H)$ (resp. $(p\mathcal{G}_H, w'_H)$ forms a group under compositions of mappings. **(iii)** We recall that $\mathcal{G}_H := \beta_{(2)}ch(H;\kappa^2|H) \cup$ *con-*$\beta_{(2)}ch(H;\kappa^2|H)$ (cf. Proof of (ii) above). Since $1_H \in \beta_{(2)}ch(H;\kappa^2|H)$, we have the following: $(\cdot 1)$ $\beta_{(2)}ch(H;\kappa^2|H)$ is a nonempty subset of the group $(\mathcal{G}_H, w_H)$, where $w_H : \mathcal{G}_H \times \mathcal{G}_H \to \mathcal{G}_H$ is the binary operation (cf. (ii) above). Let $f, g \in \beta_{(2)}ch(H;\kappa^2|H)$. Then, we see that $(\cdot 2)$ $w_H(f, g^{-1}) = g^{-1} \circ f \in \beta_{(2)}ch(H;\kappa^2|H)$. Therefore, by $(\cdot 1)$ and $(\cdot 2)$, it is shown that $\beta_{(2)}ch(H;\kappa^2|H)$ is a subgroup of $\mathcal{G}_H$ (cf. (ii) above). **(iii)$'$** We recall that $p\mathcal{G}_H := p.\beta_{(2)}ch(H;\kappa^2|H) \cup con\text{-}p.\beta_{(2)}ch(H;\kappa^2|H)$ (cf. Proof of (ii)$'$ above). We see that $1_H \in p.\beta_{(2)}ch(H;\kappa^2|H)$. Indeed, for a set $B \in \beta_{(2)}O(H)$ and $F_1 \in p.\beta_{(2)}C(H)$, we have the following: $1_H(B) = (1_H)^{-1}(B) = B \in \beta_{(2)}O(H)$ and $1_H(F_1) = (1_H)^{-1}(F_1) = F_1 \in p.\beta_{(2)}C(H)$; and so 1_H and $(1_H)^{-1}$ are both $\beta_{(2)}O(H)$-irresolute and they are $p.\beta_{(2)}C(H)$-irresolute. Thus, by Defintion 5.21(i)$'$, it is obtained that $1_H \in p.\beta_{(2)}ch(H;\kappa^2|H)$. Then, we have the following: $(\cdot 1)' p.\beta_{(2)}ch(H;\kappa^2|H)$ is a nonempty subset of the group $(p\mathcal{G}_H, w'_H)$, where $w'_H : p\mathcal{G}_H \times p\mathcal{G}_H \to p\mathcal{G}_H$ is the binary operation (cf. Proof of (ii)$'$ above). Next, we claim $(\cdot 2)'$ below. Let $f, g \in p.\beta_{(2)}ch(H;\kappa^2|H)$. Then, since f, f^{-1}, g and g^{-1} are all $\beta_{(2)}O(H)$-irresolute and they are $p.\beta_{(2)}C(H)$-irresolute, $g^{-1} \circ f$ and $(g^{-1} \circ f)^{-1} = f^{-1} \circ g$ are both $\beta_{(2)}O(H)$-irresolute and they are $p.\beta_{(2)}C(H)$-irresolute bijections. Thus, we prove that: $(\cdot 2)' w'_H(f, g^{-1}) = g^{-1} \circ f \in p.\beta_{(2)}ch(H;\kappa^2|H)$ (cf. Definition 5.21(i)$'$). Therefore, by $(\cdot 1)'$ and $(\cdot 2)'$ above, it is obtained that $p.\beta_{(2)}ch(H;\kappa^2|H)$ is a subgroup of $p\mathcal{G}_H$ (cf. (ii)$'$ above). **(iv) (resp. (iv)$'$)** We see that the identity function $1_{\mathbb{Z}^2} : (\mathbb{Z}^2, \kappa^2) \to (\mathbb{Z}^2, \kappa^2)$ is a homeomorphism and so $1_{\mathbb{Z}^2} \in h(\mathbb{Z}^2;\kappa^2) \neq \emptyset$. By (i) (resp. (i)$'$) above and its proof, it is known that $\beta_{(2)}ch(\mathbb{Z}^2;\kappa^2)$ (resp. $p.\beta_{(2)}ch(\mathbb{Z}^2;\kappa^2)$) forms a group with the binary operation $\eta_{\mathbb{Z}^2}$ (resp. $\eta'_{\mathbb{Z}^2}$) defined by $\eta_{\mathbb{Z}^2}(a, b) = b \circ a$ (resp. $\eta'_{\mathbb{Z}^2}(a, b) = b \circ a$) for every $a, b \in \beta_{(2)}ch(\mathbb{Z}^2;\kappa^2)$ (resp. $p.\beta_{(2)}ch(\mathbb{Z}^2;\kappa^2)$). And, using Theorem 5.23(ii) (resp. (ii)$'$), we recall that $h(\mathbb{Z}^2;\kappa^2) \subseteq \beta_{(2)}ch(\mathbb{Z}^2;\kappa^2)$ (resp. $h(\mathbb{Z}^2;\kappa^2) \subseteq p.\beta_{(2)}ch(\mathbb{Z}^2;\kappa^2)$). Then, we have the following: $\eta_{\mathbb{Z}^2}(f, g^{-1}) = g^{-1} \circ f$ (resp. $\eta'_{\mathbb{Z}^2}(f, g^{-1}) = g^{-1} \circ f$) $\in h(\mathbb{Z}^2;\kappa^2)$ for any $f, g \in h(\mathbb{Z}^2;\kappa^2)$. Therefore, it is proved that $h(\mathbb{Z}^2;\kappa^2)$ is a subgroup of $\beta_{(2)}ch(\mathbb{Z}^2;\kappa^2)$ (resp. $p.\beta_{(2)}ch(\mathbb{Z}^2;\kappa^2)$). And so, using (iii) (resp. (iii)$'$) above, it is obtained that $h(\mathbb{Z}^2;\kappa^2)$ is also a subgroup of $\mathcal{G}_{\mathbb{Z}^2}$ (resp. $p\mathcal{G}_{\mathbb{Z}^2}$) (cf. the proof of (ii) or (iii) (resp. (ii)$'$ or (iii)$'$) for the notation). □

Notation 5.26 The present notations are applied to Example 5.27 below. Let $H := U((0,0))$. And $U((0,0))$ is denoted abbreviately by U (i.e., $U := U((0,0))$). We define the following functions and two families of functions, **(·1)** $\rho_{45} : (U;\kappa^2|U) \to (U;\kappa^2|U)$ is defined by $\rho_{45}((0,0)) := (0,0)$, $\rho_{45}(p^{(i)}) := y^{(i)}$, $\rho_{45}(y^{(i)}) := p^{(i+1)}$ for each $i \in \{1,2,3,4\}$ with $p^{(5)} := p^{(1)}$ (cf. $(*)$ of line 5 from the top of the present subsection (IV), or (III-2) of the subsection (III)), **(·2)** $\rho_{0\times 90} := 1_U$ (the identity function on U) and $\rho_{k\times 90} := \rho_{(k-1)\times 90} \circ (\rho_{45} \circ \rho_{45})$ for each $k \in \{1,2,3\}$, **(·3)**$\rho_{1\times 45} := \rho_{45}, \rho_{m\times 45} := \rho_{90} \circ \rho_{(m-2)\times 45}$ (for $m = 3,5,7$) and **(·4)** $\mathcal{R}_{45} := \{\rho_{m\times 45}, (\rho_{m\times 45})^{-1} | m \in \{1,3,5,7\}\}$, $\mathcal{R}_{90} := \{1_U, \rho_{k\times 90}, (\rho_{k\times 90})^{-1} | k \in \{1,2,3\}\}$.

Example 5.27 Let $H := U((0,0))$ and $U := U((0,0))$. We have the following examples.

(i) $\{\rho_{45}, (\rho_{45})^{-1}\} \subseteq con\text{-}p.\beta_{(2)}ch(U;\kappa^2|U)$ (cf. Corollary 5.11(ii)).

(ii) (1) $\{\rho_{90}, (\rho_{90})^{-1}\} \subseteq \beta_{(2)}ch(U;\kappa^2|U)$,**(2)** $\{\rho_{90}, (\rho_{90})^{-1}\} \subseteq p.\beta_{(2)}ch(U;\kappa^2|U)$.

In general, we have that:

(i)$'$ $\mathcal{R}_{45} \subseteq con\text{-}p.\beta_{(2)}ch(U;\kappa^2|U)$ (cf. Corollary 5.11(ii)).

(ii)$'$ (1)$'$ $\mathcal{R}_{90} \subseteq \beta_{(2)}ch(U;\kappa^2|U)$,

(2)$'$ $\mathcal{R}_{90} \subseteq p.\beta_{(2)}ch(U;\kappa^2|U)$ (cf. Notation 5.26). Hence we have the following:

(iii) (1) $\mathcal{R}_{45} \cup \mathcal{R}_{90} \subseteq \beta_{(2)}ch(U;\kappa^2|U) \cup con\text{-}p.\beta_{(2)}ch(U;\kappa^2|U)$,

(2) $\mathcal{R}_{45} \cup \mathcal{R}_{90} \subseteq p.\beta_{(2)}ch(U;\kappa^2|U) \cup con\text{-}p.\beta_{(2)}ch(U;\kappa^2|U)$. The proofs are omitted on the present paper (cf. the detailed proofs are shown by the following pre-print; The detailed Example 5.27 [34]).

Acknowledgements. The authors would like express their gratitude to the referee and the editor.

References

[1] M. E. Abd El-Monsef, S.N. El-Deeb and R.A. Mahmoud, β-open sets and β-continuous mappings, *Bull. Fac. Assuit Univ.*, **12** (1983), 77- 90.

[2] D. Andrijević, Semi-preopen sets, *Mat. Vesnik*, **38**, (1986), 24-32.

[3] D. Andrijević, On b-open sets, *Mat. Bech.*,**8**, (1996),59- 64.

[4] S.C. Arora, S.Tahiliani and H. Maki, On π generalized β-closed sets in topological spaces II, *Sci.Math. Jpn.*, **71** (2010), no.1, 43- 54; online **e-2009**, 637- 648.

[5] M. Caldas, Weak and strong forms of irresolute maps, *Internat. J. Math., Math. Sci.*, **23**(4) (2000), 253 - 259.

[6] M. Caldas and S. Jafari, Weak and strong forms of β-irresolute maps, *Arabian J. Sci. Engrg.*, **31**1A(2006), 31-39.

[7] M. Caldas and S. Jafari, Some properties of contra-β-continuous functions, *Mem. Fac. Sci. Kochi Univ. Series A. Math.*, **22** (2001), 19-28.

[8] S.G.Crossley and S.K.Hildebrand, Semi-topological properties, *Fund Math.*, **74** (1972), 233-254.

[9] R. Devi, K. Bhuvaneswari and H. Maki, Weak forms on $g\rho$-closed sets, where $\rho \in \{\alpha, \alpha^*, \alpha^{**}\}$, and digital planes, *Mem. Fac. Sci.Kochi Univ.* (*Math.*), **25** (2004), 37-54.

[10] R.Devi, V.Kokilavani and H.Maki, More on $g^{\#}\alpha$-closed sets in digital planes, *Sci. Math. Japon.*, **70**(1)(2009), 53-62: online **e-2009**,301-310.

[11] R.Devi, S.N.Rajappriya, K.Muthkumarraswamy and H.Maki,ξ-closed sets in topological spaces and digital planes, *Sci. Math. Japon.*, **64**(3)(2006),533-549:online **e-2006**,615-631.

[12] J. Dontchev, Chracterization of some peculiar topological spaces via A- and B-sets, *Act. Math. Hungar*, **69**(1-2) (1995), 67- 71.

[13] J. Dontchev, Contra-continuous functions and strongly S-closed spaces, *Internat. J. Math. Soc.*, **19** (1996), 303- 310.

[14] J. Dontchev, Contra semi-continuous functions, *Math. Panonica*, **10** (2) (1999), 159- 168.

[15] M. Fujimoto, H. Maki, T. Noiri and S. Takigawa, The digital palne is quasi- submaximal, *Quest. Answers Gen. Topology*, **22** (2004), 163- 168.

[16] M. Fujimoto, S. Takigawa, J. Dontchev, T. Noiri and H. Maki, The topological structure and groups digital n-spaces,*Kochi J. Math.*, **1**(2006), 31-55.

[17] S. Jafari and T. Noiri, On β-quasi-irresolute functions,*Mem. Fac. Sci. Kochi Univ. Math. Ser.*, **21**(2000), 53-62.

[18] S. Jafari and T. Noiri, Contra-β-continuous functions between topological spaces, *Iran. Int. J. Sci.*, **2**(2001), pp. 153 - 167.

[19] S. Jafari and T. Noiri, Properties of β-connected spaces, *Acta Math. Hungar.*, **101**(2)(2003), 227-236.

[20] T.Y. Kong, R. Kopperman and P.R. Meyer, A topological approach to digital topology, *Amer. Math. Monthly*, **98**(1991), 901- 917.

[21] K. Kopperman, Digital topology, *Section J-4 in Encyclopedia of General Topology;edited by K.P.Hart, J.Nagata and J.E.Vaughan*, Elsever(2004), 428-432.

[22] V. Kovalevsky and R.Kopperman, Some topology-based image processing algorithms, *Annales of the New York Academy of Sciences, papers on General Topology and Its applications*, **728**(1994), 174- 182.

[23] E. D. Khalimsky, Applcations of connected ordered topological spaces in topology, *Conference of Math. Departments of Povolsia*, (1970).

[24] E. D. Khalimsky, Topological structures in computer sciences, *J.Appl. Math. Simulation*, **1**(1)(1987), 25-40.

[25] E. D. Khalimsky, R. Kopperman and P. R. Meyer, Computer graphics and connected topologies on finite ordered sets, *Topology Appl.*, **36**(1990), 1-17.

[26] N. Levine, Semi-open sets and semi-continuity in topolological spaces, *Amer. Math. Monthly*, **70**(1963), 36-41.

[27] H. Maki, S. Takigawa, M. Fujimoto, P.Sundaram and M. Sheik John, Remarks on ω-closed sets in Sundaram-Sheik John's sense of digital n-spaces, *Sci. Math. Jpn.*, **77** (3) (2014), 317-337; (Online version, e-2014 (no. 27); (2014-11), 123-143).

[28] H.Maki and A.I.El-Maghrabi, Contra-γ-irresolute mappings and related groups, *Scientiae Mathematicae Japonicae*, **80**(3)(2017), 233-245.

[29] Erik Melin, Extension of continuous functions in digital spaces with the Khalimsky topology, *Topology Appl.*, **153**(2005), 27-29.

[30] Erik Melin, Continuous extension in topological digital spaces, *Applied General Topology*, **9**(2008), 51-61.

[31] Fumie Nakaoka, Fumikazu Tamari and Haruo Maki,On generalized digital lines, *Scientiae Mathematicae Japonicae*, **80**(1) (2017), 15-34: online **e-2017**.

[32] A.A. Nasef and T. Noiri, Some weak forms of almost continuity, *Acta Math. Hinga.*, **74**(3) (1997), 211-219.

[33] A.A. Nasef and H. Maki, On some maps concerning gp-closed sets and related groups, *Sci. Math. Jpn.*, **71**(1) (2010), 55- 81: Online **e-2009**, 649-675.

[34] A.A. Nasef, H. Maki and A.El Fattah El Atik, The detailed Example 5.27, *pre-print version*, (2020), 1-4.

[35] O.Njastad, On some classes of nearly open sets, *Pacific J. Math.*, **15**(1965), 961-970.

[36] T. Noiri, Remarks on semiopen mapping, *Bull. Calcutta Math. Soc.*, **65**(1973), 197 -201.

[37] T. Noiri, On δ-continuous functions, *J. Korean Math. Soc.*, **16** (1979/80), no.2, 161- 166.

[38] T. Noiri, Modified open and closed sets (semi-open sets etc), *Section A-2 in Encyclopedia of General Topology;edited by K.P.Hart, J.Nagata and J.E.Vaughan*, Elsever(2004), 8-10.

[39] K.Nono, R.Devi, M.Devipriya, K.Muthhunkumaraswamy and H.Maki, On $g^{\#}\alpha$-closed sets and the digital plane, *Bull. Fukuoka Univ. Ed. Part III*, **53**(2004), 15-24.

[40] S. Tahiliani, On some maps concerning β-closed sets and related groups, *Gloval J. Pure and Applied Math.*, **15**(2)(2019), 161-168.

[41] S. Takigawa, M. Ganster, H. Maki, T. Noiri and M. Fujimoto, The digital n-space is quasi-submaximal, *Quest. Answers Gen. Topology*, **26**(2008), 45- 52.

[42] S. Takigawa, Characterizations of preopen sets and semi-open sets in digital n-space, this was lectured at " the 14^{th} meetings on topological spaces theory and its applications", August 22-23, 2009, Fukuoka University Seminar House; one can read the resume from its proceeding (collection of abstracts), p.43-44.

[43] J. Tong, On decomposition of continuity in topological spaces, *Acta. Math. Hungar*, **54**(1989), 51- 55.

Communicated by *Yasunao Hattor*i

Arafa A. Nasef; Department of Physics and Engineering Mathematics
Faculty of Engineering, Kafrelsheikh University, EGYPT
e-mail:nasefa50@yahoo.com

Haruo Maki; 2-10-13 Wakagidai, Fukutsu-shi, Fukuoka-ken, 811-3221, JAPAN
e-mail:makih@pop12.odn.ne.jp

Abd El Fattah A. El Atik; Department of Mathematics, Faculty of Science
Tanta University, Tanta, EGYPT
e-mail:aelatik@science.tanta.edu.eg