Dynamical system on a parabolic and elliptic Gelfand-type equation

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Abstract. We consider the parabolic equation with exponential nonlinearity and the corresponding elliptic equation. First, we study the set of stationary solution and its spectral property. Next we show that the solution of parabolic equation blows up in finite time for the initial value satisfying a positive integrand condition by the Kaplan method. Finally we find a global solution for the negative initial value by upper-lower solution method and for the two dimensional domain by the Trudinger-Moser inequality, respectively. By the global boundedness and the existence of Lyapunov function, we treat its dynamical properties of the omega limit set.

1 Introduction We consider the parabolic equation

$$(1)\qquad \begin{cases} u_t = \Delta u + \lambda\,(e^u - 1) & x \in \Omega,\ \ t \in (0, T_{u_0}), \\ u(x,t) = 0 & x \in \partial\Omega,\ \ t \in (0, T_{u_0}), \\ u(x,0) = u_0(x) & x \in \Omega \end{cases}$$

and the associated elliptic equation

$$(2)\qquad \begin{cases} \Delta v + \lambda\,(e^v - 1) = 0 & x \in \Omega, \\ v(x) = 0 & x \in \partial\Omega, \end{cases}$$

where $\lambda > 0$, Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ for $n \in \mathbb{N}$ and T_{u_0} denotes the maximal existing time for the solution of (1) with $u(x,0) = u_0(x)$. In the case of $n = 1$, we suppose that $\Omega = (0,1)$. The corresponding nonlocal parabolic and elliptic problems with the Neumann boundary condition are given by

$$(3)\qquad \begin{cases} u_t = \Delta u + \lambda\Big(\frac{e^u}{\int_\Omega e^u\,dx} - \frac{1}{|\Omega|}\Big) & x \in \Omega,\ \ t \in (0, T_{u_0}), \\ \frac{\partial u}{\partial \nu}(x,t) = 0 & x \in \partial\Omega,\ \ t \in (0, T_{u_0}), \\ u(x,0) = u_0(x) & x \in \Omega \end{cases}$$

and

$$(4)\qquad \begin{cases} \Delta v + \lambda\Big(\frac{e^v}{\int_\Omega e^v\,dx} - \frac{1}{|\Omega|}\Big) = 0 & x \in \Omega, \\ \frac{\partial v}{\partial \nu}(x) = 0 & x \in \partial\Omega, \end{cases}$$

respectively, where $|\Omega|$ is the measure of Ω in $\mathbb{R}^n$ and $\nu(x)$ is the outer unit normal vector at $x \in \partial\Omega$. (3) and (4) for $n = 1$ are investigated in [12, 17]. We have already obtained the results of the elliptic properties such as the structure of set of stationary solutions and the monotonicity of the Morse index. In the parabolic problem, for any $\lambda > 0$ and $u_0 \in H$ with an appropriate space H, (3) admits a unique global solution. However, by the lack of comparison principle we do not know whether the Morse-Smale property holds or not for

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the nonlocal problem.
Nowadays, it seems that there are not enough studies for (1) and (2) except [1]. The aim of this paper is to study (1) and (2), respectively. The solution set of (2) has already been investigated in [1]. To introduce the known results obtained in [1] and state our theorems, we denote the m-th eigenvalue and eigenfunction of $-\Delta$ in Ω with the Dirichlet boundary condition by μ^m and ϕ^m normalized as $\|\phi^m\|_2 = 1$ for $m \in \mathbb{N}$, respectively, where $\|\cdot\|_p$ is the standard L^p norm in Ω with $p \in [1, \infty]$. For the sake of convenience, we set $\mu^0 = 0$. We define the solution set $\mathcal{C}$ by

$$\mathcal{C} \equiv \Big\{(\lambda, v) \in \mathbb{R}^+ \times \big(C^2(\Omega) \cap C_0(\overline{\Omega})\big) \;\Big|\; v = v(x) \text{ solves (2) for } \lambda > 0\Big\},$$

where $\mathbb{R}^+ = \{x \mid x > 0\}$ and

$$C_0(\overline{\Omega}) \equiv \Big\{v \in C(\overline{\Omega}) \;\Big|\; v(x) = 0 \text{ on } x \in \partial\Omega\Big\}$$

endowed with the L^∞ norm. In [1], they derived the necessary condition for the existence of positive classical solution of (2). Together with their results, we have the following proposition:

Proposition 1 (Cf. Proposition 1.2 in [1]) *Let $\Omega \subset \mathbb{R}^n$ be a star-shaped domain with respect to the origin with C^2- boundary $\partial\Omega$ for $n \in \mathbb{N}$. There exists λ_0 which depends only on Ω satisfying $\lambda_0 \geq 0$ for $n = 1, 2$ and $\lambda_0 > 0$ for $n \geq 3$. Then we have following*:
(i) *if $(\lambda, v) \in \mathcal{C}$ satisfies $v > 0$ in Ω, then $\lambda \in (\lambda_0, \mu^1)$,*
(ii) *if $(\lambda, v) \in \mathcal{C}$ satisfies $v < 0$ in Ω, then $\lambda > \mu^1$,*
(iii) *if $n \geq 3$ and $\lambda \in (0, \lambda_0)$, then $(\lambda, v) \in \mathcal{C}$ satisfies $v = 0$ in Ω.*

If $(\lambda, v) \in \mathcal{C}$ is a classical solution, the Morse index $i = i(\lambda, v)$ is defined by the number of negative eigenvalues ν of

$$\begin{cases} \Delta\psi + \lambda e^v \psi = -\nu\psi & x \in \Omega, \\ \psi(x) = 0 & x \in \partial\Omega, \\ \|\psi\|_2 = 1. \end{cases} \tag{5}$$

First of all, we introduce results of the stationary solution. It is clear that (2) has a trivial solution $(\lambda, v) = (\lambda, 0)$ for any $\lambda > 0$. The second proposition is concerned with the bifurcation from the trivial solution and Morse index around the bifurcation point. The result for (2) is a little bit similar to that for (4). The difference is the value of the Morse index on the branch of the nontrivial solution set. We prove the existence of nontrivial solution by the bifurcation theory [2]. We compute Morse index by the exchange of eigenvalues [11, 12, 13].

Proposition 2 *Let $\Omega = (0, 1)$. Then we have $\mu^m = (m\pi)^2$ and $i(\lambda, 0) = m - 1$ for $\lambda \in (\mu^{m-1}, \mu^m]$ with $m \in \mathbb{N}$. Two continua $\mathcal{S}_m^\pm \subset \mathcal{C}$ of nontrivial solution bifurcate at $(\lambda, v) = (\mu^m, 0)$. Furthermore*

$$i(\lambda, v) = \begin{cases} 2k - 2 & \text{for } (\lambda, v) \in \mathcal{S}_m^- \text{ and } m = 2k - 1, \\ 2k - 1 & \text{for } (\lambda, v) \in \mathcal{S}_m^+ \text{ and } m = 2k - 1, \\ 2k - 1 & \text{for } (\lambda, v) \in \mathcal{S}_m^\pm \text{ and } m = 2k \end{cases}$$

holds for sufficiently close to the bifurcation point $(\mu^m, 0)$, where $k \in \mathbb{N}$.

In [1], they studied the bifurcation diagram and computed the bound for Morse index globally, not locally around a bifurcation point. If Ω is a unit ball and the solution is positive and radially symmetric, they establish the existence of singular solution, multiple existence of the regular solution and bound for its Morse index. Their results are similar to those of the well-known Gelfand problem

$$\Delta v + \lambda e^v = 0.$$

As mentioned in [6, 15], they also derived the bending result of the solution set for $n \in [3, 9]$. Next, we consider the case where the solution blows up in finite time. Thanks to the convexity of $e^v - 1$, we can apply the Kaplan method [8] to obtain following two blow-up conditions of u_0.

Proposition 3 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$ for $n \in \mathbb{N}$. For $u_0 \in C_0(\overline{\Omega})$, the solution of (1) blows up in finite time on the condition that*
(i) $\int_\Omega u_0(x)\phi^1(x)\,dx > 2\left\|\phi^1\right\|_1\left(\mu^1 - \lambda\right)/\lambda$ holds for $0 < \lambda < \mu^1$,
(ii) $\int_\Omega u_0(x)\phi^1(x)\,dx > 0$ holds for $\lambda \geq \mu^1$.

The first global existence result is concerned with the nonpositive solution. For the nonpositive initial value, we establish the global solution by constructing a lower-upper solution pair.

Theorem 1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$ for $n \in \mathbb{N}$. For $u_0 \in C_0(\overline{\Omega})$ with $u_0(x) \leq 0$ for any $x \in \Omega$, we have $T_{u_0} = +\infty$ and*

$$u \in C\big([0, +\infty); C_0(\overline{\Omega})\big) \cap C^1\big((0, +\infty); C_0(\overline{\Omega})\big)$$

satisfying $u \leq 0$ in $\Omega \times [0, +\infty)$. If $\lambda < \mu^1$, then

$$\|u(\cdot, t)\|_{H_0^1} \to 0$$

as $t \to +\infty$, where $\|w\|_{H_0^1} = \|\nabla w\|_2$ for $w \in H_0^1(\Omega)$.

We introduce the main theorems on the global existence for $\Omega \subset \mathbb{R}^2$ and $\Omega = (0, 1)$, respectively. For small parameter and initial value, we construct the global solution by the Lyapunov function

$$L_\lambda(u) \equiv \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \lambda\int_\Omega (e^u - u)\,dx, \tag{6}$$

Sobolev and Trudinger-Moser inequalities.

Theorem 2 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial\Omega$. For any $\lambda > 0$ and $u_0 \in H_0^1(\Omega)$ satisfying*

$$\left(C_{TM}^2 + \frac{2|\Omega|}{\mu^1}\right)\lambda^2 + \|u_0\|_{H_0^1}^2 < 4\pi(\log 4\pi - 1), \tag{7}$$

we have $T_{u_0} = +\infty$ and

$$u \in C\big([0, +\infty); H_0^1(\Omega)\big) \cap C^1\big((0, +\infty); L^2(\Omega)\big),$$

where $C_{TM} > 0$ is a constant which depends only on Ω coming from the Trudinger-Moser inequality. Moreover, there is some $\lambda_1 > 0$ such that for any $\lambda < \lambda_1$, we have

$$\|u(\cdot, t)\|_{H_0^1} \to 0$$

as $t \to +\infty$.

Theorem 3 *Let $\Omega = (0,1)$. If we replace (7) by*

$$2\left(e^{2eC_S^2} + \frac{1}{\pi^2}\right)\lambda^2 + \|u_0\|_{H_0^1}^2 < e\log 2,$$

then the conclusion of Theorem 2 is still true, where $C_S > 0$ is an embedding constant which depends only on Ω coming from $H_0^1(\Omega) \subset C(\overline{\Omega})$.

In the last result, we derive the dynamical properties. The Lyapunov function (6) plays an important role in arguing the convergence problem.

Proposition 4 (Cf. Theorem 2.1 in [4]) *Under the same hypotheses as Theorems 2 or 3, $\omega(u_0)$ is invariant, non-empty, compact and connected in $H_0^1(\Omega)$. Moreover $\omega(u_0)$ is a single point in $H_0^1(\Omega)$.*

This paper is composed of 5 sections. In Section 2, we show Propositions 1 and 2. We obtain the stationary solution by a bifurcation theory and compute the Morse index. In Section 3, we obtain some differential inequalities by the energy method and solve them to show Proposition 3. In Section 4, we decompose a solution of (1) into that of the heat equation with the non-zero initial value and the nonlinear heat equation with the zero initial value. Then we construct a lower and upper solution, which leads us to the proof of Theorem 1. In Section 5, we use the Lyapunov function and Trudinger-Moser inequality. Then we derive the H^1 estimate, which gives us the proof of Theorems 2 and 3. Finally, we derive the compactness of the orbit, which prove that the global solution converges to a stationary solution. By the existence of the Lyaounov function, we can prove Proposition 4.

2 Stationary solution First, we consider the condition on a parameter when a positive, negative or trivial solution exists. We use the Kaplan method [8] and Pohožaev identity [16]. If Ω is a ball, similar results to those in Proposition 1 are obtained in [1]. Next, we apply a bifurcation theory in [2], obtain a curve of solution (λ, v) and parametrize the solution $(\lambda, v) = (\lambda(s), v(\cdot, s))$ and the eigenpair $(\nu, \psi) = (\nu(s), \psi(\cdot, s))$, respectively. To compute the Morse index, we consider the signs of $\nu'(s)$ and $\nu''(s)$ at bifurcation points in the same way as [11, 12, 13]. Owing to the boundary condition, the proof for (2) is more complicated than that for (4) as proven in [12]. However, for completeness we prove it.

We prepare the Pohožaev identity in [16].

Lemma 1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2-boundary $\partial\Omega$ for $n \in \mathbb{N}$. Let $f(v) \in C(\mathbb{R})$. Suppose that $v \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies*

$$\begin{cases} \Delta v + f(v) = 0 & x \in \Omega, \\ v(x) = 0 & x \in \partial\Omega. \end{cases}$$

Then the identity

$$\frac{1}{2}\int_{\partial\Omega} (x \cdot \nu)\left(\frac{\partial v}{\partial \nu}\right)^2 d\omega + \frac{n-2}{2}\int_{\Omega} f(v)v\,dx = n\int_{\Omega} F(v)\,dx$$

holds, where $d\omega$ is the area element of $\partial\Omega$ with standard metric, ν is the outer unit normal vector at x and

$$F(v) = \int_0^v f(p)\,dp.$$

Proof of Proposition 1 By μ and ϕ, we denote the first eigenvalue μ^1 and corresponding eigenfunction ϕ^1 of $-\Delta$ in a star-shaped Ω with the Dirichlet boundary condition. Then we have $\phi > 0$ in Ω. First of all, we obtain relations between μ and λ stated in (i) and (ii) of the proposition. Multiplying (2) by $\phi > 0$ and integrating it over Ω, we have

$$-\mu \int_\Omega \phi v \, dx + \lambda \int_\Omega \phi \left(e^v - 1\right) \, dx = 0$$

and then

$$(\lambda - \mu) \int_\Omega \phi v \, dx < 0$$

for $v \not\equiv 0$ by $\exp x - 1 \geq x$ for $x \in \mathbb{R}$. If a solution $v > 0$ in Ω, then we have $\lambda < \mu$. For $v < 0$, we have $\lambda > \mu$. Next we obtain a lower bound λ_0 given in (i) and (ii). In the case of $n = 1, 2$, since

$$\int_\Omega |\nabla v|^2 \, dx = \lambda \int_\Omega \left(e^v - 1\right) v \, dx$$

holds, we conclude that $\lambda_0 \geq 0$ for $v \not\equiv 0$ by $(\exp x - 1)\, x \geq 0$ for all $x \in \mathbb{R}$. Thus we concentrate on the case of $n \geq 3$. Applying Lemma 1 to $f(v) = \lambda\left(e^v - 1\right)$ and $F(v) = \lambda\left(e^v - v - 1\right)$ and noting that Ω is star-shaped, we find

$$\frac{n-2}{2} \lambda \int_\Omega \left(e^v - 1\right) v \, dx \leq n\lambda \int_\Omega \left(e^v - v - 1\right) \, dx$$

and then

$$\begin{aligned}
\frac{n-2}{4} \int_\Omega |\nabla v|^2 \, dx &= \left(\frac{n-2}{2} - \frac{n-2}{4}\right) \lambda \int_\Omega \left(e^v - 1\right) v \, dx \\
&\leq n\lambda \int_\Omega \left(e^v - v - 1\right) \, dx - \frac{n-2}{4} \lambda \int_\Omega \left(e^v - 1\right) v \, dx.
\end{aligned}$$

Putting

$$g(x) = n\left(e^x - x - 1\right) - \frac{n-2}{4}\left(e^x - 1\right) x = ne^x - \frac{3n+2}{4} x - n - \frac{n-2}{4} e^x x,$$

we have $g(1) > 0$,

$$g'(x) = \frac{3n+2}{4} e^x - \frac{3n+2}{4} - \frac{n-2}{4} e^x x$$

and $\lim_{x \to +\infty} g(x) = \lim_{x \to +\infty} g'(x) = -\infty$. Hence there exists $\xi > 0$ which depends only on n such that $g(x) \leq 0$ for $x \geq \xi$. Since for ε, $\kappa \in [0, 1]$,

$$\begin{aligned}
g(x) &\leq n\left(e^x - x - 1\right) \\
&\leq n \int_0^1 \frac{d}{d\varepsilon} \left(e^{\varepsilon x} - \varepsilon x\right) \, d\varepsilon \\
&= nx \int_0^1 \left(e^{\varepsilon x} - 1\right) \, d\varepsilon \\
&= nx \int_0^1 \int_0^1 \frac{d}{d\kappa} \left(e^{\kappa \varepsilon x}\right) \, d\kappa \, d\varepsilon \\
&= n\varepsilon x^2 \int_0^1 \int_0^1 e^{\kappa \varepsilon x} \, d\kappa \, d\varepsilon \\
&\leq nx^2 \int_0^1 \int_0^1 e^{\kappa \varepsilon x} \, d\kappa \, d\varepsilon \\
&\leq \begin{cases} nx^2 e^x & \text{for } x \geq 0, \\ nx^2 & \text{for } x \leq 0 \end{cases}
\end{aligned}$$

holds for all $x \in \mathbb{R}$, we have

$$\begin{aligned}
&\frac{n-2}{4}\int_\Omega |\nabla v|^2 \, dx \\
&\le \lambda \int_{\Omega\cap\{v\le 0\}} g(v)\, dx + \lambda \int_{\Omega\cap\{0\le v\le \xi\}} g(v)\, dx + \lambda \int_{\Omega\cap\{v\ge \xi\}} g(v)\, dx \\
&\le \lambda n \int_{\Omega\cap\{v\le 0\}} v^2\, dx + \lambda n \int_{\Omega\cap\{0\le v\le \xi\}} v^2 e^v\, dx \\
&\le \lambda n \left(e^\xi + 1\right) \int_\Omega v^2\, dx \\
&\le \frac{\lambda n}{\mu^1}\left(e^\xi + 1\right) \int_\Omega |\nabla v|^2\, dx
\end{aligned}$$

thanks to the Poincaré inequality

$$\|v\|_2 \le \frac{1}{\sqrt{\mu^1}} \|\nabla v\|_2 \tag{8}$$

for all $v \in H_0^1(\Omega)$, which implies that

$$\lambda_0 \equiv \frac{\mu^1 (n-2)}{4n\left(e^\xi + 1\right)} < \mu^1 \quad \text{and} \quad 0 < \lambda_0 \le \lambda$$

for a nontrivial solution v. Finally we show the last claim (iii). Suppose that $v(x)$ is a nontrivial solution of (2). Then for $0 < \lambda < \lambda_0$, we have

$$\lambda_0 \int_\Omega |\nabla v|^2 \, dx \le \lambda \int_\Omega |\nabla v|^2 \, dx < \lambda_0 \int_\Omega |\nabla v|^2 \, dx,$$

which implies that $v \equiv 0$. □

Proof of Proposition 2 An easy calculation yields

$$(\mu^m, \phi^m) = \left((m\pi)^2, \sqrt{2}\sin m\pi x\right)$$

for $m \in \mathbb{N}$. At $(\lambda, v) = (\lambda, 0)$, (5) has the k-th eigenvalue $\nu^k = \mu^k - \lambda$ and the corresponding eigenfunction $\psi^k = \phi^k$ for $k \in \mathbb{N}$. Hence, we have a simple eigenvalue $\nu^m = 0$ at $(\lambda, v) = (\mu^m, 0)$ and $i(\lambda, v) = m - 1$ for $(\lambda, v) = (\lambda, 0)$ with $\mu^{m-1} < \lambda \le \mu^m$ with $m \in \mathbb{N}$. The first part of proposition is proved. We will show that the nontrivial solutions bifurcate from $(\lambda, v) = (\mu^m, 0)$. We define $\mathcal{X} = C^2(\overline{\Omega}) \cap C_0(\overline{\Omega})$, $\mathcal{Y} = C(\overline{\Omega})$ and a mapping $F : \mathbb{R}^+ \times \mathcal{X} \to \mathcal{Y}$ by

$$F(\lambda, v) = \Delta v + \left(\lambda + \mu^m\right)\left(e^v - 1\right)$$

for $m \in \mathbb{N}$. Then $F(\lambda, 0) = 0$ and the Fréchet derivative is given as

$$F_v(\lambda, v)[w] = \Delta w + \left(\lambda + \mu^m\right) e^v w$$

for $w \in \mathcal{X}$. Since

$$F_v(0,0)[w] = \Delta w + \mu^m w,$$

the kernel of $F_v(0,0)$ is spanned by $w_0 = \phi^m$. We have

$$F_{\lambda v}(\lambda, v)[\Lambda, w] = \Lambda e^v w,$$

which implies that $F_{\lambda v}(0,0)[\Lambda, w_0]$ does not belong to the range of $F_v(0,0)$. Hence applying Theorem 1.7 in [2] to this setting, we obtain two continua $\mathcal{S}_m^{\pm}$ of solutions (λ, v) of (2) bifurcating from $(\lambda, v) = (\mu^m, 0)$ satisfying

$$\mathcal{S}_m^{+} = \left\{ \big(\lambda(s), v(\cdot, s)\big) \mid \lim_{s\to+0} \big(\lambda(s), v(\cdot, s)\big) = \big(\mu^m, 0\big) \text{ and } s \in (0, \alpha) \right\}$$

and

$$\mathcal{S}_m^{-} = \left\{ \big(\lambda(s), v(\cdot, s)\big) \mid \lim_{s\to-0} \big(\lambda(s), v(\cdot, s)\big) = \big(\mu^m, 0\big) \text{ and } s \in (-\alpha, 0) \right\}$$

in $\mathbb{R}^+ \times \mathcal{X}$ with some $\alpha > 0$. Moreover the mapping

$$s \in (-\alpha, \alpha) \mapsto \big(\lambda(s), v(\cdot, s)\big) \in \mathbb{R}^+ \times \mathcal{X}$$

belongs to $C^2(-\alpha, \alpha)$ and $v(\cdot, s)$ is expressed as

$$v(\cdot, s) = s\phi^m(\cdot) + s\rho(\cdot, s)$$

for a function $\rho(\cdot, s) : (-\alpha, \alpha) \to \mathcal{Z}$ with C^2 dependence in s and $\rho(\cdot, 0) = 0$, where $\mathcal{Z}$ is a complement of the kernel of $F_v(0,0)$. We set

$$\mathcal{C}_m = \mathcal{S}_m^{-} \cup \big\{(\mu^m, 0)\big\} \cup \mathcal{S}_m^{+}.$$

The bifurcation result is established. Finally, we will compute the Morse index. At $\big(\lambda(s), v(\cdot, s)\big) \in \mathcal{C}_m$, it follows from a perturbation theory in [9] that the k-th eigenpair $\big(\nu_m^k, \psi_m^k\big) = \big(\nu_m^k(s), \psi_m^k(\cdot, s)\big)$ is C^2 dependence in s. A simple computation yields

$$\nu_m^k(0) = \big(k^2 - m^2\big)\pi^2 \qquad \text{and} \qquad \psi_m^k(x, 0) = \phi^k(x) = \sqrt{2}\sin k\pi x$$

for $\big(\lambda, v\big) = \big(\mu^m, 0\big) \in \mathcal{C}_m$. Under these notations, we have

$$\dot{v}(x, 0) = \phi^m(x) = \psi_m^m(x, 0) = \sqrt{2}\sin m\pi x,$$

where $\dot{v}$ stands for $(d/ds)v(\cdot, s)$ and $\dot{v}(\cdot, 0) = (d/ds)v(\cdot, s)|_{s=0}$. Now we have the following lemmas:

Lemma 2 *For $k \in \mathbb{N}$, we have the following*:

$$\begin{aligned}
I_{k,1} &\equiv \int_0^1 \psi_m^k(x, 0)\, dx = \frac{\sqrt{2}}{k\pi}\left\{1 - (-1)^k\right\},\\
I_{k,2} &\equiv \int_0^1 \left\{\psi_m^k(x, 0)\right\}^2 dx = 1,\\
I_{k,3} &\equiv \int_0^1 \left\{\psi_m^k(x, 0)\right\}^3 dx = \frac{4\sqrt{2}}{3k\pi}\left\{1 - (-1)^k\right\},\\
I_{k,4} &\equiv \int_0^1 \left\{\psi_m^k(x, 0)\right\}^4 dx = \frac{3}{2}.
\end{aligned}$$

Lemma 3 *If m is even and $\dot{\nu}_m^m(0) = \dot{\lambda}(0) = 0$, then we have*

$$J_m \equiv \int_0^1 \left\{\psi_m^m(x, 0)\right\}^2 \dot{\psi}_m^m(x, 0)\, dx = -\frac{5}{6}.$$

Proof Differentiating (5) with respect to s and putting $s = 0$ and $k = m$, then we have

$$\Delta\dot{\psi}_m^m(x,0) + \mu^m \dot{\psi}_m^m(x,0) = -\mu^m \left\{\psi_m^m(x,0)\right\}^2$$

because of $\nu_m^m(0) = \dot{\nu}_m^m(0) = \dot{\lambda}(0) = 0$ and

$$\Delta\dot{\psi}_m^m(x,0) + (m\pi)^2\,\dot{\psi}_m^m(x,0) = (m\pi)^2\,(\cos 2m\pi x - 1)\,.$$

Solving this ordinary differential equation with $\dot{\psi}_m^m(0,0) = \dot{\psi}_m^m(1,0) = 0$ under the restriction

$$\frac{d}{ds}\int_0^1 \left\{\psi_m^m(x,s)\right\}^2\,dx = 2\int_0^1 \psi_m^m(x,s)\dot{\psi}_m^m(x,s)\,dx = 0$$

at $s = 0$ and noting that m is even, we obtain

$$\dot{\psi}_m^m(x,0) \quad = \quad \frac{4}{3}\cos m\pi x - 1 - \frac{1}{3}\cos 2m\pi x = \frac{4}{3}\cos m\pi x - \frac{4}{3} + \frac{1}{3}\left\{\psi_m^m(x,0)\right\}^2,$$

which yields the conclusion along with $I_{m,2}$ and $I_{m,4}$ in Lemma 2. □

We return back to the proof of Proposition 2. Differentiating (2) twice and (5) once with respect to s, we have

$$\Delta\ddot{v} + \ddot{\lambda}(e^v - 1) + 2\dot{\lambda}e^v\dot{v} + \lambda e^v\dot{v}^2 + \lambda e^v\ddot{v} = 0$$

and

$$\Delta\dot{\psi} + \dot{\lambda}e^v\psi + \lambda e^v\dot{v}\psi + \lambda e^v\dot{\psi} = -\dot{\nu}\psi - \nu\dot{\psi}.$$

Putting $s = 0$ and $k = m$, multiplying by $\psi_m^m(x,0)$ and integrating them over $(0,1)$, we have

$$2\dot{\lambda}(0)\int_0^1 \left\{\psi_m^m(x,0)\right\}^2\,dx + \mu^m\int_0^1 \left\{\psi_m^m(x,0)\right\}^3\,dx = 0$$

and

$$\left(\dot{\lambda}(0) + \dot{\nu}_m^m(0)\right)\int_0^1 \left\{\psi_m^m(x,0)\right\}^2\,dx + \mu^m\int_0^1 \left\{\psi_m^m(x,0)\right\}^3\,dx = 0.$$

Hence, $I_{m,2}$ and $I_{m,3}$ in Lemma 2 yield

$$\dot{\lambda}(0) = \dot{\nu}_m^m(0) = -\frac{2\sqrt{2}}{3}m\pi\left\{1 - (-1)^m\right\} = \begin{cases} 0 & \text{for even } m, \\ -\frac{4\sqrt{2}}{3}m\pi & \text{for odd } m. \end{cases}$$

Henceforth, we assume that m is even. Differentiating (2) three times and (5) twice with respect to s, putting $s = 0$ and $k = m$, multiplying by $\psi_m^m(x,0)$, integrating them over $(0,1)$ and eliminating $\int_0^1 \left\{\psi_m^m(x,0)\right\}^2 \ddot{v}(x,0)\,dx$, we have

$$2(m\pi)^2 I_{m,4} + 6(m\pi)^2 J_m = -3\ddot{\nu}_m^m(0) I_{m,2}$$

and hence

$$\ddot{\nu}_m^m(0) = \frac{2(m\pi)^2}{3} > 0$$

by Lemmas 2 and 3. We conclude that

$$\begin{cases} \nu_m^m(0) = 0 \quad \text{and} \quad \dot{\nu}_m^m(0) < 0 & \text{for odd } m, \\ \nu_m^m(0) = \dot{\nu}_m^m(0) = 0 \quad \text{and} \quad \ddot{\nu}_m^m(0) > 0 & \text{for even } m. \end{cases}$$

In the case of $m = 2k - 1$, we have

$$\begin{cases} \nu_m^m(s) > 0 & \text{for sufficiently small } s < 0, \\ \nu_m^m(0) = 0, & \\ \nu_m^m(s) < 0 & \text{for sufficiently small } s > 0 \end{cases}$$

and $i(\lambda(0), 0) = i\left(\mu^{2k-1}, 0\right) = 2k - 2$. Hence we have

$$i(\lambda(s), v(\cdot, s)) = \begin{cases} 2k - 2 & \text{for sufficiently small } s < 0, \\ 2k - 1 & \text{for sufficiently small } s > 0. \end{cases}$$

In the case of $m = 2k$, we have $\nu_m^m(s) > 0$ for sufficiently small $|s| > 0$, $\nu_m^m(0) = 0$ and $i(\lambda(0), 0) = i\left(\mu^{2k}, 0\right) = 2k - 1$. Hence we have

$$i(\lambda(s), v(\cdot, s)) = 2k - 1$$

for sufficiently small $|s| > 0$, which completes the proof. □

Remark 1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$ for $n \geq 2$. If we assume that the eigenvalue μ^m is simple for all $m \in \mathbb{N}$, the conclusion of bifurcation in Proposition 2 is still true. However, the relation of Morse index is open. The bifurcation problems where the zero eigenvalue is double are also considered for the equations other than* (2)*. In* [3]*, the authors obtain four bifurcation curves and compute their Morse indices in a square in Theorem 1.1. For a disk with Neumann boundary condition, in* [10]*, the author gives a sufficient condition for a branch of non-trivial solution not to have a secondary bifurcation point in Theorem B and applies these results including the simple eigenvalue case in Theorem C.*

3 Blow-up

Let $u_0 \in C_0(\overline{\Omega})$. We transform (1) into the integral equation

$$u(t) = e^{-At}u_0 + \lambda \int_0^t e^{-A(t-s)}\left(e^{u(s)} - 1\right) ds$$

and establish a time-local solution

$$u \in C\left([0, T_{u_0}); C_0(\overline{\Omega})\right) \cap C^1\left((0, T_{u_0}); C_0(\overline{\Omega})\right)$$

by an abstract theory of evolution equation. Here, we extend $A \equiv -\Delta$ to be a self-adjoint positive operator in $C_0(\overline{\Omega})$ with the domain

$$\mathcal{D}(A) = \left\{u \in C_0(\overline{\Omega}), \quad Au \in C_0(\overline{\Omega})\right\}$$

and write the semi-group generated by A as e^{-At}. In order to prove the proposition by Kaplan's method [8], we integrate the solution multiplied by the first eigenfunction ϕ^1 of A and differentiate it with respect to t. Then we get the differential inequalities and solve them.

Proof of Proposition 3 We set

$$k = \left\|\phi^1\right\|_1 \qquad \text{and} \qquad a(t) = \int_\Omega u(x, t)\phi^1(x)\, dx$$

for all $t \in (0, T_{u_0})$. We have

$$a' = -\mu^1 a + \lambda \int_\Omega \left(e^u - 1\right)\phi^1(x)\,dx > \left(\lambda - \mu^1\right) a$$

by $\exp x - 1 \geq x$ for $x \in \mathbb{R}$ and integrate this inequality to obtain

$$a > e^{\left(\lambda-\mu^1\right)t} a(0) = e^{\left(\lambda-\mu^1\right)t} \int_\Omega u_0(x)\phi^1(x)\,dx > 0$$

for all $t \in (0, T_{u_0})$. Next the Jensen inequality [20] and positivity of a imply that

$$\begin{aligned} a' &= -\mu^1 a + \lambda \int_\Omega e^u \phi^1\,dx - \lambda k \\ &> -\mu^1 a + \lambda k \Big(\exp \frac{a}{k} - 1 \Big) \\ &> \left(\lambda - \mu^1\right) a + \frac{\lambda}{2k} a^2 \end{aligned}$$

by $\exp x \geq 1 + x + x^2/2$ for $x \geq 0$. Hence putting $p(t) \equiv 1/a(t)$ for all $t \in (0, T_{u_0})$, we have

$$p' + \left(\lambda - \mu^1\right) p < -\frac{\lambda}{2k}.$$

Multiplying $\exp\left(\lambda - \mu^1\right) t$, and integrating this differential inequality with respect to t, we obtain

$$0 < p < \begin{cases} p_0 e^{-\left(\lambda-\mu^1\right)t} + \frac{\lambda}{2k(\lambda-\mu^1)} \left(e^{-\left(\lambda-\mu^1\right)t} - 1\right) & \text{for } \lambda \neq \mu^1, \\ p_0 - \frac{\lambda}{2k} t & \text{for } \lambda = \mu^1, \end{cases}$$

where $p_0 = 1/a(0)$. Let

$$T \equiv \begin{cases} \frac{-1}{\mu^1-\lambda} \log \left(1 - \frac{2p_0 k\left(\mu^1-\lambda\right)}{\lambda}\right) & \text{for } 0 < \lambda < \mu^1, \\ \frac{2k}{\lambda} p_0 & \text{for } \lambda = \mu^1, \\ \frac{1}{\lambda-\mu^1} \log \left(1 + \frac{2p_0 k\left(\lambda-\mu^1\right)}{\lambda}\right) & \text{for } \lambda > \mu^1. \end{cases}$$

Then since the assumption of (i) in Proposition 3 is equivalent to

$$\frac{1}{p_0} > \frac{2k\left(\mu^1 - \lambda\right)}{\lambda}$$

for $0 < \lambda < \mu^1$, we have

$$0 < 1 - \frac{2p_0 k\left(\mu^1 - \lambda\right)}{\lambda} < 1.$$

Hence T is well-defined and we find $0 < T < +\infty$ such that $p(t) \to +0$ as $t \to T$. The same is true of the case of $\lambda \geq \mu^1$. Finally, we have

$$+\infty = \lim_{t\to T} \frac{1}{p(t)} = \lim_{t\to T} a(t) = \lim_{t\to T} \int_\Omega u(x,t)\phi^1(x)\,dx \leq k \lim_{t\to T} \sup_{x\in\overline{\Omega}} u(x,t),$$

which leads us to the proof of proposition. □

4 Negative global solution We decompose (1) into the heat equation with $u_0 \leq 0$ and the nonlinear equation with $w_0 \equiv 0$. First, we begin with the fundamental lemmas.

Lemma 4 *Let $u_0 \in C_0(\overline{\Omega})$ be $u_0(x) \leq 0$ for any $x \in \Omega$. The function*

$$w_1(x,t) \equiv e^{-At}u_0 \in C\big([0,+\infty);C_0(\overline{\Omega})\big) \cap C^1\big((0,+\infty);C_0(\overline{\Omega})\big)$$

solves

$$\begin{cases} w_t = \Delta w & x \in \Omega, \ \ t > 0, \\ w(x,t) = 0 & x \in \partial\Omega, \ \ t > 0, \\ w(x,0) = u_0(x) & x \in \Omega \end{cases}$$

and satisfies $w_1 \leq 0$ in $\Omega \times [0,+\infty)$.

Lemma 5 *Let $\lambda > 0$. The function*

$$w_2(x,t) \equiv \int_0^t e^{-A(t-s)}\,(-\lambda)\;ds \in C\big([0,+\infty);C_0(\overline{\Omega})\big) \cap C^1\big((0,+\infty);C_0(\overline{\Omega})\big)$$

solves

$$\begin{cases} w_t = \Delta w - \lambda & x \in \Omega, \ \ t > 0, \\ w(x,t) = 0 & x \in \partial\Omega, \ \ t > 0, \\ w(x,0) = 0 & x \in \Omega \end{cases}$$

and satisfies $w_2 \leq 0$ in $\Omega \times [0,+\infty)$.

We solve the nonlinear equation by constructing the lower-upper solution pair.

Proposition 5 *Let w_1 be a solution obtained in Lemma 4. For any $\lambda > 0$, there exists a unique solution*

$$w_3 \in C\big([0,+\infty);C_0(\overline{\Omega})\big) \cap C^1\big((0,+\infty);C_0(\overline{\Omega})\big)$$

of

$$\begin{cases} w_t = \Delta w + \lambda\left(e^{w_1+w} - 1\right) & x \in \Omega, \ \ t > 0, \\ w(x,t) = 0 & x \in \partial\Omega, \ \ t > 0, \\ w(x,0) = 0 & x \in \Omega \end{cases}$$

satisfying $w_3 \leq 0$ in $\Omega \times [0,+\infty)$.

Proof Since $(0, w_2)$ is an upper and lower solution pair, the statement follows from [18].

Proof of Theorem 1 Setting $u = w_1 + w_3$, we have a unique solution in an element of the desired space satisfying $u \leq 0$ in $\Omega \times \mathbb{R}^+$. Note that $u(\cdot,t) \in C^2(\overline{\Omega}) \cap C_0(\overline{\Omega}) \subset H^2(\Omega) \cap H_0^1(\Omega)$ for all $t \geq 1$. Applying the Poincaré inequality (8), we have

$$\mu^1 \int_\Omega |\nabla u|^2\;dx \leq \int_\Omega (\Delta u)^2\;dx$$

for $u \in H^2(\Omega) \cap H_0^1(\Omega)$ and finally

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2\;dx &= -\int_\Omega (\Delta u)^2\;dx + \lambda \int_\Omega e^u\,|\nabla u|^2\;dx \\ &\leq -\int_\Omega (\Delta u)^2\;dx + \lambda \int_\Omega |\nabla u|^2\;dx \\ &\leq -\left(\mu^1 - \lambda\right)\int_\Omega |\nabla u|^2\;dx \end{aligned}$$

for $t \geq 1$, which yields

$$\|u(\cdot,t)\|_{H_0^1}^2 \leq e^{-2\left(\mu^1-\lambda\right)(t-1)}\,\|u(\cdot,1)\|_{H_0^1}^2 \to 0 \tag{9}$$

as $t \to +\infty$. □

5 Global solution In this section, we concentrate on $\Omega \subset \mathbb{R}^2$ and apply the Trudinger-Moser inequality to our problem. We establish the global solution for sufficiently small parameter and initial value. To obtain the estimate in the H_0^1 space, we extend $B = -\Delta$ to be a self-adjoint positive operator in $X = L^2(\Omega)$ with the domain $\mathcal{D}(B) = H^2(\Omega) \cap H_0^1(\Omega)$ and write the semi-group generated by B as e^{-Bt}. For $n = 1$, we can also derive the similar estimates. We start with the Trudinger-Moser inequality and an easy lemma.

Proposition 6 ([14, 19]) *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial\Omega$. Then there exists $C_{TM} > 0$ which depends only on Ω such that*

$$\int_\Omega e^u \, dx \le C_{TM} \exp\left(\frac{1}{16\pi} \int_\Omega |\nabla u|^2 \, dx \right)$$

holds for all $u \in H_0^1(\Omega)$.

Lemma 6 *Let $a, b, k > 0$. We define*

$$f(t) = ae^{kt} + b - t$$

for $t \ge 0$. If

$$b + \frac{1}{k} \log a < -\frac{1}{k} (\log k + 1)$$

holds, then there exist $t_1 \in (0, t_0)$ and $t_2 \in (t_0, +\infty)$ such that $f(t) \ge 0$ is equivalent to $0 \le t \le t_1$ or $t \ge t_2$, where t_0, t_1 and t_2 satisfy $t_0 = (1/k) \log (1/ak)$ and $f'(t_0) = f(t_1) = f(t_2) = 0$.

Proof First, we have $f(0) = a + b > 0$ and $f'(t) = ake^{kt} - 1$. Note that

$$t_0 = \frac{1}{k} \log \frac{1}{ak} > b + \frac{1}{k} > 0$$

from the assumption. We find $f'(t_0) = ake^{kt_0} - 1 = 0$ and

$$f(t_0) = ae^{kt_0} + b - t_0 = b + \frac{1}{k} \log a + \frac{1}{k} (\log k + 1) < 0,$$

which completes the proof. □

Proof of Theorem 2 Note that $e^u \in L^1(\Omega)$ for $u \in H_0^1(\Omega)$ by Proposition 6. Hence the Lyapunov function $L_\lambda(u)$ defined in (6) is well-defined for $u \in H_0^1(\Omega)$. Since

$$\frac{d}{dt} L_\lambda(u) = - \|u_t(t)\|_2^2 \le 0$$

holds, $L_\lambda(u)$ is the Lyapunov function for (1) and we have

$$L_\lambda(u) = L_\lambda(u_0) - \int_0^t \|u_t(s)\|_2^2 \, ds \le L_\lambda(u_0),$$

which yields

$$\frac{1}{2} \|\nabla u\|_2^2 \le \lambda \int_\Omega e^u \, dx - \lambda \int_\Omega u \, dx + \frac{1}{2} \|u_0\|_{H_0^1}^2 + \lambda \int_\Omega (u_0 - e^{u_0}) \, dx.$$

Since

$$\lambda \int_\Omega e^u\, dx = C_{TM}\lambda \times \frac{1}{C_{TM}} \int_\Omega e^u\, dx \leq \frac{1}{2} C_{TM}^2 \lambda^2 + \frac{1}{2C_{TM}^2} \left(\int_\Omega e^u\, dx \right)^2$$

holds by the Young inequality, we have

$$\begin{aligned} \frac{1}{2}\|\nabla u\|_2^2 &< \frac{1}{2}C_{TM}^2\lambda^2 + \frac{1}{2C_{TM}^2}\left(\int_\Omega e^u\, dx\right)^2 + \lambda\sqrt{|\Omega|}\,\|u\|_2 + \frac{1}{2}\|u_0\|_{H_0^1}^2 \\ &\leq \frac{1}{2}C_{TM}^2\lambda^2 + \frac{1}{2}\exp\left(\frac{1}{8\pi}\|\nabla u\|_2^2\right) + \frac{\lambda\sqrt{|\Omega|}}{\sqrt{\mu^1}}\|\nabla u\|_2 + \frac{1}{2}\|u_0\|_{H_0^1}^2 \end{aligned}$$

owing to $\exp x > x$ for $x \in \mathbb{R}$, the Trudinger-Moser inequality (Proposition 6) and the Poincaré inequality (8). Again the Young inequality yields

$$\frac{\lambda\sqrt{|\Omega|}}{\sqrt{\mu^1}}\|\nabla u\|_2 = \frac{\sqrt{2}\lambda\sqrt{|\Omega|}}{\sqrt{\mu^1}} \times \frac{1}{\sqrt{2}}\|\nabla u\|_2 \leq \frac{|\Omega|}{\mu^1}\lambda^2 + \frac{1}{4}\|\nabla u\|_2^2, \tag{10}$$

which implies that

$$\frac{1}{2}\|\nabla u\|_2^2 \leq \frac{1}{2}C_{TM}^2\lambda^2 + \frac{1}{2}\exp\left(\frac{1}{8\pi}\|\nabla u\|_2^2\right) + \frac{|\Omega|}{\mu^1}\lambda^2 + \frac{1}{4}\|\nabla u\|_2^2 + \frac{1}{2}\|u_0\|_{H_0^1}^2$$

and that

$$\|u\|_{H_0^1}^2 \leq 2\exp\left(\frac{1}{8\pi}\|u\|_{H_0^1}^2\right) + 2\left(C_{TM}^2 + \frac{2|\Omega|}{\mu^1}\right)\lambda^2 + 2\|u_0\|_{H_0^1}^2.$$

Hence for $f(t) = ae^{kt} + b - t$ with

$$a = 2, \quad b = 2\left(C_{TM}^2 + \frac{2|\Omega|}{\mu^1}\right)\lambda^2 + 2\|u_0\|_{H_0^1}^2 \quad \text{and} \quad k = \frac{1}{8\pi},$$

then we have $f(\|u\|_{H_0^1}^2) \geq 0$ for all $t \geq 0$. Then the assumption

$$b + \frac{1}{k}\log a < -\frac{1}{k}(\log k + 1)$$

in Lemma 6 is satisfied. In fact

$$2\left(C_{TM}^2 + \frac{2|\Omega|}{\mu^1}\right)\lambda^2 + 2\|u_0\|_{H_0^1}^2 + 8\pi\log 2 < 8\pi(\log 8\pi - 1)$$

holds by (7). We can apply Lemma 6 to obtain $\|u\|_{H_0^1}^2 \leq t_1$ or $\|u\|_{H_0^1}^2 \geq t_2$. Now that we have

$$\|u_0\|_{H_0^1}^2 < 4\pi(\log 4\pi - 1) < 8\pi\log 4\pi = t_0$$

along with (7), we find

$$\|u\|_{H_0^1}^2 \leq t_1 < t_0 = 8\pi\log 4\pi \tag{11}$$

so long as the local solution exists. Hence we have a global solution in $H_0^1(\Omega)$. Next, we define X^α as the domain of B^α for $\alpha \geq 0$ with graph norm $\|u\|_{X^\alpha} = \|B^\alpha u\|_2$ for $u \in X^\alpha$.

Then we derive the estimate in $X^{1/2+\varepsilon} = H^{1+2\varepsilon}(\Omega)$ for $\varepsilon \in (0, 1/2)$ by the smoothing effect of the heat equation. In fact, we have for $t \geq 1$

$$\begin{aligned}
&\|u\|_{X^{1/2+\varepsilon}} \\
&= \left\| B^{\frac{1}{2}+\varepsilon} u \right\|_2 \\
&\leq \left\| B^{\varepsilon} e^{-Bt} B^{\frac{1}{2}} u_0 \right\|_2 + \lambda \int_0^t \left\| B^{\frac{1}{2}+\varepsilon} e^{-B(t-s)} \left(e^{u(s)} - 1 \right) \right\|_2 ds \\
&\leq C_1 t^{-\varepsilon} e^{-C_2 t} \|u_0\|_{H_0^1} + \lambda C_3 \int_0^t (t-s)^{-\left(\frac{1}{2}+\varepsilon\right)} e^{-C_2(t-s)} \|e^u - 1\|_2 \, ds \\
&\leq C_1 \sqrt{t_0} + \lambda C_3 \int_0^t (t-s)^{-\left(\frac{1}{2}+\varepsilon\right)} e^{-C_2(t-s)} \left(\left\| e^{2u} \right\|_1^{\frac{1}{2}} + \|1\|_2 \right) ds \\
&\leq C_1 \sqrt{t_0} + \lambda C_3 \int_0^t (t-s)^{-\left(\frac{1}{2}+\varepsilon\right)} e^{-C_2(t-s)} \left\{ C_{TM}^{\frac{1}{2}} \exp\left(\frac{1}{8\pi} \|\nabla u\|_2^2 \right) + |\Omega|^{\frac{1}{2}} \right\} ds \\
&< C_1 \sqrt{t_0} + \frac{\lambda C_3}{C_2^{\frac{1}{2}-\varepsilon}} \left\{ \sqrt{C_{TM}} \exp\left(\frac{t_0}{8\pi} \right) + \sqrt{|\Omega|} \right\} \Gamma\left(\frac{1}{2} - \varepsilon \right)
\end{aligned}$$

by (11), where $\Gamma(p)$ is a gamma function defined by

$$\Gamma(p) = \int_0^{+\infty} e^{-x} x^{p-1} dx$$

for $p > 0$ and henceforth in this proof we will denote by C_i a positive constant which depends only on Ω and ε, where $i \in \mathbb{N}$. Since we have $H^{1+2\varepsilon}(\Omega) \subset C(\overline{\Omega})$ with the Sobolev embedding constant $C_4 > 0$, we find

$$\|u\|_\infty \leq C_4 \|u\|_{X^{\frac{1}{2}+\varepsilon}} < C_5$$

for $t \geq 1$. Hence the estimate similar to (9) is given as

$$\|u(\cdot,t)\|_{H_0^1}^2 < e^{-2\left(\mu^1 - \lambda \exp C_5\right)(t-1)} \|u(\cdot,1)\|_{H_0^1}^2 \to 0$$

for $\lambda < \lambda_1 \equiv \mu^1 / \exp C_5$ as $t \to +\infty$. □

Proof of Theorem 3 First, we note that $\mu^1 = \pi^2$. In the same manner, Lyapunov function yields

$$\begin{aligned}
\frac{1}{2} \|\nabla u\|_2^2 &\leq \lambda \int_0^1 e^{|u|} \, dx + \lambda \|1\|_2 \|u\|_2 + \frac{1}{2} \|u_0\|_{H_0^1}^2 \\
&\leq \lambda e^{C_S \|u\|_{H_0^1}} + \frac{\lambda}{\pi} \|\nabla u\|_2 + \frac{1}{2} \|u_0\|_{H_0^1}^2 \\
&\leq \lambda e^{C_S \|u\|_{H_0^1}} + \frac{1}{4} \|\nabla u\|_2^2 + \frac{\lambda^2}{\pi^2} + \frac{1}{2} \|u_0\|_{H_0^1}^2
\end{aligned}$$

owing to $H^1(0,1) \subset C([0,1])$ with the Sobolev embedding constant $C_S > 0$, (8) and (10). Since we have

$$C_S \|u\|_{H_0^1} = \sqrt{2e} C_S \times \frac{1}{\sqrt{2e}} \|u\|_{H_0^1} \leq e C_S^2 + \frac{1}{4e} \|u\|_{H_0^1}^2,$$

we find

$$\begin{aligned}\frac{1}{2}\|\nabla u\|_2^2 &\leq \lambda e^{eC_S^2}\exp\left(\frac{1}{4e}\|u\|_{H_0^1}^2\right)+\frac{1}{4}\|\nabla u\|_2^2+\frac{\lambda^2}{\pi^2}+\frac{1}{2}\|u_0\|_{H_0^1}^2\\ &= \sqrt{2}\lambda e^{eC_S^2}\times\frac{1}{\sqrt{2}}\exp\left(\frac{1}{4e}\|u\|_{H_0^1}^2\right)+\frac{1}{4}\|\nabla u\|_2^2+\frac{\lambda^2}{\pi^2}+\frac{1}{2}\|u_0\|_{H_0^1}^2\\ &\leq \lambda^2 e^{2eC_S^2}+\frac{1}{4}\exp\left(\frac{1}{2e}\|u\|_{H_0^1}^2\right)+\frac{1}{4}\|\nabla u\|_2^2+\frac{\lambda^2}{\pi^2}+\frac{1}{2}\|u_0\|_{H_0^1}^2\end{aligned}$$

again by the Young inequality and

$$\exp\left(\frac{1}{2e}\|u\|_{H_0^1}^2\right)+4\left(e^{2eC_S^2}+\frac{1}{\pi^2}\right)\lambda^2+2\|u_0\|_{H_0^1}^2-\|u\|_{H_0^1}^2\geq 0,$$

which completes the proof by applying Lemma 6 for

$$a=1,\quad b=4\left(e^{2eC_S^2}+\frac{1}{\pi^2}\right)\lambda^2+2\|u_0\|_{H_0^1}^2\quad\text{and}\quad k=\frac{1}{2e}.$$

□

Proof of Proposition 4. Since the embedding $X^{1/2+\varepsilon}\subset H_0^1(\Omega)$ is compact, the orbit $\cup_{t\geq 1}\{u(\cdot,t)\}$ is relatively compact in $H_0^1(\Omega)$. Hence the omega limit set $\omega(u_0)$ is invariant, non-empty, compact and connected in $H_0^1(\Omega)$ by Theorem 5.1.8 in [7]. Again by Corollary 7.2.2 in [7] and the existence of the Lyapunov function, we have $\omega(u_0)\subset\mathcal{C}$. Moreover, thanks to the regularity of nonlinear term (e^u-1) of (1), $\omega(u_0)$ is a single point by Theorems 1.2 in [5] or 11.4.3 in [7].

□

Remark 2 *In the proposition, we impose the same hypotheses as Theorems 2 or 3, which is not needed. If the global soluiton exists, then the conclusion is true by Theorem 2.1 in* [4].

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