Putting Quantum MV algebras on the "map"

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In memoriam William W. McCune

Received July 27,2021

Abstract

In this paper, we clarify mainly some aspects concerning the quantum MV (QMV) algebras as non-lattice generalizations of MV algebras. We redefine the QMV algebras as involutive m-BE algebras and we introduce three generalizations: the pre-MV (PreMV), the metha-MV (MMV) and the orthomodular (OM) algebras. We prove that the antisymmetric QMV algebras - but also the antisymmetric PreMV and antisymmetric MMV algebras - coincide with the MV algebras, while the antisymmetric OM algebras are generalizations of the MV algebras. We introduce also the transitive QMV, PreMV, MMV, OM algebras and finally we put the QMV and the transitive QMV algebras on the same "map" with the MV algebras. The *transitive antisymmetric orthomodular algebra*, a proper generalization of MV algebra inside the class of m-BCK algebras, is pointed out. Many examples are provided.

Keywords: m-BE algebra, m-aBE algebra, m-pre-BCK algebra, m-BCK algebra, MV algebra, quantum MV algebra, orthomodular lattice, orthomodular algebra, pre-MV algebra, metha-MV algebra

MSC 2020: 06D35, 03G12, 06F99

1 Introduction

The *algebraists* work usually with the commutative additive groups and with the positive (right) cone of a partially-ordered commutative group $(G, \leq, +, -, 0)$, where there are essentially a sum $\oplus = +$ and an element 0. Sometimes, the negative (left) cone is needed also, where there are essentially a product $\odot = +$ and an element $1 = 0$. They work with algebras that have associated an (pre-order) order relation, which usually does not appear explicitely in the definitions. The presence of the (pre-order) order relation implies the presence of the (generalized) duality principle. Thus, each algebra has a dual one, the (pre-order) order relation has a dual one. We have given names to the dual algebras [15], [17], [19]: "left" algebra and "right" algebra, names connected with the left-continuity of a t-norm and with the right-continuity of a t-conorm, respectively. Hence, the algebraists usually work with the commutative *right-unital magmas*.

By contrary, the *logicians* work with the logic of *truth*, where the *truth* is represented by 1, and there is essentially one implication; we could name this logic "left-logic". One can imagine also a "right-logic", as a logic of *false*, where the *false* is represented by 0. Hence, the logicians usually work with the commutative *left-algebras of logic* (or the *algebras of left-logic*).

*Partially supported by Simons Foundation Collaboration Grant 359872

Summarizing, for algebraists, the appropriate algebras are the unital magmas, not the algebras of logic, and among the unital magmas, the appropriate algebras are the right-algebras. For logicians, by contrary, the appropriate algebras are the algebras of logic, not the unital magmas, and among the algebras of logic, the appropriate algebras are the left-algebras. This explains why, for examples, the MV algebras were initially introduced as right-unital magmas, while the Wajsberg algebras were initially introduced as left-algebras of logic.

In this paper, *regarding from (algebras of) logic side, we shall work with left-algebras (left-unital magmas), in principal*, therefore, the unital magmas will be defined multiplicatively, in principal.

Thus, the commutative algebraic structures connected directly or indirectly with classical/non-classical logics belong to two parallel "worlds":

1. the "world" of *left-algebras of logic*, where there are essentially one implication, $\rightarrow$ (two, in the non-commutative case), and an element 1 (that can be the last element); the algebras $(A, \rightarrow, 1)$, verifying the basic property (M): $1 \rightarrow x = x$, are called *left-M algebras* [17], [19]; among the M algebras with additional operations, there are the algebras $(A, \rightarrow, 0, 1)$ (where a negation can be defined by: $x^- = x \rightarrow 0$), or $(A, \rightarrow, ^-, 1)$, with $1^- = 0$, where 1 is the *last element*, verifying (or not) (Ex) (Exchange): $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$; an internal binary relation can be defined by: $x \leq y \stackrel{def.}{\Longleftrightarrow} x \rightarrow y = 1$ ($\leq$ can be a pre-order, an order, or even a lattice order); algebras belonging to this "world" are [17], [19]: the bounded MEL, BE and aBE, pre-BCK algebras, BCK algebras, bounded BCK algebras, BCK(P) algebras, Hilbert algebras, Wajsberg algebras, implicative-Boolean algebras, etc. A "Big map" (hierarchy of algebras of logic) is presented in ([19], Figure 1).

2. the "world" of *left-algebras*, where there are essentially a product, $\odot$, and an element 1 (that can be the last element); the algebras $(A, \odot, 1)$, verifying the corresponding basic properties (PU): $1 \odot x = x$ and (Pcomm): $x \odot y = y \odot x$, are called *commutative left-unital magmas*; among the commutative left-unital magmas with additional operations, there are the algebras [19] $(A, \odot, ^-, 1)$, with $1^- = 0$, where 1 is the *last element*, verifying (or not) (Pass) (associativity of product): $x \odot (y \odot z) = (x \odot y) \odot z$; an internal binary relation can be defined by: $x \leq_m y \Longleftrightarrow x \odot y^- = 0$ ($\leq_m$ can be a pre-order, an order, or even a lattice order), where 'm' comes from 'magma'; algebras belonging to this "world" are [17], [19]: the m-MEL, m-BE and m-aBE, m-pre-BCK algebras, m-BCK algebras, pocrims, (bounded) lattices, residuated lattices, BL algebras, MTL algebras, NM algebras, MV algebras, Boolean algebras, etc. A corresponding "Big map" (hierarchy of algebras) is presented in ([19], Figure 10) - see Figure 1.

MV algebras were introduced in 1958 by C.C. Chang [4], as a model of $\aleph_0$-valued Łukasiewicz logic. Chang's definition of MV algebras has 17 axioms. There is a huge literature concerning the MV algebras; we mention only a reference book, [3].

Between the two parallel " worlds" there are some connections, as for examples: the equivalence between BCK(P) algebras and pocrims, in the non-involutive case, and the definitional equivalence between Wajsberg algebras and MV algebras, in the involutive case ($(x^-)^- = x$). In [19], the two general Theorems 9.1 and 9.3 connect the two 'worlds' in the involutive case, by the inverse maps Φ ($x \odot y \stackrel{def.}{=} (x \rightarrow y^-)^-$) and Ψ ($x \rightarrow y \stackrel{def.}{=} (x \odot y^-)^-$) (Theorem 9.1 is for algebras *with last element*, while Theorem 9.3 is for algebras *without last element*); recall, for examples, that $\leq \Longleftrightarrow \leq_m$, that (M) $\Longleftrightarrow$ (PU) + (Pcomm), (Ex) $\Longleftrightarrow$ (Pass) etc. These theorems can be used to prove the *definitionally equivalence* (d.e.) between the analogous involutive (left-) algebras from the two "worlds" simply by choosing appropriate definitions of these algebras; for examples, one can prove the d.e. between implicative-Boolean algebras and Boolean algebras, between involutive BCK algebras and (involutive) m-BCK algebras etc..

Beside the classical and non-classical logics, there exist the quantum logics. Examples of algebraic structures connected with quantum logics (= quantum structures/algebras) are the bounded implicative (implication) lattices, the De Morgan algebras, the ortholattices, the orthomodular lattices, the quantum MV algebras (a better name is perhaps *quantum-MV algebras*, because they are generalizations of MV algebras, and not particular cases of MV algebras, i.e. they are not MV algebras that are 'quantum'), etc.

Quantum-MV algebras (or QMV algebras) were introduced by Roberto Giuntini in [7] (see also [8], [9], [10], [11], [12], [13], [6]), as non-lattice theoretic generalizations of MV algebras and as non-idempotent generalizations of orthomodular lattices. Cf. [6], from an algebraic point of view, MV algebras and QMV algebras share a common set of axioms, which S. Gudder [14] has called *supplement algebra* (*S algebra*). An MV algebra is an S-algebra verifying the axiom (MV), while an QMV algebra is an S algebra verifying the axiom (QMV), which is weaker than (MV).

Orthomodular lattices (particular ortholattices) generalize the Boolean algebras. They have arisen, cf. [28], "in the study of quantum logic, that is, the logic which supports quantum mechanics and which does not conform to classical logic. As noted by Birkhoff and von Neumann in 1936 [2], the calculus of propositions in quantum logic "is formally indistinguishable from the calculus of linear subspaces [of a Hilbert space] with respect to set products, linear sums and orthogonal complements" in the role of *and*, *or* and *not*, respectively. This has led to the study of the closed subspaces of a Hilbert space, which form an orthomodular lattice in contemporary terminology. As often happens in algebraic logic, the study of orthomodular lattices has tremendously developed, both for their interest in logic and for their own sake, see Kalmbach [26]".

The connections between algebras of logic/algebras and quantum algebras were not very clear. But, in papers [19], [23], we established important connections, by redefining equivalently the bounded involutive lattices and De Morgan algebras as involutive *m-MEL algebras* and the ortholattices, the MV and the Boolean algebras as involutive *m-BE algebras*, verifying some properties, and then putting all of them on the involutive "Big map"; thus, we have proved that the quantum algebras: bounded involutive lattices, De Morgan algebras and the ortholattices belong, in fact, to the "world" of *left-algebras* (involutive unital magmas).

In this paper, we clarify, mainly, some aspects concerning the quantum-MV algebras as non-lattice generalizations of MV algebras. We put MV algebras and quantum-MV algebras on the involutive "Big map", thus proving that quantum-MV algebras also belong, in fact, to the "world" of *left-algebras* (involutive left-unital magmas). We continue here the research from [23], [24], based on [19], in the "world" of involutive *left-algebras* of the form $(A, \odot, ^-, 1)$ verifying (Pass), with $1^- = 0$, 1 *being the last element*. This paper, like [19], [23], [24], presents the facts in the same *unifying way*, which consists in fixing unique names for the defining properties, making lists of these properties and then using them for defining the different algebras and for obtaining results.

The paper is organized as follows. In Section 2 (**Preliminaries**), we recall the original definitions of quantum MV (QMV) algebras and of orthomodular lattices and also the definitions of MV algebras, m-MEL, m-BE, m-pre-BCK, m-BCK algebras and some results from [19]. In Section 3 (**Redefining the QMV algebras**), we introduce the operation $\wedge_m^B$ (beside $\wedge_m^M$) and the binary relation $\leq_m^B$ (beside $\leq_m^M$) and prove that $\leq_m \Longleftrightarrow \leq_m^B$. We redefine the QMV algebras as involutive m-BE algebras verifying the property (Pqmv), just as we have redefined in [19] the MV algebras as involutive m-BE algebras verifying the property ($\wedge_m$-comm). We prove that (Pqmv) is equivalent with only two properties, (Pmv) and (Pq) (Theorem 3.19); we prove that (Pq) is equivalent with (Pom), the property characterizing the orthomodular lattices among the ortholattices - this being the core of the paper (Theorem 3.26), by its difficulty. We also prove that, if (Pom) holds, then (Pmv) is equivalent with (Δ_m), the largest non-antisymmetric generalization of ($\wedge_m$-comm) (Theorem 3.32); thus, finally, (Pqmv) is equivalent with (Δ_m) and (Pom). In Section 4 (**Three generalizations of QMV algebras**), we introduce three generalizations of QMV algebras: the *pre-MV (PreMV) algebras*, the *metha-MV (MMV) algebras* and the *orthomodular (OM) algebras*, and study their transitive and/or antisymmetric members. We clarify the connection between QMV and MV algebras, by proving that MV algebras coincide with the antisymmetric QMV algebras - but also with the antisymmetric preMV and with the antisymmetric MMV algebras - (Corollary 4.10), result enabling us to put QMV algebras and MV algebras on the same "map". We point out the *transitive QMV (tQMV) algebra*, a particular QMV algebra, and the *transitive antisymmetric orthomodular (taOM) algebra*, a proper generalization of MV algebra inside the class of m-BCK algebras. Finally, we put MV algebras and QMV, tQMV algebras on the same "map" (the involutive "Big map"); there is one open problem. In Section 5 (**Concluding remarks and future work**), we present conclusions and future work. In Section 6 (**Examples**), we present nine examples of the involved algebras.

Some of the proofs and of the examples were found by the powerful computer program *Prover9/Mace4* (version dec. 2007) created by William W. McCune (1953 - 2011). Therefore, we dedicate this paper to his memory.

2 Preliminaries

2.1 The original definition of quantum-MV algebras. The property (Wom)

Definition 2.1 ([6], Definition 2.1.1) [14]

A *supplement algebra*, or an *S algebra* for short, is an algebra $\mathcal{M} = (M, \oplus, ^-, 0, 1)$ consisting of a nonempty set M, two constant elements $0, 1$ in M, a unary operation $^-$ and a binary operation $\oplus$ on M satisfying the following axioms: for all $x, y, z \in M$,
(S1) $x \oplus y = y \oplus x$, (S2) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$, (S3) $x \oplus x^- = 1$, (S4) $x \oplus 0 = x$, (S5) $(x^-)^- = x$ (or $x^= = x$), (S6) $0^- = 1$, (S7) $x \oplus 1 = 1$.

On every S algebra, the following operations can be introduced: $x \odot y := (x^- \oplus y^-)^-$, $\quad x \sqcap y := (x \oplus y^-) \odot y$, $\quad x \sqcup y := (x \odot y^-) \oplus y$.

QMV algebras were introduced by Roberto Giuntini [7], as S algebras satisfying additionally five axioms. The equivalence of the five axioms with the next axiom (QMV) was proved in [12].

Definition 2.2 ([6], Definition 2.3.1) A *quantum MV algebra*, or a *QMV algebra* for short, is an S algebra $\mathcal{M} = (M, \oplus, ^-, 0, 1)$ satisfying: for all $x, y, z \in M$,
(QMV) $x \oplus ((x^- \sqcap y) \sqcap (z \sqcap x^-)) = (x \oplus y) \sqcap (x \oplus z)$.

Note that QMV algebras were originally defined as right-algebras (see more on left- and right-algebras in [15], [17], [19]).

The most used definition of MV algebras is the following:

Definition 2.3 [3] An *MV algebra* is an algebra $\mathcal{A} = (A, \oplus, ^-, 0)$ satisfying: for all $x, y, y \in A$,
(MV1) $x \oplus 0 = x$, (MV2) $x \oplus y = y \oplus x$, (MV3) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$, (MV4) $x \oplus 1 = 1$, where $1 \overset{def.}{=} 0^-$, (MV5) $(x^-)^- = x$ (or $x^= = x$), (MV6) = (MV) $(x^- \oplus y)^- \oplus y = (y^- \oplus x)^- \oplus x$.

Note that MV algebras were defined as right-algebras.

Definition 2.4 (See [28], [5]) An *ortholattice* is an algebra $\mathcal{A} = (A, \wedge, \vee, ^-, 0, 1)$ such that the reduct $(A, \wedge, \vee, 0, 1)$ is a bounded lattice and the unary operation $^-$ satisfies: for all $x, y \in A$,
(DN) $\quad (x^-)^- = x$ (or $x^= = x$) (Double Negation),
(DeM1) $\quad (x \vee y)^- = x^- \wedge y^-$ (De Morgan law 1) and, dually,
(DeM2) $\quad (x \wedge y)^- = x^- \vee y^-$ (De Morgan law 2),
and the complementation laws:
(m-WRe) $\quad x \wedge x^- = 0$ (noncontradiction principle) and, dually,
(m-VRe) $\quad x \vee x^- = 1$ (excluded middle principle).

Definition 2.5 An *orthomodular lattice* is an ortholattice $(A, \wedge, \vee, ^-, 0, 1)$ which satisfies the *orthomodular law*: for all $x, y \in A$,
(OML) $x \leq y \Longrightarrow x \vee (x^- \wedge y) = y$.

Note that property (OML) is not an identity, but there are many identities equivalent to (OML) within the class of ortholattices [28], as for example:

Proposition 2.6 *([28], Corollary 4.10.3) The following identity characterizes orthomodular lattices among ortholattices:*
(Wom) $(x \wedge y) \vee ((x \wedge y)^- \wedge x) = x$.

Note that orthomodular lattices were originally defined as left-algebras.

The dual of (Wom) is:
(Vom) $(x \vee y) \wedge ((x \vee y)^- \vee x) = x$,
where 'W' comes from 'wedge' (the LaTeX command for the meet, $\wedge$) and 'V' comes from 'vee' (the LaTeX command for the join, $\vee$).

2.2 The "Big map" of algebras. The involutive m-BE algebras

Recall from [19] the following:

Let $\mathcal{A}^L = (A^L, \odot, ^- = ^{-^L}, 1)$ be an algebra of type $(2, 1, 0)$ and define $0 \overset{def.}{=} 1^-$. Define an *internal* binary relation $\leq_m$ on A^L by: for all $x, y \in A^L$,
(m-dfrelP) $\quad x \leq_m y \overset{def.}{\Longleftrightarrow} x \odot y^- = 0$.

Consider the following list **m-A** of basic properties that can be satisfied by $\mathcal{A}^L$ [19]:

(PU) $1 \odot x = x = x \odot 1$ (unit element of product, the *identity*),
(Pcomm) $x \odot y = y \odot x$ (commutativity of product),
(Pass) $x \odot (y \odot z) = (x \odot y) \odot z$ (associativity of product);
(Neg1-0) $1^- = 0$,
(Neg0-1) $0^- = 1$;
(m-An) $(x \odot y^- = 0$ and $y \odot x^- = 0) \Longrightarrow x = y$ (antisymmetry),
(m-B) $[(x \odot y^-)^- \odot (x \odot z)] \odot (y \odot z)^- = 0$,
(m-BB) $[(z \odot x)^- \odot (y \odot x)] \odot (y \odot z^-)^- = 0$,
(m-*) $x \odot y^- = 0 \Longrightarrow (z \odot y^-) \odot (z \odot x^-)^- = 0$,
(m-**) $x \odot y^- = 0 \Longrightarrow (x \odot z) \odot (y \odot z)^- = 0$,
(m-L) $x \odot 0 = 0$ (last element),
(m-Re) $x \odot x^- = 0$ (reflexivity),
(m-Tr) $(x \odot y^- = 0$ and $y \odot z^- = 0) \Longrightarrow x \odot z^- = 0$ (transitivity),
etc.,
where 'm' comes from 'magma'.

Dually, let $\mathcal{A}^R = (A^R, \oplus, ^- = ^{-^R}, 0)$ be an algebra of type $(2, 1, 0)$ and define $1 \stackrel{def.}{=} 0^-$. Define an *internal* binary relation $\geq_m$ on A^R by: for all $x, y \in A^R$,

(m-dfrelS) $x \geq_m y \stackrel{def.}{\Longleftrightarrow} x \oplus y^- = 1$.

The list of dual properties is omitted.

Recall from [19] the definitions of the algebras needed in this paper (the dual ones are omitted):

Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an algebra of type $(2, 1, 0)$ through this paper. Define $0 \stackrel{def.}{=} 1^-$ (hence (Neg1-0) holds) and suppose that $0^- = 1$ (hence (Neg0-1) holds too). We say that $\mathcal{A}^L$ is a [19]:
- *left-m-MEL algebra*, if (PU), (Pcomm), (Pass), (m-L) hold;
- *left-m-BE algebra*, if (PU), (Pcomm), (Pass), (m-L), (m-Re) hold;
- *left-m-pre-BCK algebra*, if (PU), (Pcomm), (Pass), (m-L), (m-Re) and (m-BB) hold;
- *left-m-BCK algebra*, if (PU), (Pcomm), (Pass), (m-L), (m-Re), (m-An) and (m-BB) hold.

Denote by **m-MEL**, **m-BE**, ..., **m-BCK** these classes of left-algebras, respectively.

We say that $\mathcal{A}^L$ is [19] *reflexive*, if $\leq_m$ is reflexive (i.e. (m-Re) holds); *transitive*, if $\leq_m$ is transitive (i.e. (m-Tr) holds); *antisymmetric*, if $\leq_m$ is antisymmetric (i.e. (m-An) holds). If **X** is a class of algebras, we shall denote by **tX** (**aX**, **atX**=**taX**) the subclass of all transitive (antisymmetric, transitive and antisymmetric, respectively) algebras of **X**.

A hierarchy of classes of such algebras will be represented by a kind of Hasse-type diagram, where the algebras are represented as follows:

- *reflexive* algebras by ◯
- *antisymmetric* algebras by ∘
- *transitive* algebras by •
- *reflexive* and *antisymmetric* algebras by ⊚
- *reflexive* and *transitive* algebras by ⊙
- *ordered* algebras by ●

and a class of algebras which does not verify (m-Re), (m-An), (m-Tr), by □.

In ([19], Figure 10) - see next Figure 1 - the "Big map", connecting the commutative left-unital magmas, including these new algebras, was drawn.

We say that an algebra is *involutive*, if it verifies (DN). If **X** is a class of algebras, we shall denote by $\mathbf{X}_{(DN)}$ the subclass of all involutive algebras of **X**. By ([19], Theorem 6.12), in any involutive m-BE algebra we have the equivalences: (m-BB) ⇔ (m-B) ⇔ (m-**) ⇔ (m-*) ⇔ (m-Tr).

Note that: $\mathbf{m\text{-}pre\text{-}BCK}_{(DN)} = \mathbf{pre\text{-}m\text{-}BCK}_{(DN)}$ $(= \mathbf{m\text{-}tBE}_{(DN)})$.

Any left-m-BCK algebra is involutive, by ([19], Theorem 6.13). We write: $\mathbf{m\text{-}BCK} = \mathbf{m\text{-}BCK}_{(DN)}$ $(= \mathbf{m\text{-}taBE}_{(DN)})$. Note that an (involutive) m-BCK algebra satisfies all the properties in the list **m-A** of properties and, additionally, (DN) and other properties.

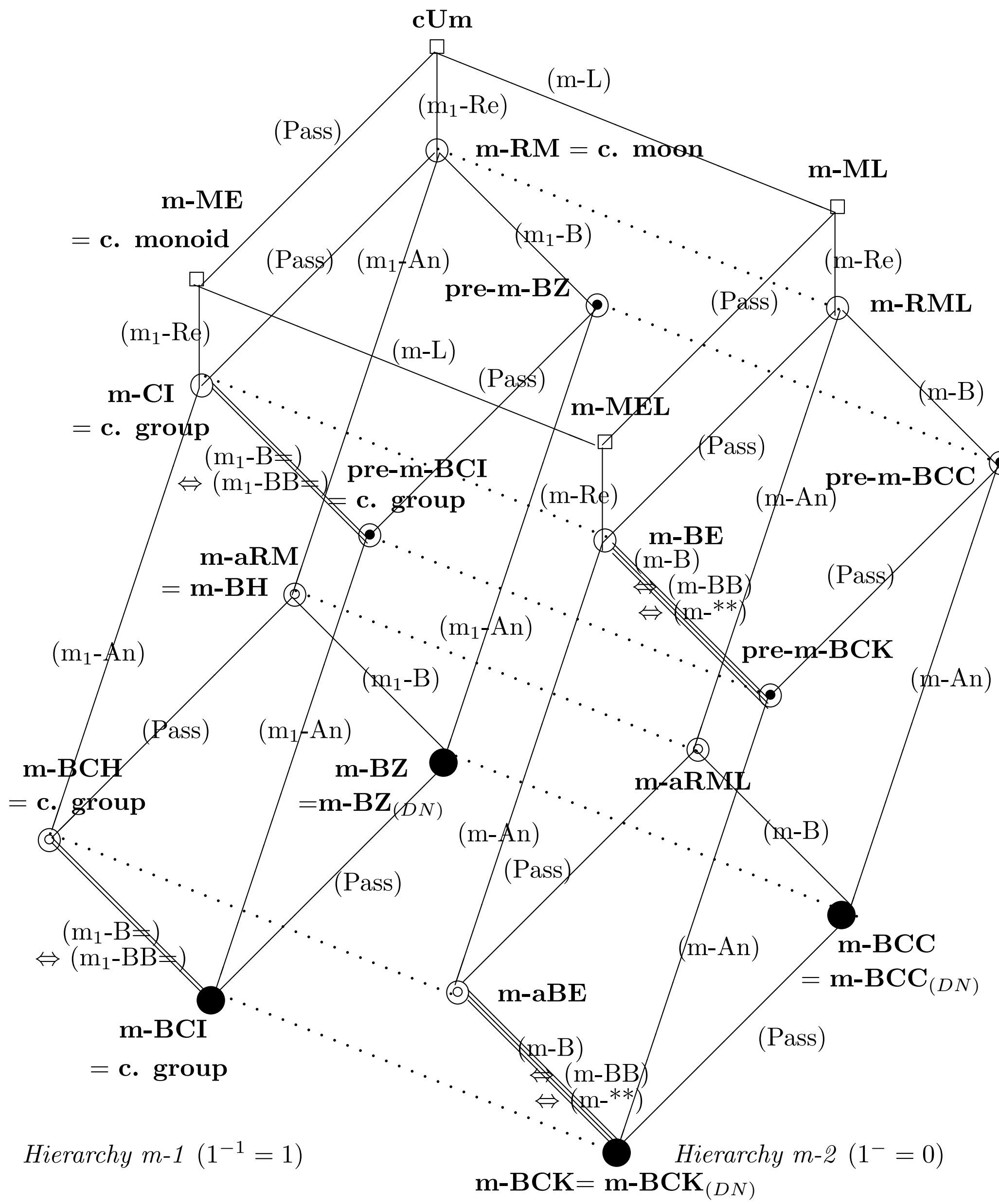


Figure 1: The “Big map”: hierarchies m-1 and m-2 of commutative unital magmas (**cUm**)

Note that the binary relation $\leq_m$ is only **reflexive** in **m-BE**$_{(DN)}$, it is a **pre-order** in **m-pre-BCK**$_{(DN)}$ and it is an **order** in **m-BCK**.

2.2.1 Involutive m-MEL algebras

Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-MEL algebra. Because of the axiom (DN), we have introduced in [23] the new operation sum, $\oplus$, the dual of product, $\odot$, by: for all $x, y \in A^L$,

$$(1) \qquad x \oplus y \stackrel{def.}{=} (x^- \odot y^-)^-.$$

Hence,

$$(2) \qquad x \odot y = (x^- \oplus y^-)^-.$$

Then, $(A^L, \oplus, ^-, 0)$ is an involutive right-m-MEL algebra. We have: $x \leq_m y \Longleftrightarrow y \geq_m x$.

Beside the old, natural binary relation $\leq_m$ and its dual $\geq_m$, we have introduced in [23] a new binary relation:
(m-dfP) $x \ \leq_m^P \ y \stackrel{def.}{\Longleftrightarrow} x \odot y = x$ and, dually,
(m-dfS) $x \ \geq_m^S \ y \stackrel{def.}{\Longleftrightarrow} x \oplus y = x$.
By ([23], Proposition 3.11), $\leq_m^P$ is antisymmetric and transitive and $0 \leq_m^P x \leq_m^P 1$, for any x.

With the notations from this subsection, Definition 2.3 of MV algebras becomes [19]:

Definition 2.7
(i) A *left-MV algebra* is an algebra $\mathcal{A}^L = (A^L, \odot, ^- = ^{-^L}, 1)$ of type $(2, 1, 0)$ verifying (PU), (Pcomm), (Pass), (m-L), (DN) and:
($\wedge_m$-comm) $(x^- \odot y)^- \odot y = (y^- \odot x)^- \odot x$.
(i') Dually, a *right-MV algebra* is an algebra $\mathcal{A}^R = (A^R, \oplus, ^- = ^{-^R}, 0)$ of type $(2, 1, 0)$ verifying (SU), (Scomm), (Sass), (m-L^R), (DN) and:
($\vee_m$-comm) $(x^- \oplus y)^- \oplus y = (y^- \oplus x)^- \oplus x$.

We recall the following important remark, which was the motivation of paper [19]:
(i) The left-MV algebra is just the involutive left-m-MEL algebra verifying ($\wedge_m$-comm).
(i') Dually, the right-MV algebra is just the involutive right-m-MEL algebra verifying ($\vee_m$-comm).

Denote by **MV** the class of all left-MV algebras and by **MV**^{R} the class of all right-MV algebras.

Proposition 2.8 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-MEL algebra. Then:*
$(\wedge_m - comm) \iff (\vee_m - comm)$.

Proof. Routine. □

2.2.2 Involutive m-BE algebras. The property (Pom)

Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then, $(A^L, \oplus, ^-, 0)$ is an involutive right-m-BE algebra.

Remark 2.9 An *S algebra* (Definition 2.1) is just an *involutive right-m-BE algebra.*

Remarks 2.10 *(The dual one is omitted)*
(i) Since ($\wedge_m$-comm) implies (m-Re), by ([19], (mB1)), it follows that **any left-MV algebra is in fact an involutive left-m-BE algebra verifying ($\wedge_m$-comm)**.
(ii) Since ($\wedge_m$-comm) implies also (m-An) and (m-BB) ($\Leftrightarrow \ldots \Leftrightarrow$ (m-Tr)), by ([19], (mB2), (mCBN1)), respectively, i.e. we have:

$$(3) \qquad (\wedge_m - comm) \implies (m - An) \ + \ (m - Tr),$$

it follows that **any left-MV algebra is in fact a left-m-BCK algebra**, *i.e. we have:*

$$\mathbf{MV} \subset \mathbf{m-BCK} = \mathbf{m-BCK_{(DN)}}(= \mathbf{m-taBE_{(DN)}}).$$

(iii) Moreover, by ([19], Theorem 6.21), the class of left-MV algebras is d.e. with the class of $\wedge_m$-commutative (involutive) left-m-BCK algebras (i.e. left-m-BCK algebras verifying ($\wedge_m$-comm)).

In ([19], Figure 8), the connections between m-BE algebras, m-BCK algebras, MV algebras, ortholattices and Boolean algebras were established, thus putting MV algebras, ortholattices and Boolean algebras on the "map" (the right side of the involutive "Big map").

We have redefined equivalently the ortholattices (Definition 2.4) in [19], [23] as follows (Definition 2):

(i) A *left-ortholattice* is a (involutive) left-m-BE algebra $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ verifying:
(m-Pimpl) $[(x \odot y^-)^- \odot x^-]^- = x$.

(i') Dually, a *right-ortholattice* is a (involutive) right-m-BE algebra $\mathcal{A}^R = (A^R, \oplus, ^-, 0)$ verifying:
(m-Simpl) $[(x \oplus y^-)^- \oplus x^-]^- = x$.

It follows that an orthomodular lattice (Definition 2.5) can be redefined equivalently as follows (Definition 2):

(i) An *orthomodular left-lattice* is a (involutive) left-m-BE algebra $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ verifying (m-Pimpl) and (Pom) ((Wom) becomes (Pom)), where:
(Pom) $\quad (x \odot y) \oplus ((x \odot y)^- \odot x) = x$.

(i') Dually, an *orthomodular right-lattice* is a (involutive) right-m-BE algebra $\mathcal{A}^R = (A^R, \oplus, ^-, 0)$ verifying (m-Simpl) and (Som) ((Vom) becomes (Som)), where:
(Som) $\quad (x \oplus y) \odot ((x \oplus y)^- \oplus x) = x$.

Thus, **(Pom) is the property characterizing the orthomodular left-lattices among the left-ortholattices** (Definitions 2). It will play a major role in QMV algebras.

3 Redefining the QMV algebras

Remark 3.1 *Starting from the equality from the property*

$$(\wedge_m - comm) \qquad (x^- \odot y)^- \odot y = (y^- \odot x)^- \odot x,$$

verified by a left-MV algebra, we could introduce two different 'twin' operations, $\wedge_m^M$ ('M' comes from 'MV algebra') and $\wedge_m^B$ ('B' comes from 'Boolean algebra'), by: for all x, y:
$x \wedge_m^M y = (x^- \odot y)^- \odot y$ *and* $x \wedge_m^B y = (y^- \odot x)^- \odot x$.
Then, ($\wedge_m$-comm) would mean: either
($\wedge_m^M$*-comm*) $\quad x \wedge_m^M y = y \wedge_m^M x$ *or*
($\wedge_m^B$*-comm*) $\quad x \wedge_m^B y = y \wedge_m^B x$.

In left-MV algebras, the two operations $\wedge_m^M$ and $\wedge_m^B$ are equal, but in general, in an involutive left-m-MEL algebra, they are different; but note that: $x \wedge_m^M y = y \wedge_m^B x$, which means, in the finite case, that the matrix of $\wedge_m^B$ is the transposed matrix of that of $\wedge_m^M$ and vice-versa.

Following the above Remark 3.1, we shall introduce in an involutive left-m-MEL algebra $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ the following operations:

$$x \wedge_m^M y \overset{def.}{=} (x^- \odot y)^- \odot y \overset{(Pcomm)}{=} y \odot (y \odot x^-)^- \quad \textit{and, dually,} \tag{4}$$

$$x \vee_m^M y \overset{def.}{=} (x^- \wedge_m^M y^-)^- = [(x \odot y^-)^- \odot y^-]^- = (x \odot y^-) \oplus y = (x^- \oplus y)^- \oplus y = y \oplus (y \oplus x^-)^- \tag{5}$$

and

$$x \wedge_m^B y \overset{def.}{=} (y^- \odot x)^- \odot x \overset{(Pcomm)}{=} x \odot (x \odot y^-)^- = x \odot (x \to y) = y \wedge_m^M x \quad \textit{and, dually,} \tag{6}$$

$$x \vee_m^B y \overset{def.}{=} (x^- \wedge_m^B y^-)^- = [(y \odot x^-)^- \odot x^-]^- = (y \odot x^-) \oplus x = (y^- \oplus x)^- \oplus x = x \oplus (x \oplus y^-)^- = y \vee_m^M x. \tag{7}$$

Note that the dual operations $\wedge_m^M$, $\vee_m^M$ are just $\sqcap$, $\sqcup$, respectively, recalled in subsection 2.1.
In what follows, we shall present only the properties of $\wedge_m^M$ and $\vee_m^M$.

Beside the old, natural binary relation $\leq_m$, and its dual $\geq_m$, and the binary relation $\leq_m^P$, and its dual $\geq_m^S$, we introduce two binary relations, the old $\leq_m^M$ (see [6]) and the new $\leq_m^B$: for all $x, y \in A^L$,
(m-dfWM) $x \ \leq_m^M \ y \stackrel{def.}{\Longleftrightarrow} x \wedge_m^M y = x$ and, dually,
(m-dfVM) $x \ \geq_m^M \ y \stackrel{def.}{\Longleftrightarrow} x \vee_m^M y = x$,
and
(m-dfWB) $x \ \leq_m^B \ y \stackrel{def.}{\Longleftrightarrow} x \wedge_m^B y = x \ (\Longleftrightarrow y \wedge_m^M x = x)$ and, dually,
(m-dfVB) $x \ \geq_m^B \ y \stackrel{def.}{\Longleftrightarrow} x \vee_m^B y = x \ (\Longleftrightarrow y \vee_m^M x = x)$.

Lemma 3.2 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. We have:*
(1) $\quad x \odot y^- = 0 \Longleftrightarrow x \odot (x \odot y^-)^- = x$ *and, dually,*
(1') $\quad x \oplus y^- = 1 \Longleftrightarrow x \oplus (x \oplus y^-)^- = x$.

Proof. (1): Suppose that $x \odot y^- = 0$; then, $(x \odot y^-)^- = 1$, hence $x \odot (x \odot y^-)^- = x \odot 1 = x$.
Conversely, suppose that $x \odot (x \odot y^-)^- = x$; then,
$x \odot y^- = (x \odot (x \odot y^-)^-) \odot y^- \stackrel{(Pcomm),(Pass)}{=} (x \odot y^-) \odot (x \odot y^-)^- \stackrel{(m-Re)}{=} 0$.
(1'): Suppose that $x \oplus y^- = 1$; then, $(x \oplus y^-)^- = 0$, hence $x \oplus (x \oplus y^-)^- = x \oplus 0 = x$.
Conversely, suppose that $x \oplus (x \oplus y^-)^- = x$; then,
$x \oplus y^- = (x \oplus (x \oplus y^-)^-) \oplus y^- \stackrel{(Scomm),(Sass)}{=} (x \oplus y^-) \oplus (x \oplus y^-)^- \stackrel{(m-Re^R)}{=} 1$. □

Proposition 3.3 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. We have:*
(1) $\quad x \leq_m y \Longleftrightarrow x \leq_m^B y$ *and, dually*
(1') $\quad x \geq_m y \Longleftrightarrow x \geq_m^B y$.
(2) If ($\wedge_m$-comm) holds (i.e. $x \wedge_m^M y = y \wedge_m^M x$), then

$$x \leq_m y \ (\Longleftrightarrow x \leq_m^B y) \ \Longleftrightarrow x \leq_m^M y.$$

(2') If ($\wedge_m$-comm) holds, then ($\vee_m$-comm) holds (i.e. $x \vee_m^M y = y \vee_m^M x$) and

$$x \geq_m y \ (\Longleftrightarrow x \geq_m^B y) \ \Longleftrightarrow x \geq_m^M y.$$

Proof. (1): By above Lemma 3.2 (1),
$x \leq_m y \stackrel{def.}{\Longleftrightarrow} x \odot y^- = 0 \Longleftrightarrow x \odot (x \odot y^-)^- = x \Longleftrightarrow x \wedge_m^B y = x \stackrel{def.}{\Longleftrightarrow} x \leq_m^B y$.
(1'): By above Lemma 3.2 (1'),
$x \geq_m y \stackrel{def.}{\Longleftrightarrow} x \oplus y^- = 1 \Longleftrightarrow x \oplus (x \oplus y^-)^- = x \Longleftrightarrow x \vee_m^B y = x \stackrel{def.}{\Longleftrightarrow} x \geq_m^B y$.
(2): By above (1),
$x \leq_m^M y \stackrel{def.}{\Longleftrightarrow} x \wedge_m^M y = x \stackrel{(\wedge_m-comm)}{\Longleftrightarrow} y \wedge_m^M x = x \Longleftrightarrow x \wedge_m^B y = x \stackrel{def.}{\Longleftrightarrow} x \leq_m^B y \Longleftrightarrow x \leq_m y$.
(2'): By Proposition 2.8 and above (1'). □

Remark 3.4 *The equivalence $\leq_m \Longleftrightarrow \leq_m^B$ implies that $\leq_m$ is an order relation if and only if $\leq_m^B$ is an order relation. But, it does not imply that if $\leq_m$ is a lattice order w.r. to say $\wedge, \vee$, then $\leq_m^B$ is a lattice order too with respect to $\wedge_m^B, \vee_m^B$.*

Proposition 3.5 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then,*

$$(x \leq_m^B y \Longleftrightarrow) \ x \leq_m y \Longleftrightarrow y \geq_m x \ (\Longleftrightarrow y \geq_m^B x).$$

Proof. By Proposition 3.3 and ([23], Proposition 3.10). □

Proposition 3.6 *(See ([6], Proposition 2.1.2), in dual case)*
Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-MEL algebra. We have:

(8) $$x \wedge_m^M 1 = x = 1 \wedge_m^M x, \quad x \wedge_m^M 0 = 0,$$

$$x \vee_m^M 0 = x = 0 \vee_m^M x, \quad x \vee_m^M 1 = 1, \tag{9}$$

$$(x \vee_m^M y)^- = x^- \wedge_m^M y^- \quad (De\ Morgan\ law\ 1), \tag{10}$$

$$(x \wedge_m^M y)^- = x^- \vee_m^M y^- \quad (De\ Morgan\ law\ 2)\ and,\ hence, \tag{11}$$

$$x \wedge_m^M y = (x^- \vee_m^M y^-)^-. \tag{12}$$

Proposition 3.7 *(See ([6], Proposition 2.1.2), in dual case)*
Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. We have:

$$if \quad x \odot y = 1, \quad then \quad x = y = 1, \tag{13}$$

$$if \quad x \wedge_m^M y = 1, \quad then \quad x = y = 1, \tag{14}$$

$$0 \wedge_m^M x = 0, \tag{15}$$

$$1 \vee_m^M x = 1, \tag{16}$$

$$x \wedge_m^M x = x, \quad x \vee_m^M x = x, \tag{17}$$

$$if \quad x \leq_m^M y, \quad then \quad y \wedge_m^M x = x, \tag{18}$$

$$if \quad x \leq_m^M y, \quad then \quad x \leq_m y. \tag{19}$$

Proposition 3.8 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. We have:*

$$x \oplus x^- = 1, \quad i.e. \quad (m - Re^R) \quad holds; \tag{20}$$

$$x \odot (y \wedge_m^M x^-) = 0, \tag{21}$$

$$x \odot (x^- \wedge_m^M y) = 0, \tag{22}$$

$$(y \vee_m^M x) \wedge_m^M x = x, \tag{23}$$

$$(y \wedge_m^M x) \vee_m^M x = x, \tag{24}$$

$$if \quad x \leq_m^M y, \quad then \quad x \vee_m^M y = y, \tag{25}$$

$$x \vee_m^M y = y \iff x \odot y^- = 0 \quad (\iff x \leq_m y), \tag{26}$$

$$(x \odot y) \vee_m^M x = x, \tag{27}$$

$$x \wedge_m^M (x \odot y) = x \odot y, \tag{28}$$

$$x \wedge_m^M (y \wedge_m^M x) = y \wedge_m^M x. \tag{29}$$

Proof. (20): $x \oplus x^- \overset{(1)}{=} (x^- \odot x)^- \overset{(Pcomm)}{=} (x \odot x^-)^- \overset{(m-Re)}{=} 0^- \overset{(Neg0-1)}{=} 1$.

(21): $x \odot (y \wedge_m^M x^-) = x \odot [(y^- \odot x^-)^- \odot x^-]$
$\overset{(Pcomm)}{=} x \odot [x^- \odot (y^- \odot x^-)^-] \overset{(Pass)}{=} (x \odot x^-) \odot (y^- \odot x^-)^-$
$\overset{(m-Re)}{=} 0 \odot (y^- \odot x^-)^- \overset{(Pcomm)}{=} (y^- \odot x^-)^- \odot 0 \overset{(m-L)}{=} 0$.

(22): $x \odot (x^- \wedge_m^M y) = x \odot [(x \odot y)^- \odot y] = x \odot [y \odot (x \odot y)^-] = (x \odot y) \odot (x \odot y)^- \overset{(m-Re)}{=} 0$.

(23): $(y \vee_m^M x) \wedge_m^M x \overset{(5)}{=} [(y \odot x^-)^- \odot x^-]^- \wedge_m^M x \overset{(4),(DN)}{=} (((y \odot x^-)^- \odot x^-) \odot x)^- \odot x$
$\overset{(Pass),(Pcomm)}{=} ((y \odot x^-)^- \odot (x \odot x^-))^- \odot x \overset{(m-Re)}{=} ((y \odot x^-)^- \odot 0)^- \odot x \overset{(m-L)}{=} 0^- \odot x = 1 \odot x = x$.

(24): $(y \wedge_m^M x) \vee_m^M x \overset{(4)}{=} [(y^- \odot x)^- \odot x] \vee_m^M x \overset{(5)}{=} ([(y^- \odot x)^- \odot x] \odot x^-) \oplus x$
$\overset{(Pass)}{=} ((y^- \odot x)^- \odot (x \odot x^-)) \oplus x \overset{(m-Re)}{=} ((y^- \odot x)^- \odot 0) \oplus x \overset{(m-L)}{=} 0 \oplus x \overset{(SU)}{=} x$.

(25): Suppose $x \leq_m^M y$, i.e. $x \wedge_m^M y = x$; then, $x \vee_m^M y = (x \wedge_m^M y) \vee_m^M y \overset{(24)}{=} y$.

(26): If $y = x \vee_m^M y \overset{(5)}{=} (x \odot y^-) \oplus y$, then $y^- = ((x \odot y^-) \oplus y)^- = (x \odot y^-)^- \odot y^-$. Then,
$x \odot y^- = x \odot [(x \odot y^-)^- \odot y^-] \overset{(Pcomm),(Pass)}{=} (x \odot y^-) \odot (x \odot y^-)^- \overset{(m-Re)}{=} 0$.
Conversely, if $x \odot y^- = 0$, then $x \vee_m^M y \overset{(5)}{=} (x \odot y^-) \oplus y = 0 \oplus y \overset{(SU)}{=} y$.

(27: $(x \odot y) \vee_m^M x = [[(x \odot y) \odot x^-]^- n \odot x^-]^- \overset{(m-Re),(m-L)}{=} [0^- \odot x^-]^- = (x^-)^- = x$.

(28): $x \wedge_m^M (x \odot y) = [x^- \odot (x \odot y)]^- \odot (x \odot y) \overset{(m-Re),(m-L)}{=} 0^- \odot (x \odot y) = 1 \odot (x \odot y) = x \odot y$.

(29): $x \wedge_m^M (y \wedge_m^M x) = x \wedge_m^M (x \odot (x \odot y^-)^-) = (x \odot (x \odot y^-)^-) \odot (x^- \odot (x \odot (x \odot y^-)^-))^-$
$\overset{(Pass)}{=} (x \odot (x \odot y^-)^-) \odot ((x^- \odot x) \odot (x \odot y^-)^-)^-$
$\overset{(Pcomm),(m-Re),(m-L)}{=} (x \odot (x \odot y^-)^-) \odot 0^- \overset{(Neg0-1),(PU)}{=} x \odot (x \odot y^-)^- = y \wedge_m^M x$. $\square$

Corollary 3.9 *(See ([6], Corollary 2.1.3))*
Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then, the binary relation $\leq_m^M$ is reflexive and antisymmetric and $0 \leq_m^M x \leq_m^M 1$, for all $x \in A^L$, where $0 \overset{def.}{=} 1^-$.

3.1 Redefining the QMV algebras as involutive m-BE algebras

Following the original definition (Definition 2.2) of QMV algebras, the definition of involutive m-BE algebras and Remark 2.9, we obtain the following redefinition of QMV algebras as involutive m-BE algebras, which helps us to put them on the "map" (the involutive "Big map"):

Definitions 3.10
(i) A *left-quantum-MV algebra*, or a *left-QMV algebra* for short, is an involutive left-m-BE algebra $\mathcal{A}^L = (A^L, \odot, ^- = ^{-_L}, 1)$ verifying the following axiom: for all $x, y, z \in A^L$,
(Pqmv) $\quad x \odot [(x^- \vee_m^M y) \vee_m^M (z \vee_m^M x^-)] = (x \odot y) \vee_m^M (x \odot z)$.
(i') A *right-quantum-MV algebra*, or a *right-QMV algebra* for short, is an involutive right-m-BE algebra (= S algebra) $\mathcal{A}^R = (A^R, \oplus, ^- = ^{-_R}, 0)$ verifying the following dual axiom: for all $x, y, z \in A^R$,
(Sqmv) = (QMV) $\quad x \oplus [(x^- \wedge_m^M y) \wedge_m^M (z \wedge_m^M x^-)] = (x \oplus y) \wedge_m^M (x \oplus z)$.

We shall denote by **QMV** the class of all left-QMV algebras and by $\mathbf{QMV}^R$ the class of all right-QMV algebras.

Proposition 3.11 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then:*

$$(Pqmv) \Longleftrightarrow (Sqmv).$$

Proof. Suppose (Pqmv) holds; then, $x \oplus [(x^- \wedge_m^M y) \wedge_m^M (z \wedge_m^M x^-)]$
$\overset{(1)}{=} (x^- \odot [(x^- \wedge_m^M y) \wedge_m^M (z \wedge_m^M x^-)])^- \overset{(11)}{=} (x^- \odot [(x^- \wedge_m^M y)^- \vee_m^M (z \wedge_m^M x^-)^-])^-$
$\overset{(11)}{=} (x^- \odot [(x \vee_m^M y^-) \vee_m^M (z^- \vee_m^M x)])^- \overset{(Pqmv)}{=} ((x^- \odot y^-) \vee_m^M (x^- \odot z^-))^-$
$\overset{(10)}{=} (x^- \odot y^-)^- \wedge_m^M (x^- \odot z^-)^- \overset{(1)}{=} (x \oplus y) \wedge_m^M (x \oplus z)$, i.e. (Sqmv) holds.
Suppose (Sqmv) holds; then, $x \odot [(x^- \vee_m^M y) \vee_m^M (z \vee_m^M x^-)]$

$\overset{(2)}{=} (x^- \oplus [(x^- \vee_m^M y) \vee_m^M (z \vee_m^M x^-)])^- \overset{(10)}{=} (x^- \oplus [(x^- \vee_m^M y)^- \wedge_m^M (z \vee_m^M x^-)^-])^-$
$\overset{(10)}{=} (x^- \oplus [(x \wedge_m^M y^-) \wedge_m^M (z^- \wedge_m^M x)])^- \overset{(Sqmv)}{=} ((x^- \oplus y^-) \wedge_m^M (x^- \oplus z^-))^-$
$\overset{(11)}{=} (x^- \oplus y^-)^- \vee_m^M (x^- \oplus z^-)^- \overset{(2)}{=} (x \odot y) \vee_m^M (x \odot z)$, i.e. (Pqmv) holds. □

Corollary 3.12 Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a left-QMV algebra. Then, $(A^L, \oplus, ^-, 0)$ is a right-QMV algebra.

Proof. By ([23], Corollary 4.3) and Proposition 3.11. □

Proposition 3.13 *(See ([6], Proposition 2.3.2), in dual case)*
Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a left-QMV algebra. We have:

(30) $$x \odot (y \vee_m^M x^-) = x \odot y,$$

$$(Pmv) \quad x \odot (x^- \vee_m^M y) = x \odot y,$$
$$(Pq) \quad x \odot [y \vee_m^M (z \vee_m^M x^-)] = (x \odot y) \vee_m^M (x \odot z);$$

(31) $$x \odot y \ \leq_m^M \ x, \quad i.e. \quad (x \odot y) \wedge_m^M x = x \odot y,$$

(32) $$x \ \leq_m^M \ x \oplus y, \quad i.e. \quad x \wedge_m^M (x \oplus y) = x,$$

(33) $$x \wedge_m^M y \ \leq_m^M \ y, \quad i.e. \quad (x \wedge_m^M y) \wedge_m^M y = x \wedge_m^M y,$$

(34) $$y \ \leq_m^M \ x \vee_m^M y, \quad i.e. \quad y \wedge_m^M (x \vee_m^M y) = y,$$

(35) $$x \odot [y \vee_m^M (x \odot z)^-] = (x \odot y) \vee_m^M (x \odot (x \odot z)^-),$$

(36) $$x \vee_m^M (y \wedge_m^M x) = x,$$

(37) $$x \ \leq_m^M \ y \Longrightarrow y \vee_m^M x = y,$$

(38) $$x \ \leq_m^M \ y \Longrightarrow y^- \ \leq_m^M \ x^- \quad (order-reversibility\ of\ ^-),$$

(39) $$x \ \leq_m^M \ y \Longrightarrow x \oplus z \ \leq_m^M \ y \oplus z \quad (monotonicity\ of\ \oplus),$$

(40) $$x \ \leq_m^M \ y \Longrightarrow x \odot z \ \leq_m^M \ y \odot z \quad (monotonicity\ of\ \odot),$$

(41) $$(x \wedge_m^M y) \wedge_m^M z = (x \wedge_m^M y) \wedge_m^M (y \wedge_m^M z),$$

(42) $$(x \vee_m^M y) \vee_m^M z = (x \vee_m^M y) \vee_m^M (y \vee_m^M z),$$

(43) $$x \odot y = x \odot y \odot (x \oplus y),$$

(44) $$(x^- \odot y) \wedge_m^M (y^- \odot x) = 0.$$

Remarks 3.14 *(i) Concerning (33), note that $x \wedge_m^M y \ \not\leq_m^M \ x$. For example, in the left-QMV algebra from Example 6.4, $a \wedge_m^M c \ \not\leq_m^M \ a$. Indeed, $a \wedge_m^M c = c$, while $(a \wedge_m^M c) \wedge_m^M a = c \wedge_m^M a = a \neq c$.*

(ii) Concerning (34), note that $x \ \not\leq_m^M \ x \vee_m^M y$. For example, in the left-QMV algebra from Example 6.4, $a \ \not\leq_m^M \ a \vee_m^M c$. Indeed, $a \vee_m^M c = c$, while $a \wedge_m^M (a \vee_m^M c) = a \wedge_m^M c = c \neq a$.

Recall now the following well known property (prel) (prelinearity) from a bounded residuated lattice $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ [15]:
(prel) $\quad (x \rightarrow y) \vee (y \rightarrow x) = 1$.

Here, we shall consider that $\vee \overset{def.}{=} \vee_m^B$, which is no more a lattice operation, therefore the property will be denoted by (prel$_m$):
(prel$_m$) $\quad (x \rightarrow y) \vee_m^B (y \rightarrow x) = 1$,
where $x \rightarrow y \overset{def.}{=} (x \odot y^-)^-$ (see the map Ψ from [19]).

Note that, in MV algebras, (prel$_m$) and (prel) coincide.

Then, we have the following result:

Corollary 3.15 *Any left-QMV algebra verifies the property (prel$_m$).*

Proof. $(x \rightarrow y) \vee_m^B (y \rightarrow x) = (x \odot y^-)^- \vee_m^B (y \odot x^-)^-$
$= (y \odot x^-)^- \vee_m^M (x \odot y^-)^- = ((y \odot x^-) \wedge_m^M (x \odot y^-))^- \overset{(Pcomm),(44)}{=} 0^- = 1$. $\square$

Proposition 3.16 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a left-QMV algebra. We have:*

(45) $$x \vee_m^M y \;\leq_m^M x \oplus y,$$

(46) $$x \odot y \;\leq_m^M x \wedge_m^M y.$$

Proof. (45): Since $x \odot y^- \;\leq_m^M\; x$, by (31), then $x \vee_m^M y = (x \odot y^-) \oplus y \;\leq_m^M\; x \oplus y$, by (39).

(46): Since $x^- \odot y \;\leq_m^M\; x^-$, by (31), then $x \;\leq_m^M\; (x^- \odot y)^-$, by (38) and (DN); hence, $x \odot y \;\leq_m^M$ $(x^- \odot y)^- \odot y = x \wedge_m^M y$, by (40). $\square$

Proposition 3.17 *(See ([6], Proposition 2.3.5), in dual case)*
Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a left-QMV algebra. We have:

(47) $$x \wedge_m^M ((x \oplus y) \wedge_m^M z) = x \wedge_m^M z \quad (\textit{absorption law } 1),$$

(48) $$x \vee_m^M ((x \odot y) \vee_m^M z) = x \vee_m^M z \quad (\textit{absorption law } 2),$$

(49) $$x \;\leq_m^M\; z^-,\; y \;\leq_m^M\; z^-,\; x \oplus z = y \oplus z \implies x = y \quad (\textit{cancellation law } 1),$$

(50) $$z^- \;\leq_m^M\; x,\; z^- \;\leq_m^M\; y,\; x \odot z = y \odot z \implies x = y \quad (\textit{cancellation law } 2),$$

(51) $$x \;\leq_m^M\; y \implies x \wedge_m^M z \;\leq_m^M\; y \wedge_m^M z \quad (\textit{monotonicity of } \wedge_m^M),$$

(52) $$x \;\leq_m^M\; y \implies x \vee_m^M z \;\leq_m^M\; y \vee_m^M z \quad (\textit{monotonicity of } \vee_m^M),$$

(53) $$x \;\leq_m^M\; y,\; y \;\leq_m^M\; z \implies x \;\leq_m^M\; z \quad (\textit{transitivity of } \;\leq_m^M\;).$$

Corollary 3.18 *(See ([6], page 157))*
Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a left-QMV algebra. The binary relation $\leq_m^M$ is an order relation.

We shall prove next **the first very important result of this paper**, Theorem 3.19, saying that axiom (Pqmv) is equivalent to only two properties, the properties (Pmv) and (Pq) from Proposition 3.13:
(Pmv) $\quad x \odot (x^- \vee_m^M y) = x \odot y$,
(Pq) $\quad x \odot [y \vee_m^M (z \vee_m^M x^-)] = (x \odot y) \vee_m^M (x \odot z)$.
Recall that, cf. ([6], Proposition 2.3.4), Giuntini proved that axiom (Pqmv) is equivalent to the properties (Pmv), (35), (36), (41) and (44).

Theorem 3.19 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then,*

$$(Pqmv) \iff (Pmv) \, + \, (Pq).$$

Proof. By Proposition 3.13, the axioms of $\mathcal{A}^L$ and (Pqmv) imply (Pmv) and (Pq). To prove the converse, assume that (Pmv) and (Pq) are satisfied by $\mathcal{A}^L$. Then:

$x \odot [(x^- \vee_m^M y) \vee_m^M (z \vee_m^M x^-)]$

$\overset{(5)}{=} x \odot [((x^- \vee_m^M y) \odot (z \vee_m^M x^-)^-) \oplus (z \vee_m^M x^-)]$

$\overset{(10),(DN)}{=} x \odot [((x^- \vee_m^M y) \odot (z^- \wedge_m^M x)) \oplus (z \vee_m^M x^-)]$

$\overset{(4),(DN)}{=} x \odot [((x^- \vee_m^M y) \odot ((z \odot x)^- \odot x)) \oplus (z \vee_m^M x^-)]$

$\overset{(Pass),(Pcomm)}{=} x \odot [((x \odot (x^- \vee y)) \odot (z \odot x)^-) \oplus (z \vee_m^M x^-)]$

$\overset{(Pmv)}{=} x \odot [((x \odot y) \odot (z \odot x)^-) \oplus (z \vee_m^M x^-)]$

$\overset{(Pcomm),(Pass)}{=} x \odot [(y \odot ((z \odot x)^- \odot x)) \oplus (z \vee_m^M x^-)]$

$\overset{(4),(DN)}{=} x \odot [(y \odot (z^- \wedge_m^M x)) \oplus (z \vee_m^M x^-)]$

$\overset{(10),(DN)}{=} x \odot [(y \odot (z \vee_m^M x^-)^-) \oplus (z \vee_m^M x^-)]$

$\overset{(5)}{=} x \odot [y \vee_m^M (z \vee_m^M x^-)]$

$\overset{(Pq)}{=} (x \odot y) \vee_m^M (x \odot z)$; thus, (Pqmv) holds. □

Proposition 3.20 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a left-QMV algebra. Then, (Pom) holds.*

Proof. Take $y = 1$ in (Pqmv). □

Proposition 3.21 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a left-QMV algebra verifying (G) ($x \odot x = x$). Then:*
(1) $\leq_m^P$ is reflexive also, hence it is an **order relation**.
(2) We have the equivalence:

$$(x \odot y = x \Longleftrightarrow) \; x \leq_m^P y \Longleftrightarrow x \leq_m^M y \; (\Longleftrightarrow x \wedge_m^M y = x).$$

Proof. (1): $x \leq_m^P x \Longleftrightarrow x \odot x = x$, that is true by (G).

(2): Suppose $x \leq_m^P y$, i.e. $x \odot y = x$. Then, by (31), $x = x \odot y \leq_m^M y$. Conversely, suppose $x \leq_m^M y$. Then, by (40), we have: $x \overset{(G)}{=} x \odot x \leq_m^M y \odot x \overset{(Pcomm)}{=} x \odot y$, and since we also have, by (31), that $x \odot y \leq_m^M x$, we obtain, by antisymmetry of $\leq_m^M$ (by Corollary 3.9), that $x \odot y = x$, i.e. $x \leq_m^P y$. □

Remarks 3.22 *In a left-QMV algebra $\mathcal{A}^L = (A^L, \odot, ^-, 1)$:*
- the initial binary relation, $\leq_m$ ($x \leq_m y \Longleftrightarrow x \odot y^- = 0$) ($\leq_m \Longleftrightarrow \leq_m^B$), is **only reflexive** *((m-Re) holds, by definition of m-BE algebra);*
- the binary relation $\leq_m^M$ ($x \leq_m^M y \Longleftrightarrow x \wedge_m^M y = x$) is an **order**, *by Corollary 3.18, but not a lattice order with respect to $\wedge_m^M$, $\vee_m^M$, since $x \wedge_m^M y \neq y \wedge_m^M x$;*
- the binary relation $\leq_m^P$ ($x \leq_m^P y \Longleftrightarrow x \odot y = x$) is **only antisymmetric and transitive**, *by ([23], Proposition 3.11).*

In a left-QMV algebra verifying (G), $\leq_m^M$ and $\leq_m^P$ are order relations and $\leq_m^M \Longleftrightarrow \leq_m^P$.

3.2 The equivalence between (Pq) and (Pom)

Consider now the properties:
(Pq) $x \odot [y \vee_m^M (z \vee_m^M x^-)] = (x \odot y) \vee_m^M (x \odot z)$ and
(Pom) $(x \odot y) \oplus ((x \odot y)^- \odot x) = x$ or, equivalently, $x \vee_m^M (x \odot y) = x$, which characterizes the orthomodular lattices among ortholattices.

Proposition 3.23 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then,*

$$(Pq) \implies (Pom).$$

Proof. In (Pq) take $y := 1$ to obtain: $x = x \vee_m^M (x \odot z)$, i.e. (Pom). $\Box$

The converse result, the next Proposition 3.25 (saying that (Pom) implies (Pq)), was proved by *Prover9* in about an hour, only after changing the basic Prover9 options *order* from 'lpo' to 'kbo' and *eq-defs* from 'unfold' to 'fold' and after removing those axioms of the algebra containing 0, 1. The proof by *Prover9* had the length 54 (i.e. there were 54 steps); after proving the 54 steps from the chain of length 54, we have gouped the steps into the following Lemma 3.24 and Proposition 3.25.

Lemma 3.24 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-MEL algebra. We have:*

$$x^- \oplus (y \odot x) = y \vee_m^M x^-, \tag{54}$$

$$x \oplus (y \odot (z \odot x^-)) = (y \odot z) \vee_m^M x, \tag{55}$$

$$x^- \oplus ((y \odot x) \vee_m^M z) = y \vee_m^M (x^- \oplus z), \tag{56}$$

$$(x \odot y) \oplus (z \odot (x^- \wedge_m^M y)) = (z \odot y) \vee_m^M (x \odot y), \tag{57}$$

Proof. (54): $y \vee_m^M x^- = x^- \oplus (y \odot x)$, by definition and (DN).

(55): $x \oplus (y \odot (z \odot x^-)) \stackrel{(Pass)}{=} x \oplus ((y \odot z) \odot x^-) = (y \odot z) \vee_m^M x$.

(56): The left side: $x^- \oplus ((y \odot x) \vee_m^M z) \stackrel{(55)}{=} x^- \oplus (z \oplus (y \odot (x \odot z^-))) \stackrel{(Pass)}{=} x^- \oplus [z \oplus ((y \odot x) \odot z^-)]$. The right side: $y \vee_m^M (x^- \oplus z) = (x^- \oplus z) \oplus (y \odot (x^- \oplus z)^-) = (x^- \oplus z) \oplus (y \odot (x \odot z^-)) \stackrel{(Pass),(Sass)}{=} x^- \oplus [z \oplus ((y \odot x) \odot z^-)]$. Hence, (56) holds.

(57): $(x \odot y) \oplus (z \odot (x^- \wedge_m^M y)) = (x \odot y) \oplus (z \odot (y \odot (x \odot y)^-)) \stackrel{(55)}{=} (z \odot y) \vee_m^M (x \odot y)$, for $X := x \odot y$, $Y := z$, $Z := y$ in (55). $\Box$

Proposition 3.25 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then,*

$$(Pom) \implies (Pq).$$

Proof. The proof has 13 steps:

$$x \wedge_m^M (y \oplus x) = x. \tag{58}$$

Indeed, $x^- \vee_m^M (y \oplus x)^- = x^- \vee_m^M (y^- \odot x^-) \stackrel{(Pom)}{=} x^-$; then, $x \wedge_m^M (y \oplus x) = (x^- \vee_m^M (y \oplus x)^-)^- = x^= \stackrel{(DN)}{=} x$.

$$x \vee_m^M (y \wedge_m^M x) = x, \tag{59}$$

$$x \wedge_m^M (y \vee_m^M x) = x. \tag{60}$$

Indeed, $x \vee_m^M (y \wedge_m^M x) = x \vee_m^M (x \odot (y \oplus x^-)) \stackrel{(Pom)}{=} x$; thus, (59) holds. (60) follows by duality.

$$(x \odot y) \wedge_m^M y = x \odot y, \tag{61}$$

$$(x \oplus y) \vee_m^M y = x \oplus y. \tag{62}$$

Indeed, $(x \odot y) \wedge_m^M y \stackrel{(Pom)}{=} (x \odot y) \wedge_m^M (y \vee_m^M (x \odot y)) \stackrel{(60)}{=} x \odot y$, with $X := x \odot y$; thus, (61) holds. (62) follows by duality.

- Now, we prove

$$(x \vee_m^M y) \odot (x^- \oplus y) = y, \tag{63}$$

(64) $$(x \wedge_m^M y) \oplus (x^- \odot y) = y.$$

Indeed, $(x \vee_m^M y) \odot (x^- \oplus y) = ((x^- \oplus y)^- \oplus y) \odot (x^- \oplus y) = y \wedge_m^M (x^- \oplus y) \stackrel{(58)}{=} y$; thus, (63) holds. (64) follows by duality.

(65) $$(x \odot y^-) \oplus (z \oplus (y \wedge_m^M x)) = z \oplus x.$$

Indeed, $(x \odot y^-) \oplus (z \oplus (y \wedge_m^M x)) \stackrel{(Scomm)}{=} (z \oplus (y \wedge_m^M x)) \oplus (x \odot y^-) \stackrel{(Sass)}{=} z \oplus ((y \wedge_m^M x) \oplus (x \odot y^-)) \stackrel{(Pcomm)}{=} z \oplus ((y \wedge_m^M x) \oplus (y^- \odot x)) \stackrel{(64)}{=} z \oplus x$.

(66) $$(x \oplus y) \vee_m^M (x \oplus (z \wedge_m^M y)) = x \oplus y.$$

Indeed, by (65), we have $(y \odot z^-) \oplus (x \oplus (z \wedge_m^M y)) = x \oplus y$; put $X := y \odot z^-$, $Y := x \oplus (z \wedge_m^M y)$; hence, we have $X \oplus Y = x \oplus y$; then, $x \oplus y = X \oplus Y \stackrel{(62)}{=} (X \oplus Y) \vee_m^M Y = (x \oplus y) \vee_m^M (x \oplus (z \wedge_m^M y))$.

(67) $$(x \oplus y) \vee_m^M (x \oplus (z \odot y)) = x \oplus y.$$

Indeed, $(x \oplus y) \vee_m^M (x \oplus (z \odot y)) \stackrel{(61)}{=} (x \oplus y) \vee_m^M (x \oplus ((z \odot y) \wedge_m^M y)) \stackrel{(66)}{=} x \oplus y$, where $Z := z \odot y$ in (66).

(68) $$x \vee_m^M ((y \odot x) \oplus (z \odot (y^- \wedge_m^M x))) = x.$$

Indeed, first, by (64), we have $(y \odot x) \oplus (y^- \wedge_m^M x) = x$; put $X := y \odot x$ and $Y := y^- \wedge_m^M x$, hence we have $X \oplus Y = x$; now, $x = X \oplus Y \stackrel{(67)}{=} (X \oplus Y) \vee_m^M (X \oplus (z \odot Y)) = x \vee_m^M ((y \odot x) \oplus (z \odot (y^- \wedge_m^M x)))$.

• Now, we prove

(69) $$x \vee_m^M ((y \odot x) \vee_m^M (z \odot x)) = x.$$

Indeed, $x \vee_m^M ((y \odot x) \vee_m^M (z \odot x)) \stackrel{(57)}{=} x \vee_m^M [(z \odot x) \oplus (y \odot (z^- \wedge_m^M x))] \stackrel{(68)}{=} x$, with $Y := z$ and $Z := y$.

• Finally, we prove (Pq), i.e. $x \odot [y \vee_m^M (z \vee_m^M x^-)] = (y \odot x) \vee_m^M (z \odot x)$.

Indeed, $x \odot [y \vee_m^M (z \vee_m^M x^-)]$
$\stackrel{(54)}{=} x \odot [y \vee_m^M (x^- \oplus (z \odot x))]$
$\stackrel{(56)}{=} x \odot [x^- \oplus ((y \odot x) \vee_m^M (z \odot x))]$
$\stackrel{(69)}{=} (x \vee_m^M [(y \odot x) \vee_m^M (z \odot x)]) \odot (x^- \oplus [(y \odot x) \vee_m^M (z \odot x)])$
$\stackrel{(63)}{=} (y \odot x) \vee_m^M (z \odot x) \stackrel{(Pcomm)}{=} (x \odot y) \vee_m^M (x \odot z)$. □

By Propositions 3.23 and 3.25, we obtain **the second very important result, the core of this paper**, by its difficulty:

Theorem 3.26 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then,*

$$(Pq) \iff (Pom).$$

Consequently, by Theorems 3.19 and 3.26, we obtain:

Theorem 3.27 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then,*

$$(Pqmv) \iff (Pmv) + (Pom).$$

3.3 The property (Δ_m)

Consider now the property introduced in ([24] 5.2.1) (the dual one is omitted):
(Δ_m) $(x \wedge_m^M y) \odot (y \wedge_m^M x)^- = 0$.

Note that (Δ_m) is the largest non-antisymmetric generalization of ($\wedge_m$-comm). It is equivalent to: $(y \odot (x^- \odot y)^-) \odot (x \odot (y^- \odot x)^-)^- = 0$.

Proposition 3.28 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then,*

$$(Pmv) \implies (\Delta_m).$$

Proof. First, note that, by replacing y with y^-, (Pmv) becomes, by (DN):
(a) $x \odot ((x^- \odot y)^- \odot y)^- = x \odot y^-$.

Now, consider (Δ_m), i.e. $(y \odot (x^- \odot y)^-) \odot (x \odot (y^- \odot x)^-)^- = 0$, and by interchanging x with y, we obtain:
(b) $(x \odot (y^- \odot x)^-) \odot (y \odot (x^- \odot y)^-)^- = 0$. We shall prove (b).
Indeed, $(x \odot (y^- \odot x)^-) \odot (y \odot (x^- \odot y)^-)^-$
$\stackrel{(Pcomm),(Pass)}{=} (x \odot ((x^- \odot y)^- \odot y)^-) \odot (x \odot y^-)^-$
$\stackrel{(a)}{=} (x \odot y^-) \odot (x \odot y^-)^- \stackrel{(m-Re)}{=} 0$. Thus, (Δ_m) holds. □

The converse of Proposition 3.28 does not hold, in general; there are examples of involutive m-BE algebras verifying (Δ_m) and not verifying (Pmv), see Example 6.3.

But, in particular, we have the following Proposition 3.31 (saying that if the involutive m-BE algebra verifies (Pom), then (Δ_m) implies (Pmv)), proved by *Prover9* in 2453 seconds, the length of the proof being 33; the proof by *Prover9* generated the proofs of the following Lemmas 3.29, 3.30 and Proposition 3.31.

Lemma 3.29 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra verifying (Pom). Then,*

$$(x \odot y)^- \odot (x \odot (x \odot y)^-)^- = x^-, \tag{70}$$

$$(x \odot (y \odot z))^- \odot [x \odot (y \odot (x \odot (y \odot z))^-)]^- = (x \odot y)^-, \tag{71}$$

$$(x \odot y^-)^- \odot [x \odot ((y \odot z)^- \odot (x \odot y^-)^-)]^- = (x \odot (y \odot z)^-)^-. \tag{72}$$

Proof. (70): From (Pom), by (Pcomm).

(71: In (70), take $X := x \odot y$ and $Y := z$ to obtain:
$((x \odot y) \odot z)^- \odot ((x \odot y) \odot ((x \odot y) \odot z)^-)^- = (x \odot y)^-$; then, by (Pass), we obtain (71).

(72): In (71), take $X := x$, $Y := (y \odot z)^-$, $Z := (y \odot (y \odot z)^-)^-$ to obtain:
(a) $(X \odot (Y \odot Z))^- \odot [X \odot (Y \odot (X \odot (Y \odot Z))^-)]^- = (X \odot Y)^-$; but,
$X \odot (Y \odot Z) = x \odot ((y \odot z)^- \odot (y \odot (y \odot z)^-)^-) \stackrel{(70)}{=} x \odot y^-$; hence, (a) becomes:
$(x \odot y^-)^- \odot [x \odot ((y \odot z)^- \odot (x \odot y^-)^-)]^- = (x \odot (y \odot z)^-)^-$, that is (72). □

Lemma 3.30 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra verifying (Δ_m). Then,*

$$x \odot ((y^- \odot x)^- \odot (y \odot (x^- \odot y)^-)^-) = 0, \tag{73}$$

$$x \odot ((x \odot y^-)^- \odot (y \odot (y \odot x^-)^-)^-) = 0, \tag{74}$$

$$x^- \odot ((x^- \odot y^-)^- \odot (y \odot (y \odot x)^-)^-) = 0. \tag{75}$$

Proof. (73): Since $x \wedge_m y = y \odot (x^- \odot y)^-$, then (Δ_m) $((x \wedge_m y) \odot (y \wedge_m x)^- = 0)$ becomes:
(a) $(y \odot (x^- \odot y)^-) \odot (x \odot (y^- \odot x)^-)^- = 0$; then, interchanging x with y in (a), we obtain:
(b) $(x \odot (y^- \odot x)^-) \odot (y \odot (x^- \odot y)^-)^- = 0$; then, by (Pass), we obtain (73).

(74): From (73), by (Pcomm).

(75): From (74), by taking $X := x^-$ and by (DN). □

Proposition 3.31 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then,*

$$(Pom) \ + \ (\Delta_m) \implies (Pmv).$$

Proof. First, we prove:

$$(x^- \odot (y \odot (y \odot x)^-)^-)^- = (x^- \odot y^-)^-. \tag{76}$$

Indeed, in (72), take $X := x^-$, $Y := y$ and $Z := (y \odot x)^-$ to obtain:
(a) $(x^- \odot y^-)^- \odot [x^- \odot ((y \odot (y \odot x)^-)^- \odot (x^- \odot y^-)^-)]^- = (x^- \odot (y \odot (y \odot x)^-)^-)^-$;
but, $x^- \odot ((y \odot (y \odot x)^-)^- \odot (x^- \odot y^-)^-) \overset{(75)}{=} 0$; hence, (a) becomes:
$(x^- \odot y^-)^- \odot [0]^- = (x^- \odot (y \odot (y \odot x)^-)^-)^-$, i.e. (76) holds, by (Neg0-1), (PU).
Next, from (76), it follows, by (DN) and (Pcomm):

$$x^- \odot ((x \odot y)^- \odot y)^- = x^- \odot y^-. \tag{77}$$

Finally, from (77), by taking $X := x^-$ and $Y := y^-$, we obtain, by (DN):
$x \odot ((x^- \odot y^-)^- \odot y^-)^- = x \odot y$, that is (Pmv). $\Box$

Resuming, by Propositions 3.28, 3.31, we obtain **the third very important result of this paper**:

Theorem 3.32 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then,*

$$(Pom) \implies ((Pmv) \Leftrightarrow (\Delta_m)).$$

Consequently, by Theorems 3.27 and 3.32, we obtain:

Theorem 3.33 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then,*

$$(Pqmv) \iff (\Delta_m) + (Pom).$$

4 Three generalizations of QMV algebras

Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra throughout this section.

4.1 The three algebras

Consider the properties:
(Pom) $(x \odot y) \oplus ((x \odot y)^- \odot x) = x$ or, equivalently, $x \vee_m^M (x \odot y) = x$ and, dually,
(Som) $(x \oplus y) \odot ((x \oplus y)^- \oplus x) = x$ or, equivalently, $x \wedge_m^M (x \oplus y) = x$;
(Pmv) $x \odot (x^- \vee_m^M y) = x \odot y$ and, dually,
(Smv) $x \oplus (x^- \wedge_m^M y) = x \oplus y$;
(Δ_m) $(x \wedge_m^M y) \odot (y \wedge_m^M x)^- = 0$ and, dually,
(∇_m) $(x \vee_m^M y) \oplus (y \vee_m^M x)^- = 1$.
We introduce the following notions:

Definitions 4.1
(i) An involutive left-m-BE algebra $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ is:
- a *left-orthomodular algebra*, or a *left-OM algebra* for short, if it verifies (Pom),
- a *left-pre-MV algebra*, or a *left-PreMV algebra* for short, if it verifies (Pmv),
- a *left-metha-MV algebra*, or a *left-MMV algebra* for short, if it verifies (Δ_m).
(i') Dually, an involutive right-m-BE algebra $\mathcal{A}^R = (A^R, \oplus, ^-, 0)$ is:
- a *right-orthomodular algebra*, or a *right-OM algebra* for short, if it verifies (Som),
- a *right-pre-MV algebra*, or a *right-PreMV algebra* for short, if it verifies (Smv),
- a *right-metha-MV algebra*, or a *right-MMV algebra* for short, if it verifies (∇_m).

We shall denote by **OM**, **PreMV**, **MMV** the classes of the corresponding left-algebras and by $\mathbf{OM}^R$, $\mathbf{PreMV}^R$, $\mathbf{MMV}^R$ the classes of the corresponding right-algebras. See Examples 6.1, 6.2, 6.3 of left-OM, left-PreMV, left-MMV algebras, respectively, and Example 6.4 of left-QMV algebra.
By Propositions 3.20, 3.13, 3.28 and Theorems 3.27, 3.33, we obtain:

Corollary 4.2 *We have:*
QMV $\subset$ **OM**, **QMV** $\subset$ **PreMV** $\subset$ **MMV**
and
QMV $=$ **PreMV** $\cap$ **OM** $=$ **MMV** $\cap$ **OM**.

Note that we can say that QMV algebras are *orthomodular PreMV algebras*, or *orthomodular MMV algebras*.

Hence, we have the situation from the Figure 2.

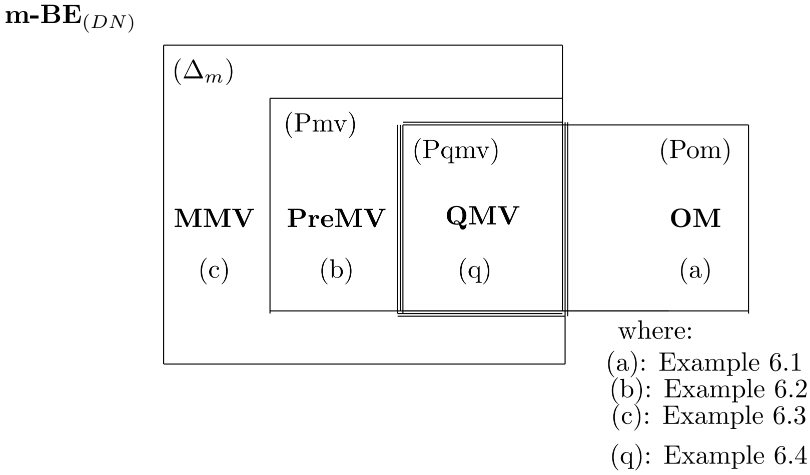


Figure 2: Resuming connections between **OM**, **PreMV**, **MMV** and **QMV**

4.2 The transitive and/or antisymmetric algebras

We shall denote by **tOM**, **tPreMV**, **tMMV**, **tQMV** the classes of the corresponding transitive left-algebras. Note that these classes of algebras are contained in the class **m-pre-BCK**$_{(DN)}$ = **m-tBE**$_{(DN)}$. See Examples 6.5, 6.6, 6.7, 6.8 of left-tOM, left–tPreMV, left-MMV, left-tQMV algebras, respectively.

By the previous Corollary 4.2, we obtain:

Corollary 4.3 *We have:*

$$\begin{array}{ccc} \mathbf{QMV} & \subset & \mathbf{OM} \\ \cup & & \cup \\ \mathbf{tQMV} & \subset & \mathbf{tOM}, \end{array} \qquad \begin{array}{ccccc} \mathbf{QMV} & \subset & \mathbf{PreMV} & \subset & \mathbf{MMV} \\ \cup & & \cup & & \cup \\ \mathbf{tQMV} & \subset & \mathbf{tPreMV} & \subset & \mathbf{tMMV} \end{array}$$

and

tQMV $=$ **tPreMV** $\cap$ **tOM** $=$ **tMMV** $\cap$ **tOM**.

Hence, we have the situation from the Figure 3.

We shall denote by **aOM**, **aPreMV**, **aMMV**, **aQMV** the classes of the corresponding antisymmetric left-algebras. Note that these classes of algebras are contained in the class **m-aBE**$_{(DN)}$.

By Corollary 4.2 again, we obtain the analogous of Corollary 4.3, which by lack of space is omitted.

We shall denote by **taOM**, **taPreMV**, **taMMV**, **taQMV** the classes of the corresponding transitive and antisymmetric left-algebras. Note that these classes of algebras are contained in the class **m-BCK** = **m-taBE**$_{(DN)}$.

By Corollary 4.3 and its analogous, we then obtain:

Corollary 4.4 *We have:*

$$\begin{array}{ccc} \mathbf{QMV} & \subset & \mathbf{OM} \\ \cup & & \cup \\ \mathbf{tQMV} & \subset & \mathbf{tOM} \\ \cup & & \cup \\ \mathbf{taQMV} & \subset & \mathbf{taOM}, \end{array} \qquad \begin{array}{ccccc} \mathbf{QMV} & \subset & \mathbf{PreMV} & \subset & \mathbf{MMV} \\ \cup & & \cup & & \cup \\ \mathbf{tQMV} & \subset & \mathbf{tPreMV} & \subset & \mathbf{tMMV} \\ \cup & & \cup & & \cup \\ \mathbf{taQMV} & \subseteq & \mathbf{taPreMV} & \subseteq & \mathbf{taMMV} \end{array}$$

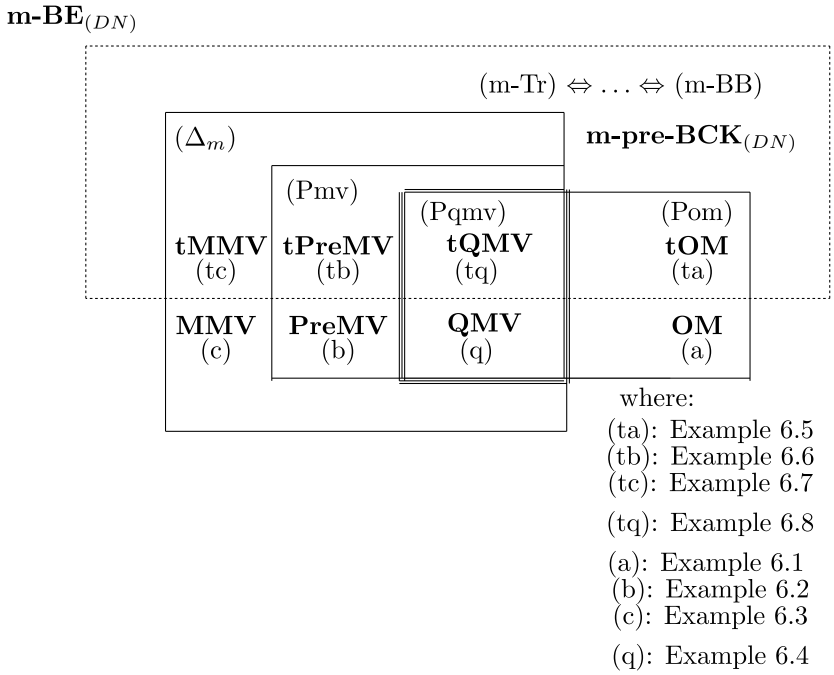


Figure 3: Resuming connections between **OM**, **PreMV**, **MMV**, **QMV** and (m-Tr)

and

$$\begin{array}{ccc} \mathbf{QMV} & \subset & \mathbf{OM} \\ \cup & & \cup \\ \mathbf{aQMV} & \subset & \mathbf{aOM} \\ \cup & & \cup \\ \mathbf{taQMV} & \subset & \mathbf{taOM}, \end{array} \qquad \begin{array}{ccccc} \mathbf{QMV} & \subset & \mathbf{PreMV} & \subset & \mathbf{MMV} \\ \cup & & \cup & & \cup \\ \mathbf{aQMV} & \subseteq & \mathbf{aPreMV} & \subseteq & \mathbf{aMMV} \\ \cup & & \cup & & \cup \\ \mathbf{taQMV} & \subseteq & \mathbf{taPreMV} & \subseteq & \mathbf{taMMV} \end{array}$$

and

$\mathbf{aQMV} = \mathbf{aPreMV} \cap \mathbf{aOM} = \mathbf{aMMV} \cap \mathbf{aOM}$,
$\mathbf{taQMV} = \mathbf{taPreMV} \cap \mathbf{taOM} = \mathbf{taMMV} \cap \mathbf{taOM}$.

4.3 The connections with the MV algebras

We know (see ([6], Example 2.3.14)) that any MV algebra is a QMV algebra: $\mathbf{MV} \subset \mathbf{QMV}$, since:

(78) $$(\wedge_m - comm) \implies (Pqmv).$$

Consequently, we have:

(79) $$(\wedge_m - comm) \implies (Pom) + (Pmv) + (\Delta_m).$$

The next Theorems 4.6, 4.7 and 4.9 say that ($\wedge_m$-comm) is equivalent with some properties.

Proposition 4.5 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then,*

$$(Pmv) + (m - An) \implies (\wedge_m - comm).$$

Proof. Suppose (m-An) holds, i.e. $X \leq_m Y$ and $Y \leq_m X$ imply $X = Y$, which mean $X \odot Y^- = 0$ and $Y \odot X^- = 0$ imply $X = Y$.

Take $X \overset{notation}{=} x \wedge_m^M y \overset{(4)}{=} (x^- \odot y)^- \odot y$ and $Y \overset{notation}{=} y \wedge_m^M x \overset{(4)}{=} (y^- \odot x)^- \odot x$.

We have: $X \odot Y^- = [(x^- \odot y)^- \odot y] \odot [y \wedge_m^M x]^-$
$\overset{(11)}{=} [(x^- \odot y)^- \odot y] \odot [y^- \vee_m^M x^-] \overset{(Pass)}{=} (x^- \odot y)^- \odot (y \odot (y^- \vee_m^M x^-))$
$\overset{(Pmv)}{=} (x^- \odot y)^- \odot (y \odot x^-) \overset{(Pcomm)}{=} (y \odot x^-) \odot (y \odot x^-)^- \overset{(m-Re)}{=} 0.$

Similarly, we have: $Y \odot X^- = (y^- \odot x)^- \odot x \odot [x \wedge_m^M y]^-$
$(y^- \odot x)^- \odot x \odot (x^- \vee_m^M y^-) \overset{(Pmv)}{=} (y^- \odot x)^- \odot (x \odot y^-) = 0.$

By (m-An), we obtain $X = Y$, i.e. ($\wedge_m$-comm) holds. □

By Proposition 4.5 and (3), (79), we obtain:

Theorem 4.6

$$(Pmv) \ + \ (m-An) \iff (\wedge_m - comm).$$

Recall again ([24] 5.2.1) saying that:

Theorem 4.7

$$(\Delta_m) \ + \ (m-An) \iff (\wedge_m - comm).$$

Proposition 4.8

$$(Pqmv) \ + \ (m-An) \implies (\wedge_m - comm).$$

Proof. By Propositions 3.13, 4.5, we obtain:
(Pqmv) + (m-An) $\Longrightarrow$ (Pmv) + (m-An) $\Longrightarrow$ ($\wedge_m$-comm). □

By Proposition 4.8 and by (3), (78), we obtain:

Theorem 4.9

$$(Pqmv) \ + \ (m-An) \iff (\wedge_m - comm).$$

By previous Theorems 4.6, 4.7, 4.9, we obtain **the fourth very important result of this paper**:

Corollary 4.10 *We have:*

$$\begin{array}{lllllll}
\mathbf{PreMV} & + & \textit{(m-An)} & = & \mathbf{MV}, & \textit{i.e.} & \mathbf{aPreMV} = \mathbf{MV}, \\
\mathbf{MMV} & + & \textit{(m-An)} & = & \mathbf{MV}, & \textit{i.e.} & \mathbf{aMMV} = \mathbf{MV}, \\
\mathbf{QMV} & + & \textit{(m-An)} & = & \mathbf{MV}, & \textit{i.e.} & \mathbf{aQMV} = \mathbf{MV}.
\end{array}$$

Remark 4.11 *By (3), we have:*
MV = **aMV** = **tMV** = **taMV**, *hence* **taPreMV** = **taMMV** = **taQMV** = **MV**.

By Corollaries 4.4, 4.10 and by Remark 4.11, we obtain:

Corollary 4.12 *We have:*

$$\begin{array}{ccccccccccc}
\mathbf{QMV} & \subset & \mathbf{OM} & & \mathbf{QMV} & \subset & \mathbf{PreMV} & \subset & \mathbf{MMV} \\
\cup & & \cup & & \cup & & \cup & & \cup \\
\mathbf{tQMV} & \subset & \mathbf{tOM} & & \mathbf{tQMV} & \subset & \mathbf{tPreMV} & \subset & \mathbf{tMMV} \\
\cup & & \cup & & \cup & & \cup & & \cup \\
\mathbf{MV} & \subset & \mathbf{taOM}, & & \mathbf{MV} & \subseteq & \mathbf{MV} & \subseteq & \mathbf{MV}
\end{array}$$

and

$$\begin{array}{ccccccccccc}
\mathbf{QMV} & \subset & \mathbf{OM} & & \mathbf{QMV} & \subset & \mathbf{PreMV} & \subset & \mathbf{MMV} \\
\cup & & \cup & & \cup & & \cup & & \cup \\
\mathbf{MV} & \subset & \mathbf{aOM} & & \mathbf{MV} & \subseteq & \mathbf{MV} & \subseteq & \mathbf{MV} \\
\cup & & \cup & & & & & & \\
\mathbf{MV} & \subset & \mathbf{taOM} & & & & & &
\end{array}$$

and

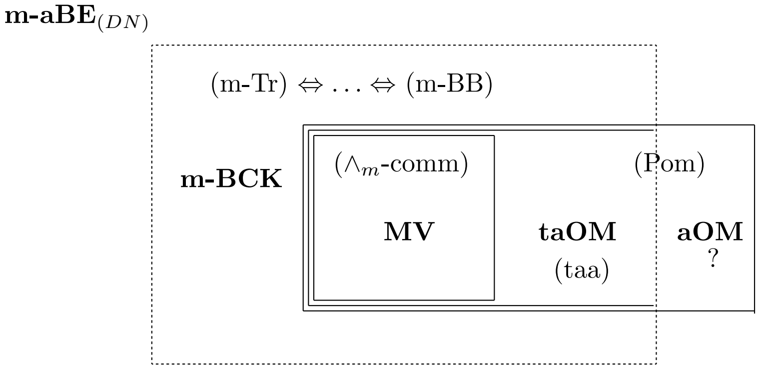
MV = **MV** $\cap$ **aOM**,
MV = **MV** $\cap$ **taOM**.

Note that *taOM algebras* are proper generalizations of MV algebras inside the class of m-BCK algebras. See Example 6.9 of left-taOM algebra.

A problem we have not been able to resolve is the following.

Open problem 4.13 Find an example of antisymmetric orthomodular algebra (aOM) which does not verify (m-Tr) ($\Longleftrightarrow \ldots$ (m-BB)), i.e. a proper element of **aOM** (using MACE4, we have searched exhaustively for an example up through and including size 20), or prove that an involutive left-m-aBE algebra satisfying (Pom) satisfies also (m-Tr) (i.e. **aOM** = **taOM**) (we have also tried to find a proof using PROVER9, but despite letting it run for several days, it was unable to find one).

Hence, we have the situation from the Figure 4.



where:
(taa): Example 6.9

Figure 4: Resuming connections between **MV**, **taOM** and **aOM**, where ? means that there is an open problem concerning **aOM**

Note that, by Theorems 4.6, 4.7, 4.9 again, we obtain:

Theorem 4.14 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-aBE algebra. Then,*

$$(\wedge_m - comm) \Longleftrightarrow (Pmv) \Longleftrightarrow (\Delta_m) \Longleftrightarrow (Pqmv).$$

Remarks 4.15 *(See Remarks 3.22) In a left-MV algebra $\mathcal{A}^L = (A^L, \odot, ^-, 1)$:*
- the initial binary relation, $\leq_m$ ($\Longleftrightarrow \leq^B_m$), is a **lattice order relation** *w.r. to $\wedge^B_m = \wedge^M_m, \vee^B_m = \vee^M_m$, since (m-Re), (m-An), (m-Tr) hold and since $\wedge^B_m$ is commutative,*
- the binary relation $\leq^M_m$ is a **lattice order relation** *w.r. to $\wedge^M_m, \vee^M_m$, by Corollary 3.18 and since $\wedge^M_m$ is commutative,*
- $\leq_m$ and $\leq^M_m$ are equivalent: $\leq_m$ ($\Longleftrightarrow \leq^B_m$) $\Longleftrightarrow \leq^M_m$, by Proposition 3.3; the lattice is distributive;
- the binary relation $\leq^P_m$ is **only antisymmetric and transitive**, *by ([23], Proposition 3.11).*

In a left-MV algebra verifying (G) ($x \odot x = x$), i.e. in a left-Boolean algebra,

$$\leq_m \ (\Longleftrightarrow \leq^B_m) \Longleftrightarrow \leq^M_m \Longleftrightarrow \leq^P_m .$$

4.4 Putting QMV and tQMV algebras on the "map"

By the previous results, we are now able to put QMV algebras and tQMV algebras (and MV algebras) on the involutive "Big map" (and, hence, on the "map") - see the Figure 5.

5 Concluding remarks and future work

In this paper, we have dug arround the structure of QMV algebras and we have obtained a decomposition of (Pqmv) into only two properties: (Pmv) and (Pq), at the begining, (Δ_m) and (Pom), at the end, where (Δ_m) is the largest non-antisymmetric generalization of ($\wedge_m$-comm) and (Pom) is the property

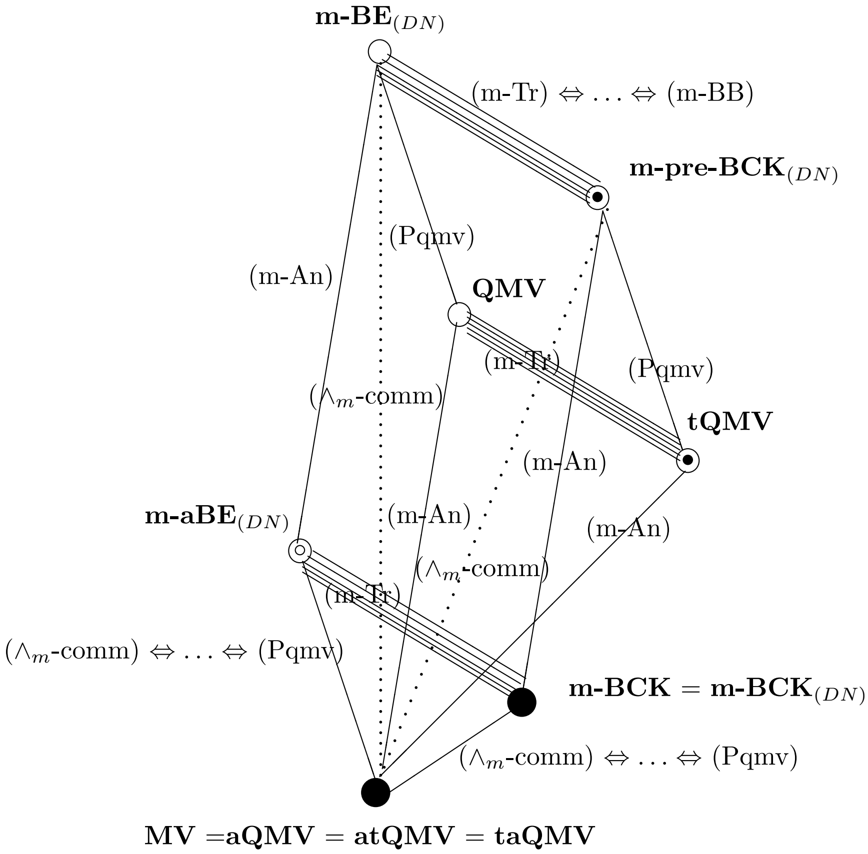


Figure 5: Putting QMV and tQMV algebras on the "map"

characterizing the orthomodular lattices among the ortholattices (Definitions 2). We have thus introduced three generalizations of the QMV algebras: two new non-antisymmetric generalizations of MV algebras, the *pre-MV (PreMV) algebras* and the *metha-MV (MMV) algebras*, and the *orthomodular (OM) algebras*. The QMV algebra is then just an *orthomodular PreMV algebra* or an *orthomodular MMV algebra*; in other words, the QMV algebra is that non-antisymmetric generalization of MV algebra that is an orthomodular algebra. We have also introduced and studied the *transitive QMV (tQMV) algebras*, the *transitive PreMV (tPreMV) algebras*, the *transitive MMV (tMMV) algebras* and the *transitive OM (tOM) algebras*. It was known that any MV algebra is a QMV algebra, but the exact connection between MV and QMV algebras it was not known. We have clarified this problem, by proving that MV algebras coincide with the antisymmetric QMV (aQMV) algebras - but also with the antisymmetric preMV (aPreMV) and with the antisymmetric MMV (aMMV) algebras. Consequently, MV algebras and QMV algebras, and also tQMV algebras, were put on the same "map" (involutive "Big map"). The *taOM algebra*, a proper generalization of MV algebra inside the class of m-BCK algebras, is put in evidence.

By putting QMV (and tQMV) algebras on the "map", we have proved again (see [19], [23]) the deep connections existing between the algebraic structures connected to the classical and non-classical logics and the algebraic structures connected to the quantum logics: they exist on the same "map", but at different levels (parallels), i.e. the QMV (and tQMV) algebras also belong to the "world" of left-algebras (involutive left-unital magmas).

The 'story' of the algebras involved in this paper is connected to the 'story' of the three/four binary relations that can be defined in such algebras:

$x \leq_m y \overset{def.}{\Longleftrightarrow} x \odot y^- = 0$, with $x \leq_m y \iff x \leq_m^B y$, $\quad x \leq_m^B y \overset{def}{\Longleftrightarrow} x \wedge_m^B y = x$,

$x \leq_m^M y \overset{def}{\Longleftrightarrow} x \wedge_m^M y = x$, with $x \wedge_m^M y = y \wedge_m^B x$, and

$x \leq_m^P y \overset{def}{\Longleftrightarrow} x \odot y = x$.

Note that the central role is played by the binary relation $\leq_m$, that determines the "parallels" and the "meridians" of the "map".

By the inverse maps Φ ($x \odot y \overset{def.}{=} (x \to y^-)^-$) and Ψ ($x \to y \overset{def.}{=} (x \odot y^-)^-$) ([19], Theorem 9.1) that connect the "world" of *algebras of logic* of the form $(A, \to, ^-, 1)$ and the "world" of *algebras* of the form $(A, \odot, ^{-}, 1)$, in the involutive case, one can obtain simply, by choosing the appropriate definitions of the algebras, the definitionally equivalent *involutive algebras of logic* corresponding to the *involutive algebras* from this paper and the corresponding examples and results. Note that, in ([19], Definition 3.29), the *implicative-ortholattices* were already introduced as involutive *BE algebras* verifying (impl) ($(x \to y) \to x = x$) and their d.e. with the ortholattices was proved in Theorem 9.2. Similarly, one can introduce now the *quantum-Wajsberg algebras*, as algebras of logic (involutive *BE algebras* verifying, say, (qw)) d. e. with the quantum-MV algebras, etc. Note that the *BE algebras* were introduced in 2006 by H.S. Kim and Y.H. Kim [25] and are intensively studied.

This research is continued by the first author: in [20], we clarify some more aspects concerning the QMV algebras as non-lattice generalizations of MV algebras by studying more deeply the OM algebras; we prove that all the properties of QMV algebras, excepting (43) and (44), are verified by the OM algebras; we study in some details the taOM algebras. In [21], we clarify some aspects concerning the QMV algebras as non-idempotent generalizations of orthomodular lattices; we introduce and study two generalizations of orthomodular lattices, the orthomodular softlattices and the orthomodular widelattices, following [24]. Finally, in [22], we study the properties (m-Pabs-i) and (WNM_m), introduced in [24], in MV algebras and in tQMV algebras.

6 Examples

We introduce the following definition: an X algebra is said to be *proper*, if it verifies the properties from its definition and does not verify the other properties from this paper, except (prel_m).

Example 6.1 Proper orthomodular algebra: OM

By a PASCAL program, we found that the algebra $\mathcal{A}^L = (A_6 = \{0, a, b, c, d, 1\}, \odot, ^{-}, 1)$, with the following tables of $\odot$ and $^{-}$ and of the additional operation $\oplus$, is an involutive left-m-BE algebra verifying (Pom) and (prel_m) and not verifying (m-B) for (a, b, a), (m-BB) for (a, a, b), (m-*) for (b, d, a), (m-**) for (a, b, a), (m-Tr) for (a, b, d), (m-An) for (a, b), (Pqmv) for $(d, d, 0)$, (Pmv) for (d, d), (Δ_m) for (a, d).

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	0	0	a
b	0	0	0	0	0	b
c	0	0	0	0	0	c
d	0	0	0	0	0	d
1	0	a	b	c	d	1

and

x	x^{-}
0	1
a	d
b	c
c	b
d	a
1	0

, with

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	1	1	1	1	1
b	b	1	1	1	1	1
c	c	1	1	1	1	1
d	d	1	1	1	d	1
1	1	1	1	1	1	1

.

Note that $\leq_m^M$ is transitive, hence $\leq_m^M$ is an order relation, by Corollary 3.9, but not a lattice order w.r. to $\wedge_m^M$, $\vee_m^M$, since $\wedge_m^M$ is not commutative.

Example 6.2 Proper PreMV algebra: PreMV

By a PASCAL program, we found that the algebra $\mathcal{A}^L = (A_6 = \{0, a, b, c, d, 1\}, \odot, ^{-}, 1)$, with the following tables of $\odot$ and $^{-}$ and of the additional operation $\oplus$, is an involutive left-m-BE algebra verifying (Pmv) (hence (Δ_m)) and (prel_m) and not verifying (m-B) for (a, d, c), (m-BB) for (a, c, a), (m-*) for (a, d, c), (m-**) for (a, d, c), (m-Tr) for (a, d, b), (m-An) for (a, c), (Pqmv) for $(a, 1, c)$, (Pom) for (a, c).

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	b	0	a
b	0	0	0	0	0	b
c	0	b	0	b	0	c
d	0	0	0	0	0	d
1	0	a	b	c	d	1

and

x	x^{-}
0	1
a	d
b	c
c	b
d	a
1	0

, with

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	1	1	1	1	1
b	b	1	c	1	c	1
c	c	1	1	1	1	1
d	d	1	c	1	1	1
1	1	1	1	1	1	1

.

Note that $\leq_m^M$ is transitive, hence $\leq_m^M$ is an order relation, by Corollary 3.9, but not a lattice order w.r. to $\wedge_m^M$, $\vee_m^M$, since $\wedge_m^M$ is not commutative.

Example 6.3 Proper MMV algebra: MMV

By a PASCAL program, we found that the algebra $\mathcal{A}^L = (A_6 = \{0, a, b, c, d, 1\}, \odot, ^-, 1)$, with the following tables of $\odot$ and $^-$ and of the additional operation $\oplus$, is an involutive left-m-BE algebra verifying (Δ_m) and not verifying (m-B) for (a, b, a), (m-BB) for (a, a, b), (m-*) for (a, b, c), (m-**) for (a, b, a), (m-Tr) for (a, b, d), (m-An) for (a, b), (Pqmv) for $(d, 0, d)$, (Pom) for (d, d), (Pmv) for (d, d), (prel$_m$) for (a, d).

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	0	0	a
b	0	0	0	0	0	b
c	0	0	0	b	0	c
d	0	0	0	0	b	d
1	0	a	b	c	d	1

and

x	x^-
0	1
a	d
b	c
c	b
d	a
1	0

, with

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	c	1	1	1	1
b	b	1	c	1	1	1
c	c	1	1	1	1	1
d	d	1	1	1	d	1
1	1	1	1	1	1	1

.

Note that $\leq_m^M$ is transitive, hence $\leq_m^M$ is an order relation, by Corollary 3.9, but not a lattice order w.r. to $\wedge_m^M$, $\vee_m^M$, since $\wedge_m^M$ is not commutative.

Example 6.4 Proper QMV algebra: QMV

By a PASCAL program, we found that the algebra $\mathcal{A}^L = (A_6 = \{0, a, b, c, d, 1\}, \odot, ^-, 1)$, with the following tables of $\odot$ and $^-$ and of the additional operation $\oplus$, is a proper left-QMV algebra, i.e. (PU), (Pcomm), (Pass), (m-L), (m-Re), (Pqmv) (hence (Pom), (Pmv), (Δ_m), (prel$_m$)), (DN) hold and it does not verify (m-B) for (a, d, c), (m-BB) for (a, c, a), (m-*) for (a, d, c), (m-**) for (a, d, c), (m-Tr) for (a, d, b), (m-An) for (a, c), ($\wedge_m$-comm) for (a, c).

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	b	0	a
b	0	0	0	0	0	b
c	0	b	0	d	0	c
d	0	0	0	0	0	d
1	0	a	b	c	d	1

and

x	x^-
0	1
a	d
b	c
c	b
d	a
1	0

, with

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	1	1	1	1	1
b	b	1	a	1	c	1
c	c	1	1	1	1	1
d	d	1	c	1	1	1
1	1	1	1	1	1	1

.

Note that $\leq_m^M$ is an order relation, by Corollary 3.18, but not a lattice order w.r. to $\wedge_m^M$, $\vee_m^M$, since $\wedge_m^M$ is not commutative.

Example 6.5 Proper transitive OM algebra : tOM

By MACE4 program, we found that the algebra $\mathcal{A}^L = (A_8 = \{0, a, b, c, d, e, f, 1\}, \odot, ^-, 1)$, with the following tables of $\odot$ and $^-$ and of the additional operation $\oplus$, is an involutive left-m-BE algebra verifying (Pom) and (m-Tr) $\Longleftrightarrow \ldots \Longleftrightarrow$ (m-BB), and also (prel$_m$), and not verifying (m-An) for (c, d), (Pqmv) for $(b, b, 0)$, (Pmv) for (b, b), (Δ_m) for (a, b).

$\odot$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	a	0	c	d	c	d	a
b	0	0	0	0	0	0	0	b
c	0	c	0	0	0	0	0	c
d	0	d	0	0	0	0	0	d
e	0	c	0	0	0	b	b	e
f	0	d	0	0	0	b	b	f
1	0	a	b	c	d	e	f	1

and

x	x^-
0	1
a	b
b	a
c	e
d	f
e	c
f	d
1	0

, with

$\oplus$	0	a	b	c	d	e	f	1
0	0	a	b	c	d	e	f	1
a	a	1	1	1	1	1	1	1
b	b	1	b	e	f	e	f	1
c	c	1	e	a	a	1	1	1
d	d	1	f	a	a	1	1	1
e	e	1	e	1	1	1	1	1
f	f	1	f	1	1	1	1	1
1	1	1	1	1	1	1	1	1

.

Note that $\leq_m^M$ is transitive, hence $\leq_m^M$ is an order relation, by Corollary 3.9, but not a lattice order w.r. to $\wedge_m^M$, $\vee_m^M$, since $\wedge_m^M$ is not commutative.

Example 6.6 Proper transitive pre-MV algebra: tPreMV

By a PASCAL program, we found that the algebra $\mathcal{A}^L = (A_6 = \{0, a, b, c, d, 1\}, \odot, ^-, 1)$, with the following tables of $\odot$ and $^-$ and of the additional operation $\oplus$, is an involutive left-m-BE algebra verifying (Pmv) (hence (Δ_m)) and (m-Tr) $\Longleftrightarrow \ldots \Longleftrightarrow$ (m-BB), (prel$_m$) and not verifying (m-An) for (b, c), (Pqmv) for $(b, 1, d)$, (Pom) for (b, d).

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	0	0	a	b
c	0	0	0	0	a	c
d	0	0	a	a	b	d
1	0	a	b	c	d	1

and

x	x^-
0	1
a	d
b	c
c	b
d	a
1	0

, with

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	c	d	d	1	1
b	b	d	1	1	1	1
c	c	d	1	1	1	1
d	d	1	1	1	1	1
1	1	1	1	1	1	1

.

Note that $\leq_m^M$ is transitive, hence $\leq_m^M$ is an order relation, by Corollary 3.9, but not a lattice order w.r. to $\wedge_m^M$, $\vee_m^M$, since $\wedge_m^M$ is not commutative.

Example 6.7 Proper transitive MMV algebra : tMMV

By a PASCAL program, we found that the algebra $\mathcal{A}^L = (A_6 = \{0, a, b, c, d, 1\}, \odot, ^-, 1)$, with the following tables of $\odot$ and $^-$ and of the additional operation $\oplus$, is an involutive left-m-BE algebra verifying (Δ_m) and (m-Tr) $\Longleftrightarrow \ldots \Longleftrightarrow$ (m-BB), and also (prel$_m$), and not verifying (m-An) for (a, b), (Pqmv) for $(b, 0, a)$, (Pom) for (b, a), (Pmv) for (b, a).

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	a	0	0	a
b	0	a	a	0	0	b
c	0	0	0	c	c	c
d	0	0	0	c	d	d
1	0	a	b	c	d	1

and

x	x^-
0	1
a	d
b	c
c	b
d	a
1	0

, with

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	a	b	1	1	1
b	b	b	b	1	1	1
c	c	1	1	d	d	1
d	d	1	1	d	d	1
1	1	1	1	1	1	1

.

Note that $\leq_m^M$ is transitive, hence $\leq_m^M$ is an order relation, by Corollary 3.9, but not a lattice order w.r. to $\wedge_m^M$, $\vee_m^M$, since $\wedge_m^M$ is not commutative.

Example 6.8 Proper transitive QMV algebra: tQMV

By a PASCAL program, we found that the algebra $\mathcal{A}^L = (A_6 = \{0, a, b, c, d, 1\}, \odot, ^-, 1)$, with the following tables of $\odot$ and $^-$ and of the additional operation $\oplus$, is a proper left-tQMV algebra, i.e. (PU), (Pcomm), (Pass), (m-L), (m-Re), (Pqmv) (hence (Pom), (Pmv), (Δ_m), (prel$_m$)), (DN), (m-Tr) $\Longleftrightarrow$ $\ldots \Longleftrightarrow$ (m-BB) hold and it does not verify (m-An) for (a, b).

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	0	0	0	b
c	0	0	0	a	b	c
d	0	0	0	b	a	d
1	0	a	b	c	d	1

and

x	x^-
0	1
a	d
b	c
c	b
d	a
1	0

, with

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	d	c	1	1	1
b	b	c	d	1	1	1
c	c	1	1	1	1	1
d	d	1	1	1	1	1
1	1	1	1	1	1	1

.

Note that $\leq_m^M$ is an order relation, by Corollary 3.18, but not a lattice order w.r. to $\wedge_m^M$, $\vee_m^M$, since $\wedge_m^M$ is not commutative.

Example 6.9 Transitive, antisymmetric OM algebra: taOM

By a PASCAL program, we found that the algebra $\mathcal{A}^L = (A_4 = \{0, a, b, 1\}, \odot, ^-, 1)$, with the following tables of $\odot$ and $^-$ and of the additional operation $\oplus$, is a transitive, antisymmetric left-orthomodular algebra (= m-BCK algebra verifying (Pom)), i.e. (PU), (Pcomm), (Pass), (m-L), (m-Re), (m-An), (DN), (m-Tr) $\Longleftrightarrow \ldots \Longleftrightarrow$ (m-BB), (Pom), but also (prel$_m$) hold and it does not verify ($\wedge_m$-comm) for (a, b), (Pqmv) for $(a, a, 0)$, (Pmv) for (a, a), (Δ_m) for (b, a).

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	b	b
1	0	a	b	1

and

x	x^-
0	1
a	b
b	a
1	0

, with

$\oplus$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	1	1
1	1	1	1	1

.

Note that $\leq_m^M$ is transitive, hence $\leq_m^M$ is an order relation, by Corollary 3.9, but not a lattice order w.r. to $\wedge_m^M$, $\vee_m^M$, since $\wedge_m^M$ is not commutative.

Note that this algebra is the NM (Nilpotent Minimum) algebra $\mathcal{F}_4$ from [15]; it will be reviewed in [20].

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Communicated by *Jair Minoro Abe*