FUZZY SCHWARZ INEQUALITY

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Abstract. In the present paper, the fuzzy Schwarz inequality in inner product spaces is derived. It is an extension of the Schwarz inequality, and is described by using a fuzzy norm and a fuzzy inner product defined by Zadeh's extension principle. The fuzzy norm of a fuzzy set is the image of the fuzzy set under the crisp norm, and it is also a fuzzy set. The fuzzy inner product between two fuzzy sets is the image of the two fuzzy sets under the crisp inner product, and it is also a fuzzy set. The Schwarz inequality evaluates the inner product between two vectors in an inner product space by norms of the two vectors. On the other hand, the fuzzy Schwarz inequality evaluates the fuzzy inner product between two fuzzy sets on an inner product space by fuzzy norms of the two fuzzy sets.

1 Introduction The concept of fuzzy sets has been primarily introduced for representing sets containing uncertainty or vagueness by Zadeh [18]. Then, fuzzy set theory has been applied in various areas such as economics, management science, engineering, optimization theory, operations research, etc. [6, 10, 14, 15, 16, 17]. Zadeh's extension principle [4, 18] provides a natural way for extending the domain of a mapping. It is an important tool in the development of fuzzy arithmetic and other areas. Let $f : X \times Y \to Z$ be a mapping, and let $\widetilde{a}$ and $\widetilde{b}$ be fuzzy sets on X and Y, respectively. In addition, let $f(\widetilde{a}, \widetilde{b})$ be the fuzzy set on Z obtained from $\widetilde{a}$ and $\widetilde{b}$ by Zadeh's extension principle. In [12], relationships between $f([\widetilde{a}]_\alpha, [\widetilde{b}]_\alpha)$ and $[f(\widetilde{a}, \widetilde{b})]_\alpha$ are investigated, where $[\widetilde{a}]_\alpha$, $[\widetilde{b}]_\alpha$, and $[f(\widetilde{a}, \widetilde{b})]_\alpha$ are the α-level sets of $\widetilde{a}$, $\widetilde{b}$, and $f(\widetilde{a}, \widetilde{b})$, respectively. A fuzzy norm and a fuzzy inner product defined by Zadeh's extension principle are proposed, and their properties are investigated in [9] and [8], respectively. We adopt them. The fuzzy norm of a fuzzy set is the image of the fuzzy set under the crisp norm, and it is also a fuzzy set. The fuzzy inner product between two fuzzy sets is the image of the two fuzzy sets under the crisp inner product, and it is also a fuzzy set.

Fuzzy normed spaces and fuzzy inner product spaces have been discussed in several papers; see, for example [13] and references therein. Fuzzy norms and fuzzy inner products in most of papers are based on axioms rather than Zadeh's extension principle, and their values are fuzzy sets for norms and inner products of crisp vectors rather than of fuzzy sets. The Schwarz inequality evaluates the inner product between two vectors in an inner product space by norms of the two vectors, and it has a long history; see, for example [3]. We consider the Schwarz inequality in fuzzy settings by using our adopted fuzzy norm and fuzzy inner product. The Schwarz inequality is derived for fuzzy matrices by using a fuzzy norm and a fuzzy inner product based on axioms rather than Zadeh's extension principle in [5], and the Schwarz inequality is derived for fuzzy integrals in [2]. Our settings such as the fuzzy norm and the fuzzy inner product are different from the previous works on the Schwarz inequality in fuzzy settings.

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In the present paper, the fuzzy Schwarz inequality in inner product spaces is derived. It is an extension of the Schwarz inequality, and is described by using the fuzzy norm and the fuzzy inner product. The fuzzy Schwarz inequality evaluates the fuzzy inner product between two fuzzy sets on an inner product space by fuzzy norms of the two fuzzy sets.

The remainder of the present paper is organized as follows. In Section 2, some notations are presented. In Section 3, we investigate relationships between level sets of fuzzy sets and level sets of another fuzzy set obtained by Zadeh's extension principle, and the fuzzy norm and the fuzzy inner product defined by Zadeh's extension principle are presented. In Section 4, the fuzzy Schwarz inequality is derived by using the fuzzy norm and the fuzzy inner product as an extension of the Schwarz inequality in inner product spaces. Finally, conclusions are presented in Section 5.

2 Preliminaries In this section, some notations are presented.

Let $\mathbb{R}$ and $\mathbb{C}$ be the set of all real numbers and the set of all complex numbers, respectively. We set $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ and $\mathbb{R}_- = \{x \in \mathbb{R} : x \leq 0\}$. For $A \subset \mathbb{R}$, we denote the interior of A by $\mathrm{int}(A)$. For $a, b \in \mathbb{R}$, we set $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$, $[a, b[= \{x \in \mathbb{R} : a \leq x < b\}$, $]a, b] = \{x \in \mathbb{R} : a < x \leq b\}$, and $]a, b[= \{x \in \mathbb{R} : a < x < b\}$.

Let X be a set. Then, $\widetilde{a} : X \to [0, 1]$ is called a fuzzy set on X, and let $\mathcal{F}(X)$ be the set of all fuzzy sets on X. For $\widetilde{a} \in \mathcal{F}(X)$ and $\alpha \in]0, 1]$, the *α-level set* of $\widetilde{a}$ is defined as

$$[\widetilde{a}]_\alpha = \{x \in X : \widetilde{a}(x) \geq \alpha\}. \tag{1}$$

For a crisp set $S \subset X$, the indicator function of S is a function $c_S : X \to \{0, 1\}$ defined as $c_S(x) = 1$ if $x \in S$, and $c_S(x) = 0$ if $x \notin S$ for each $x \in X$. A fuzzy set $\widetilde{a} \in \mathcal{F}(X)$ can be represented as

$$\widetilde{a} = \sup_{\alpha \in]0,1]} \alpha c_{[\widetilde{a}]_\alpha} \tag{2}$$

which is well-known as the *decomposition theorem* or the *representation theorem*; see, for example [4].

We consider fuzzy sets on a topological space. Let $(X, \mathbb{T})$ be a topological space. Let $\mathcal{C}(X)$ and $\mathcal{K}(X)$ be the set of all closed subsets of X and the set of all compact subsets of X, respectively. Let $\widetilde{a} \in \mathcal{F}(X)$. The fuzzy set $\widetilde{a}$ is called a *closed fuzzy set* (on X) if $[\widetilde{a}]_\alpha \in \mathcal{C}(X)$ for any $\alpha \in]0, 1]$. The fuzzy set $\widetilde{a}$ is a closed fuzzy set on X if and only if $\widetilde{a}$ is an upper semicontinuous function on X. The fuzzy set $\widetilde{a}$ is called a *compact fuzzy set* (on X) if $[\widetilde{a}]_\alpha \in \mathcal{K}(X)$ for any $\alpha \in]0, 1]$. Let $\mathcal{FC}(X)$ and $\mathcal{FK}(X)$ be the set of all closed fuzzy sets on X and the set of all compact fuzzy sets on X, respectively.

In $\mathbb{R}$, we define an order relation for crisp sets, and then define an order relation for fuzzy sets by using the order relation for crisp sets. Let $A, B \subset \mathbb{R}$. We write $A \leq B$ if $B \subset A + \mathbb{R}_+$ and $A \subset B + \mathbb{R}_-$, and write $A < B$ if $B \subset A + \mathrm{int}(\mathbb{R}_+)$ and $A \subset B + \mathrm{int}(\mathbb{R}_-)$. Then, $\leq$ is a pseudo order on $2^{\mathbb{R}}$. $B \subset A + \mathbb{R}_+$ if and only if for any $b \in B$, there exists $a \in A$ such that $a \leq b$. $A \subset B + \mathbb{R}_-$ if and only if for any $a \in A$, there exists $b \in B$ such that $a \leq b$. $B \subset A + \mathrm{int}(\mathbb{R}_+)$ if and only if for any $b \in B$, there exists $a \in A$ such that $a < b$. $A \subset B + \mathrm{int}(\mathbb{R}_-)$ if and only if for any $a \in A$, there exists $b \in B$ such that $a < b$. Let $\widetilde{a}, \widetilde{b} \in \mathcal{F}(\mathbb{R})$. We write $\widetilde{a} \preceq \widetilde{b}$ if $[\widetilde{a}]_\alpha \leq [\widetilde{b}]_\alpha$ for any $\alpha \in]0, 1]$, and write $\widetilde{a} \prec \widetilde{b}$ if $[\widetilde{a}]_\alpha < [\widetilde{b}]_\alpha$ for any $\alpha \in]0, 1]$. Then, $\preceq$ and $\prec$ are called the *fuzzy max order* and the *strict fuzzy max order*, respectively, and $\preceq$ is a pseudo order on $\mathcal{F}(\mathbb{R})$; see [7, 11].

3 Images of fuzzy sets by Zadeh's extension principle In this section, we investigate relationships between level sets of fuzzy sets and level sets of another fuzzy set obtained by Zadeh's extension principle, and the fuzzy norm and the fuzzy inner product defined by Zadeh's extension principle are presented.

Definition 1. Let X_i, $i = 1, 2, \cdots, n$ be sets, and let $\widetilde{a}_i \in \mathcal{F}(X_i)$, $i = 1, 2, \cdots, n$. Then, $\prod_{i=1}^n \widetilde{a}_i \in \mathcal{F}(\prod_{i=1}^n X_i)$ is defined as

$$\left(\prod_{i=1}^{n} \widetilde{a}_i\right)(x_1, x_2, \cdots, x_n) = \min_{i=1,2,\cdots,n} \widetilde{a}_i(x_i)$$

for each $(x_1, x_2, \cdots, x_n) \in \prod_{i=1}^n X_i$, and is called the *fuzzy product set* of $\widetilde{a}_i$, $i = 1, 2, \cdots, n$. The fuzzy product set $\prod_{i=1}^n \widetilde{a}_i$ is also represented as $\widetilde{a}_1 \times \widetilde{a}_2 \times \cdots \times \widetilde{a}_n$ or $(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n)$.

The following definition provides images of fuzzy sets under a crisp mapping by Zadeh's extension principle; see [4, 18] for Zadeh's extension principle.

Definitin 2. Let X_i, $i = 1, 2, \cdots, n$ and Y be sets, and let $f : \prod_{i=1}^n X_i \to Y$. Then, for $\widetilde{a}_i \in \mathcal{F}(X_i)$, $i = 1, 2, \cdots, n$, $f(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n) \in \mathcal{F}(Y)$ is defined as

$$f(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n)(y) = \sup_{(x_1,x_2,\cdots,x_n)\in f^{-1}(y)} \min_{i=1,2,\cdots,n} \widetilde{a}_i(x_i)$$

for each $y \in Y$, where $\sup \emptyset = 0$.

Let X be a real or complex normed space equipped with a norm $\|\cdot\|$, and set $f : X \to \mathbb{R}$ as $f(x) = \|x\|$ for each $x \in X$. For $\widetilde{a} \in \mathcal{F}(X)$, it follows that

$$f(\widetilde{a})(y) = \|\widetilde{a}\|(y) = \sup_{x\in f^{-1}(y)} \widetilde{a}(x), \quad y \in \mathbb{R} \tag{3}$$

from Definition 2. Then, $\|\widetilde{a}\| \in \mathcal{F}(\mathbb{R})$ is called the *fuzzy norm* of $\widetilde{a}$, and some properties of fuzzy norms are investigated in [9].

Let X be a real or complex inner product space equipped with an inner product $\langle\cdot,\cdot\rangle$, and set $f : X \times X \to \mathbb{K}$ as $f(x, y) = \langle x, y\rangle$ for each $x, y \in X$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. For $\widetilde{a}, \widetilde{b} \in \mathcal{F}(X)$, it follows that

$$f(\widetilde{a}, \widetilde{b})(z) = \langle\widetilde{a}, \widetilde{b}\rangle(z) = \sup_{(x,y)\in f^{-1}(z)} \min\{\widetilde{a}(x), \widetilde{b}(y)\}, \quad z \in \mathbb{K} \tag{4}$$

from Definition 2. Then, $\langle\widetilde{a}, \widetilde{b}\rangle \in \mathcal{F}(\mathbb{K})$ is called the *fuzzy inner product* between $\widetilde{a}$ and $\widetilde{b}$, and some properties of fuzzy inner products are investigated in [8].

The following theorem provides a relationship between level sets of fuzzy sets and level sets of another fuzzy set obtained by Zadeh's extension principle.

Theorem 1. *[12] Let X_i, $i = 1, 2, \cdots, n$ and Y be sets, and let $f : \prod_{i=1}^m X_i \to Y$. In addition, let $\widetilde{a}_i \in \mathcal{F}(X_i)$, $i = 1, 2, \cdots, n$. Then,*

$$[f(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n)]_\alpha = f([\widetilde{a}_1]_\alpha, [\widetilde{a}_2]_\alpha, \cdots, [\widetilde{a}_n]_\alpha) \tag{5}$$

for any $\alpha \in]0, 1]$ if and only if $y \in Y$ and $f^{-1}(y) \neq \emptyset$ imply the existence of $(x_1^*, x_2^*, \cdots, x_n^*) \in f^{-1}(y)$ such that

$$\min_{i=1,2,\cdots,n} \widetilde{a}_i(x_i^*) = \sup_{(x_1,x_2,\cdots,x_n)\in f^{-1}(y)} \min_{i=1,2,\cdots,m} \widetilde{a}_i(x_i).$$

The following theorem gives sufficient conditions for (5) in Theorem 1 to hold.

Theorem 2. *Let* $(X_i, \mathbb{T}_i)$, $i = 1, 2, \cdots, n$ *be Hausdorff spaces, and let* Y *be a* T_1*-space. Assume that* $f : \prod_{i=1}^n X_i \to Y$ *is continuous. In addition, let* $\widetilde{a}_i \in \mathcal{FK}(X_i)$, $i = 1, 2, \cdots, n$. *Then,*

$$[f(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n)]_\alpha = f([\widetilde{a}_1]_\alpha, [\widetilde{a}_2]_\alpha, \cdots, [\widetilde{a}_n]_\alpha)$$

for any $\alpha \in \,]0, 1]$.

Proof. Fix any $y \in Y$, and suppose that $f^{-1}(y) \neq \emptyset$. Then, it is sufficient to show the existence of $(x_1^*, x_2^*, \cdots, x_n^*) \in f^{-1}(y)$ such that

$$\min_{i=1,2,\cdots,n} \widetilde{a}_i(x_i^*) = \sup_{(x_1,x_2,\cdots,x_n)\in f^{-1}(y)} \left(\prod_{i=1}^n \widetilde{a}_i\right)(x_1, x_2, \cdots, x_n)$$

from Theorem 1. For any $\alpha \in \,]0, 1]$, since $[\widetilde{a}_i]_\alpha \in \mathcal{K}(X_i)$, $i = 1, 2, \cdots, n$, it follows that

$$\left[\prod_{i=1}^n \widetilde{a}_i\right]_\alpha = \prod_{i=1}^n [\widetilde{a}_i]_\alpha \in \mathcal{K}\left(\prod_{i=1}^n X_i\right)$$

from Tychonoff's theorem; see [1] for Tychonoff's theorem. Thus, it follows that

$$\prod_{i=1}^n \widetilde{a}_i \in \mathcal{FK}\left(\prod_{i=1}^n X_i\right) \subset \mathcal{FC}\left(\prod_{i=1}^n X_i\right),$$

and that $\prod_{i=1}^n \widetilde{a}_i$ is an upper semicontinuous function on $\prod_{i=1}^n X_i$. If

$$\sup_{(x_1,x_2,\cdots,x_n)\in f^{-1}(y)} \left(\prod_{i=1}^n \widetilde{a}_i\right)(x_1, x_2, \cdots, x_n) = 0,$$

then

$$\left(\prod_{i=1}^n \widetilde{a}_i\right)(x_1', x_2', \cdots, x_n') = 0 = \sup_{(x_1,x_2,\cdots,x_n)\in f^{-1}(y)} \left(\prod_{i=1}^n \widetilde{a}_i\right)(x_1, x_2, \cdots, x_n)$$

for any $(x_1', x_2', \cdots, x_n') \in f^{-1}(y)$. Suppose that

$$\sup_{(x_1,x_2,\cdots,x_n)\in f^{-1}(y)} \left(\prod_{i=1}^n \widetilde{a}_i\right)(x_1, x_2, \cdots, x_n) > 0.$$

Then, there exists $(x_1'', x_2'', \cdots, x_n'') \in f^{-1}(y)$ such that

$$\left(\prod_{i=1}^n \widetilde{a}_i\right)(x_1'', x_2'', \cdots, x_n'') > 0.$$

We set

$$\beta = \left(\prod_{i=1}^n \widetilde{a}_i\right)(x_1'', x_2'', \cdots, x_n'') > 0.$$

Then, since $x_i'' \in [\widetilde{a}_i]_\beta$, $i = 1, 2, \cdots, n$, it follows that

$$(x_1'', x_2'', \cdots, x_n'') \in f^{-1}(y) \cap \left(\prod_{i=1}^n [\widetilde{a}_i]_\beta\right).$$

Since

$$\prod_{i=1}^{n}[\widetilde{a}_i]_\beta \in \mathcal{K}\left(\prod_{i=1}^{n} X_i\right)$$

and

$$f^{-1}(y) \in \mathcal{C}\left(\prod_{i=1}^{n} X_i\right)$$

by the continuity of f, it follows that

$$f^{-1}(y) \cap \left(\prod_{i=1}^{n}[\widetilde{a}_i]_\beta\right) \in \mathcal{K}\left(\prod_{i=1}^{n} X_i\right).$$

Since

$$\left(\prod_{i=1}^{n}\widetilde{a}_i\right)(x_1, x_2, \cdots, x_n) \geq \beta$$

for any

$$(x_1, x_2, \cdots, x_n) \in f^{-1}(y) \cap \left(\prod_{i=1}^{n}[\widetilde{a}_i]_\beta\right),$$

and

$$\left(\prod_{i=1}^{n}\widetilde{a}_i\right)(x_1, x_2, \cdots, x_n) < \beta$$

for any

$$(x_1, x_2, \cdots, x_n) \in f^{-1}(y) \setminus \left(\prod_{i=1}^{n}[\widetilde{a}_i]_\beta\right),$$

we have

$$\begin{aligned}
&\sup_{(x_1,x_2,\cdots,x_n)\in f^{-1}(y)} \left(\prod_{i=1}^{n}\widetilde{a}_i\right)(x_1, x_2, \cdots, x_n) \\
&\qquad = \sup_{(x_1,x_2,\cdots,x_n)\in f^{-1}(y)\cap\left(\prod_{i=1}^{n}[\widetilde{a}_i]_\beta\right)} \left(\prod_{i=1}^{n}\widetilde{a}_i\right)(x_1, x_2, \cdots, x_n).
\end{aligned}$$

By the compactness of $f^{-1}(y) \cap (\prod_{i=1}^{n}[\widetilde{a}_i]_\beta) \neq \emptyset$ and the upper semicontinuity of $\prod_{i=1}^{n}\widetilde{a}_i$, there exists $(x_1^*, x_2^*, \cdots, x_n^*) \in f^{-1}(y) \cap (\prod_{i=1}^{n}[\widetilde{a}_i]_\beta)$ such that

$$\begin{aligned}
&\left(\prod_{i=1}^{n}\widetilde{a}_i\right)(x_1^*, x_2^*, \cdots, x_n^*) \\
&\qquad = \sup_{(x_1,x_2,\cdots,x_n)\in f^{-1}(y)\cap\left(\prod_{i=1}^{n}[\widetilde{a}_i]_\beta\right)} \left(\prod_{i=1}^{n}\widetilde{a}_i\right)(x_1, x_2, \cdots, x_n).
\end{aligned}$$

□

The following theorem gives sufficient conditions for the fuzzy set obtained by Zadeh's extension principle from other fuzzy sets to be a compact fuzzy set.

Theorem 3. *Let $(X_i, \mathbb{T}_i), i = 1, 2, \cdots, n$ be Hausdorff spaces, and let Y be a T_1-space. Assume that $f : \prod_{i=1}^n X_i \to Y$ is continuous. In addition, let $\widetilde{a}_i \in \mathcal{FK}(X_i)$, $i = 1, 2, \cdots, n$. Then, $f(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n) \in \mathcal{FK}(Y)$.*

Proof. Fix any $\alpha \in]0, 1]$. Since $\widetilde{a}_i \in \mathcal{FK}(X_i)$, $i = 1, 2, \cdots, n$, it follows that $[\widetilde{a}_i]_\alpha \in \mathcal{K}(X_i)$, $i = 1, 2, \cdots, n$, and that $\prod_{i=1}^n [\widetilde{a}_i]_\alpha \in \mathcal{K}(\prod_{i=1}^n X_i)$ from Tychonoff's theorem. From Theorem 2 and the continuity of f, if follows that $[f(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n)]_\alpha = f([\widetilde{a}_1]_\alpha, [\widetilde{a}_2]_\alpha, \cdots, [\widetilde{a}_n]_\alpha) \in \mathcal{K}(Y)$. Therefore, we have $f(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n) \in \mathcal{FK}(Y)$. □

The following theorem shows that order relations of functions imply order relations of fuzzy sets obtained by Zadeh's extension principle using the functions.

Theorem 4. *Let $(X_i, \mathbb{T}_i)$, $i = 1, 2, \cdots, n$ be Hausdorff spaces, and let $\widetilde{a}_i \in \mathcal{FK}(X_i)$, $i = 1, 2, \cdots, n$. Assume that $f, g : \prod_{i=1}^n X_i \to \mathbb{R}$ are continuous.*

(i) If $f \leq g$, then $f(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n) \preceq g(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n)$.

(ii) If $f < g$, then $f(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n) \prec g(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n)$.

Proof. We shall show only (i). (ii) can be shown in the similar way to (i).

From Theorem 2, it follows that $[f(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n)]_\alpha = f([\widetilde{a}_1]_\alpha, [\widetilde{a}_2]_\alpha, \cdots, [\widetilde{a}_n]_\alpha)$ and $[g(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n)]_\alpha = g([\widetilde{a}_1]_\alpha, [\widetilde{a}_2]_\alpha, \cdots, [\widetilde{a}_n]_\alpha)$ for any $\alpha \in]0, 1]$. Fix any $\alpha \in]0, 1]$.

First, let $z \in [g(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n)]_\alpha = g([\widetilde{a}_1]_\alpha, [\widetilde{a}_2]_\alpha, \cdots, [\widetilde{a}_n]_\alpha)$. Then, there exists $(x_1, x_2, \cdots, x_n) \in \prod_{i=1}^n [\widetilde{a}_i]_\alpha$ such that $z = g(x_1, x_2, \cdots, x_n)$. Set $y = f(x_1, x_2, \cdots, x_n) \in f([\widetilde{a}_1]_\alpha, [\widetilde{a}_2]_\alpha, \cdots, [\widetilde{a}_n]_\alpha) = [f(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n)]_\alpha$, then it follows that $y = f(x_1, x_2, \cdots, x_n) \leq g(x_1, x_2, \cdots, x_n) = z$ from the assumption. Thus, for any $z \in [g(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n)]_\alpha$, there exists $y \in [f(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n)]_\alpha$ such that $y \leq z$.

Next, let $y \in [f(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n)]_\alpha = f([\widetilde{a}_1]_\alpha, [\widetilde{a}_2]_\alpha, \cdots, [\widetilde{a}_n]_\alpha)$. Then, there exists $(x_1, x_2, \cdots, x_n) \in \prod_{i=1}^n [\widetilde{a}_i]_\alpha$ such that $y = f(x_1, x_2, \cdots, x_n)$. Set $z = g(x_1, x_2, \cdots, x_n) \in g([\widetilde{a}_1]_\alpha, [\widetilde{a}_2]_\alpha, \cdots, [\widetilde{a}_n]_\alpha) = [g(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n)]_\alpha$, then it follows that $y = f(x_1, x_2, \cdots, x_n) \leq g(x_1, x_2, \cdots, x_n) = z$ from the assumption. Thus, for any $y \in [f(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n)]_\alpha$, there exists $z \in [g(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n)]_\alpha$ such that $y \leq z$.

Therefore, we have $f(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n) \preceq g(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n)$ since $[f(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n)]_\alpha \leq [g(\widetilde{a}_1, \widetilde{a}_2, \cdots, \widetilde{a}_n)]_\alpha$ for any $\alpha \in]0, 1]$. □

4 Fuzzy Schwarz inequality In this section, the fuzzy Schwarz inequality is derived by using the fuzzy norm and the fuzzy inner product as an extension of the Schwarz inequality in inner product spaces.

Throughout this section, let X be a real or complex inner product space equipped with an inner product $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. For each $x \in X$, the norm of x is defined as

$$\|x\| = \sqrt{\langle x, x \rangle}. \tag{6}$$

The same notations $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ are used when some inner product spaces are considered.

The inner product on $X \times X$ is defined as

$$\langle (x_1, y_1), (x_2, y_2) \rangle = \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle \tag{7}$$

for each $(x_1, y_1), (x_2, y_2) \in X \times X$, and the norm on $X \times X$ is defined as

$$\|(x, y)\| = \sqrt{\langle (x, y), (x, y) \rangle} = \sqrt{\|x\|^2 + \|y\|^2} \tag{8}$$

for each $(x, y) \in X \times X$.

The following theorem provides an inequality which evaluates the inner product between two vectors in an inner product space by norms of the two vectors; see, for example [3].

Theorem 5. (Schwarz Inequality) *For any* $x, y \in X$,

$$|\langle x, y\rangle| \leq \|x\|\|y\|.$$

Moreover, equality holds in this inequality if and only if x *and* y *are linearly dependent.*

In order to derive fuzzy Schwarz inequality, we present the following lemma.

Lemma 1. *Define* $f_1 : X \times X \to \mathbb{K}$ *as* $f_1(x, y) = \langle x, y\rangle$ *for each* $(x, y) \in X \times X$, $f_2 : \mathbb{K} \to \mathbb{R}$ *as* $f_2(z) = |z|$ *for each* $z \in \mathbb{K}$, *and* $f : X \times X \to \mathbb{R}$ *as* $f(x, y) = f_2(f_1(x, y)) = |\langle x, y\rangle|$ *for each* $(x, y) \in X \times X$. *In addition, define* $g_1 : X \to \mathbb{R}$ *as* $g_1(x) = \|x\|$ *for each* $x \in X$, $g_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *as* $g_2(u, v) = uv$ *for each* $(u, v) \in \mathbb{R} \times \mathbb{R}$, *and* $g : X \times X \to \mathbb{R}$ *as* $g(x, y) = g_2(g_1(x), g_1(y)) = \|x\|\|y\|$ *for each* $(x, y) \in X \times X$. *Let* $\widetilde{a}, \widetilde{b} \in \mathcal{FK}(X)$. *Then,*

$$f(\widetilde{a}, \widetilde{b}) = f_2(f_1(\widetilde{a}, \widetilde{b})) = |\langle \widetilde{a}, \widetilde{b}\rangle|, \tag{9}$$

$$g(\widetilde{a}, \widetilde{b}) = g_2(g_1(\widetilde{a}), g_1(\widetilde{b})) = \|\widetilde{a}\|\|\widetilde{b}\| \tag{10}$$

where the second equalities in (9) and (10) are definitions.

Proof. For $A, B \subset X$, it can be shown easily that

$$f(A, B) = f_2(f_1(A, B)) = |\langle A, B\rangle|, \tag{11}$$

$$g(A, B) = g_2(g_1(A), g_1(B)) = \|A\|\|B\| \tag{12}$$

where the second equalities in (11) and (12) are definitions.

Since f_1, f_2, and f are continuous, it follows that $[f(\widetilde{a}, \widetilde{b})]_\alpha = f([\widetilde{a}]_\alpha, [\widetilde{b}]_\alpha) = f_2(f_1([\widetilde{a}]_\alpha, [\widetilde{b}]_\alpha)) = f_2([f_1(\widetilde{a}, \widetilde{b})]_\alpha) = [f_2(f_1(\widetilde{a}, \widetilde{b}))]_\alpha$ for any $\alpha \in]0, 1]$ from Theorems 2, 3, and (11). Therefore, we have $f(\widetilde{a}, \widetilde{b}) = f_2(f_1(\widetilde{a}, \widetilde{b}))$ from the decomposition theorem (2).

Since g_1, g_2, and g are continuous, it follows that $[g(\widetilde{a}, \widetilde{b})]_\alpha = g([\widetilde{a}]_\alpha, [\widetilde{b}]_\alpha) = g_2(g_1([\widetilde{a}]_\alpha), g_1([\widetilde{b}]_\alpha)) = g_2([g_1(\widetilde{a})]_\alpha, [g_1(\widetilde{b})]_\alpha) = [g_2(g_1(\widetilde{a}), g_1(\widetilde{b}))]_\alpha$ for any $\alpha \in]0, 1]$ from Theorems 2, 3, and (12). Therefore, we have $g(\widetilde{a}, \widetilde{b}) = g_2(g_1(\widetilde{a}),\ g_1(\widetilde{b}))$ from the decomposition theorem (2). $\square$

The following theorem provides an inequality which evaluates the fuzzy inner product between two fuzzy sets on an inner product space by fuzzy norms of the two fuzzy sets.

Theorem 6. (Fuzzy Schwarz Inequality) *For any* $\widetilde{a}, \widetilde{b} \in \mathcal{FK}(X)$,

$$|\langle \widetilde{a}, \widetilde{b}\rangle| \preceq \|\widetilde{a}\|\|\widetilde{b}\|.$$

Proof. Define $f, g : X \times X \to \mathbb{R}$ as $f(x, y) = |\langle x, y\rangle|$ and $g(x, y) = \|x\|\|y\|$ for each $(x, y) \in X \times X$. From Theorem 5, it follows that $f(x, y) = |\langle x, y\rangle| \leq \|x\|\|y\| = g(x, y)$ for any x, $y \in X$.

Let $\widetilde{a}, \widetilde{b} \in \mathcal{FK}(X)$. Then, it follows that $f(\widetilde{a}, \widetilde{b}) = |\langle \widetilde{a}, \widetilde{b}\rangle|$ and $g(\widetilde{a}, \widetilde{b}) = \|\widetilde{a}\|\|\widetilde{b}\|$ from Lemma 1. Since f, g are continuous, we have $|\langle \widetilde{a}, \widetilde{b}\rangle| = f(\widetilde{a}, \widetilde{b}) \preceq g(\widetilde{a}, \widetilde{b}) = \|\widetilde{a}\|\|\widetilde{b}\|$ from Theorem 4 (i). $\square$

5 Conclusions In the present paper, the fuzzy Schwarz inequality in inner product spaces was derived. It was an extension of the Schwarz inequality, and was described by using a fuzzy norm and a fuzzy inner product defined by Zadeh's extension principle. The Schwarz inequality evaluates the inner product between two vectors in an inner product space by norms of the two vectors. On the other hand, the fuzzy Schwarz inequality evaluates the fuzzy inner product between two fuzzy sets on an inner product space by fuzzy norms of the two fuzzy sets.

First, the fuzzy norm and the fuzzy inner product were defined by Zadeh's extension principle. The fuzzy norm of a fuzzy set is the image of the fuzzy set under the crisp norm, and it is also a fuzzy set. The fuzzy inner product between two fuzzy sets is the image of the two fuzzy sets under the crisp inner product, and it is also a fuzzy set. Next, sufficient conditions for the image of level sets of fuzzy sets to coincide with level sets of another fuzzy set obtained by Zadeh's extension principle were given as Theorem 2. Next, sufficient conditions for the fuzzy set obtained by Zadeh's extension principle from other fuzzy sets to be a compact fuzzy set were given as Theorem 3. Next, it was shown that order relations of functions implied order relations of fuzzy sets obtained by Zadeh's extension principle using the functions as Theorem 4. Finally, based on these results, the fuzzy Schwarz inequality was derived as Theorem 6.

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