SOME RESULTS ON DIRECT SUMS OF BANACH SPACES — A SURVEY

MIKIO KATO*, TAKUYA SOBUKAWA† AND TAKAYUKI TAMURA‡

ABSTRACT. We shall discuss three notions of direct sums of Banach spaces, Z-, ψ-, and A-direct sums, which are in fact all isometric. Weak nearly uniform smoothness, uniform non-squareness and uniform non-ℓ_1^n-ness etc. will be discussed, especially in the general A-direct sum setting. As applications some examples of Banach spaces will be presented concerning FPP as well as super-reflexivity.

1 Introduction Direct sums of Banach spaces have been often treated in the context of geometry of Banach spaces and the fixed point property (e.g. [2, 3, 6, 7, 8, 9, 10, 11, 14, 15, 16, 21, 22, 23, 25, 26, 27, 28, 29, 30, 32, 33, 36, 40, 41, 42, 43]). We shall discuss three notions of direct sums of Banach spaces.

It is known that every absolute normalized norm $\|\cdot\|_{AN}$ on $\mathbb{R}^N$ corresponds to a unique convex function ψ on the standard simplex in $\mathbb{R}^{N-1}$ (we shall mention it precisely in Section 2). So we shall write $\|\cdot\|_\psi$ for $\|\cdot\|_{AN}$ and refer to as a ψ-norm. Let $\|\cdot\|_Z$ and $\|\cdot\|_A$ be an absolute and an arbitrary norm on $\mathbb{R}^N$ respectively, which we shall call a Z-norm and an A-norm.

A Z-direct sum $(X_1 \oplus \cdots \oplus X_N)_Z$ of Banach spaces $X_1, \ldots, X_N$ is their direct sum equipped with the norm

$$\|(x_1, \ldots, x_N)\|_Z = \|(\|x_1\|, \ldots, \|x_N\|)\|_Z, \quad (x_1, \ldots, x_N) \in X_1 \oplus \cdots \oplus X_N,$$

where the norm $\|\cdot\|_Z$ in the right side is an absolute norm on $\mathbb{R}^N$. A ψ-direct sum $(X_1 \oplus \cdots \oplus X_N)_\psi$ and an A-direct sum $(X_1 \oplus \cdots \oplus X_N)_A$ are defined in the same way by means of a ψ-norm $\|\cdot\|_\psi$ and an A-norm $\|\cdot\|_A$.

In Section 2 the correspondence will be mentioned between the set AN_N of all absolute normalized norms on $\mathbb{R}^N$ and the collection Ψ_N of all convex functions satisfying certain conditions on the standard simplex Δ_N in $\mathbb{R}^{N-1}$. A couple of subclasses $\Psi_N^{(1)}$ and $\Psi_N^{(\infty)}$ of Ψ_N will be discussed, which were introduced in Kato and Tamura [29, 30] to discuss weak nearly uniform smoothness and uniform non-squareness for direct sums. These classes can be described in terms of Properties T_1^N and T_∞^N, which Dowling and Saejung [10] introduced to discuss uniform non-squareness for Z-direct sums.

In Section 3 it will be seen that any A-direct sum is isometrically isomorphic to a ψ-direct sum with some $\psi \in \Psi_N$ ([8]); therefore the direct sums stated above are all isometrically isomorphic and ψ-direct sums are general enough. In Sections 4, 5, and 6 we shall obtain A-direct sum versions of previous results.

Section 4 will deal with weak nearly uniform smoothness (WNUS-ness in short). *Every WNUS space has FPP, the fixed point property* (for nonexpansive mappings; [15, 14]). A characterization of WNUS-ness for $(X_1 \oplus \cdots \oplus X_N)_\psi$ will be presented by means of the class $\Psi_N^{(1)}$.

2010 *Mathematics Subject Classification.* 52A51, 26A51, 47H10, 46B10, 46B20.

Key words and phrases. absolute norm, convex function, Properties T_1^N and T_∞^N, direct sum of Banach spaces, weak nearly uniform smoothness, uniform non-squareness, uniform non-ℓ_1^N-ness, fixed point property.

*, †, ‡ The first and third authors, resp. the second author were supported in part by JSPS KAKENHI (C) Grant number JP26400131, resp. JSPS KAKENHI (B) Grant number JP15H03621.

In Section 5 we shall discuss uniform non-squareness (UNSQ-ness) which has been playing an important roll in geometry of Banach spaces. The starting point of our discussion is the following result in Kato-Saito-Tamura [22]: *A ψ-direct sum $X \oplus_\psi Y$ is UNSQ if and only if X and Y are UNSQ and $\psi \neq \psi_1, \psi_\infty$, where ψ_1 and ψ_∞ are the corresponding convex functions to the ℓ_1- and ℓ_∞-norms, respectively.* They [22] asked for a characterization for N Banach spaces. We shall present a sequence of partial results by Dowling-Saejung [10], Betiuk-Pilarska and Prus [2], and Dhompongsa-Kato-Tamura [8]. In [10] the following was shown: *Under the assumption $\|\cdot\|_Z$ is strictly monotone, $(X_1 \oplus \cdots \oplus X_N)_Z$ is UNSQ if and only if $X_1, \ldots, X_N$ are UNSQ and $\|\cdot\|_Z$ has Properties T_1^N and T_∞^N.* In the case $N = 3$ this assumption was dropped. More precise results are shown in [8] for ψ-direct sums in terms of $\Psi_N^{(1)}$, from which the A-direct sum versions are derived. In [2] it was shown that *$(X_1 \oplus \cdots \oplus X_N)_Z$ is UNSQ if and only if $X_1, \ldots, X_N$ and $(\mathbb{R}^N, \|\cdot\|)$ are UNSQ*, where it remains unknown when $(\mathbb{R}^N, \|\cdot\|)$ is UNSQ.

Recently Kato-Tamura [30, in preparation] obtained a characterization of UNSQ-ness for $(X_1 \oplus \cdots \oplus X_N)_\psi$ as well as A-direct sum without any additional assumption, which covers all the above-mentioned results and explains why the case $N = 3$ is successful in [10].

In Section 6 uniform non-ℓ_1^n-ness will be discussed. When $n = 2$, uniform non-ℓ_1^2-ness coincides with UNSQ-ness. Every uniformly non-ℓ_1^n space is uniformly non-ℓ_1^{n+1}. The above result for UNSQ-ness of $X \oplus_\psi Y$ ([22]) is extended to uniform non-ℓ_1^n-ness ([23]). The spaces $X \oplus_1 Y$ and $X \oplus_\infty Y$ cannot be UNSQ, while they can be uniformly non-ℓ_1^n, $n \geq 3$. We shall discuss when they are uniformly non-ℓ_1^n.

In the last Section 7 applications to FPP will be discussed. As UNSQ spaces have FPP ([16]), it is natural to ask whether every uniformly non-ℓ_1^3-space has FPP. We shall see a plenty of Banach spaces (direct sums) with FPP which is not UNSQ can be constructed. Super-reflexivity will be treated as well.

In the following $X, X_1, \ldots, X_N$ will stand for Banach spaces. Let S_X and B_X denote the unit sphere and the closed unit ball of X. Let $\mathbb{R}_+^N$ denote the set of all points in $\mathbb{R}^N$ with nonegative entries.

2 Absolute norms on $\mathbb{R}^N$ and convex functions

A norm $\|\cdot\|$ on $\mathbb{R}^N$ is called *absolute* if $\|(a_1, \cdots, a_N)\| = \|(|a_1|, \cdots, |a_N|)\|$ for all $(a_1, \cdots, a_N) \in \mathbb{R}^N$, and *normalized* if $\|(1, 0, \cdots, 0)\| = \cdots = \|(0, \cdots, 0, 1)\| = 1$. A norm $\|\cdot\|$ on $\mathbb{R}^N$ is called *monotone* provided that, if $|a_j| \leq |b_j|$ for $1 \leq j \leq N$, $\|(a_1, \ldots, a_N)\| \leq \|(b_1, \ldots, b_N)\|$. $\|\cdot\|$ is called *strictly monotone* provided it is monotone and, if $|a_j| < |b_j|$ for some $1 \leq j \leq N$, $\|(a_1, \ldots, a_N)\| < \|(b_1, \ldots, b_N)\|$. The following is known.

Lemma 2.1 (Bhatia [4], see also [30]) *A norm $\|\cdot\|$ on $\mathbb{R}^N$ is absolute if and only if it is monotone.*

We shall see that for every absolute normalized norm on $\mathbb{R}^N$ there corresponds a unique convex function ψ on a certain convex set in $\mathbb{R}^{N-1}$ ([38, 5]).

Lemma 2.2 *Let $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^N$ and define*

$$\psi(s) = \left\|(1 - \sum_{i=1}^{N-1} s_i, s_1, \ldots, s_{N-1})\right\|, \quad s = (s_1, \cdots, s_{N-1}) \in \Delta_N, \tag{2.1}$$

where $\Delta_N = \{s = (s_1, \cdots, s_{N-1}) \in \mathbb{R}^{N-1} : \sum_{i=1}^{N-1} s_i \leq 1,\ s_i \geq 0\}$. Then:

(i) *The norm $\|\cdot\|$ is normalized if and only if*

$$\psi(0, \cdots, 0) = \psi(1, 0, \cdots, 0) = \cdots = \psi(0, \cdots, 0, 1) = 1. \tag{A_0}$$

(ii) For each $1 \leq k \leq N$ the following (a) and (b) are equivalent.
(a) The norm $\|\cdot\|$ is monotone in the k-th entry, that is,

$$|x_k| \geq |y_k| \;\Rightarrow\; \|(x_1, ..., \overset{k}{\smile}{x_k}, ..., x_N)\| \geq \|(x_1, ..., \overset{k}{\smile}{y_k}, ..., x_N)\|$$

(b) The convex function ψ satisfies

$$(A_k) \qquad \psi(s_1, \ldots, s_{N-1}) \geq (1 - s_k)\psi\left(\frac{s_1}{1 - s_k}, \ldots, \overset{k-1}{\smile}{0}, \ldots, \frac{s_{N-1}}{1 - s_k}\right)$$

In the case $k = 1$, (A_1) should be understood as

$$(A_1) \qquad \psi(s_1, \cdots, s_{N-1}) \geq (1 - s_0)\psi\left(\frac{s_1}{1 - s_0}, \ldots, \frac{s_{N-1}}{1 - s_0}\right),$$

where $s_0 = 1 - \sum_{i=1}^{N-1} s_i$.

Let

$$AN_N = \{\text{all absolute normalized norms on } \mathbb{R}^N\},$$
$$\Psi_N = \{\text{all convex functions } \psi \text{ satisfying } (A_k), 0 \leq k \leq N\}.$$

Theorem 2.3 (Saito-Kato-Takahashi [38]) (i) *For any* $\|\cdot\| \in AN_N$ *let*

$$\psi(s) = \left\|(1 - \sum_{i=1}^{N-1} s_i, s_1, \ldots, s_{N-1})\right\|, \quad s = (s_1, \cdots, s_{N-1}) \in \Delta_N. \tag{2.1}$$

Then $\psi \in \Psi_N$*, that is,*

$$(A_0) \qquad \psi(0, \cdots, 0) = \psi(1, 0, \cdots, 0) = \cdots = \psi(0, \cdots, 0, 1) = 1;$$

and for each $1 \leq k \leq N$

$$(A_k) \qquad \psi(s_1, \ldots, s_{N-1}) \geq (1 - s_k)\psi\left(\frac{s_1}{1 - s_k}, \ldots, \overset{k-1}{\smile}{0}, \ldots, \frac{s_{N-1}}{1 - s_k}\right).$$

Conversely
(ii) *For any* $\psi \in \Psi_N$ *define*

$$(*) \quad \|(a_1, \cdots, a_N)\|_\psi = \begin{cases} \left(\sum_{j=1}^N |a_j|\right)\psi\left(\dfrac{|a_2|}{\sum_{j=1}^N |a_j|}, \cdots, \dfrac{|a_N|}{\sum_{j=1}^N |a_j|}\right) & \\ \qquad\qquad\qquad\qquad \textit{if } (a_1, \cdots, a_N) \neq (0, \cdots, 0), \\ 0 \qquad\qquad\qquad\qquad \textit{if } (a_1, \cdots, a_N) = (0, \cdots, 0). \end{cases}$$

Then $\|\cdot\|_\psi \in AN_N$ *and* $\|\cdot\|_\psi$ *satisfies* (2.1).

In fact, since an absolute normalized norm is monotone by Lemma 2.1, the statement (i) is a consequence of Lemma 2.2. For the assertion (ii) we refer the reader to [38]. Thus AN_N and Ψ_N correspond in a one-to-one way.

Remark 2.4 *(i) Let us see why we defined the norm* $\|\cdot\|_\psi$ *by the formula* $(*)$ *from* $\psi \in \Psi_N$. *For an arbitrary norm* $\|\cdot\|$ *on* $\mathbb{R}^N$ *let* ψ *be a convex function given by (2.1). Then the norm* $\|\cdot\|$ *is represented by means of* ψ *as follows. Let* $M = \sum_{j=1}^N |a_j|$ *for nonzero* $(a_1, \ldots, a_N) \in \mathbb{R}^N$. *Then*

$$\|(a_1, \cdots, a_N)\| = M\|(a_1/M, \cdots, a_N/M)\| = M\psi\left(\frac{|a_2|}{M}, \cdots, \frac{|a_N|}{M}\right).$$

(ii) In the case $N = 2$ *a convex funtion* ψ *on* $\Delta_2 = [0, 1]$ *belongs to* Ψ_2 *if and only if* $\max\{1-t,\, t\} \leq \psi(t) \leq 1$ *for* $0 \leq t \leq 1$, *from which* $\psi(0) = \psi(1) = 1$ *is derived. Thus if we draw the graph of a convex function* $\psi \in \Psi_2$ *in this triangle area we shall obtain an absolute normalized norm* $\|\cdot\|_\psi$ *on* $\mathbb{R}^2$.

Example 2.5 *The* ℓ_p*-norm on* $\mathbb{R}^N$,

$$\|(a_1, \ldots, a_N)\|_p = \begin{cases} \{\sum_{j=1}^N |a_j|^p\}^{1/p} & 1 \leq p < \infty, \\ \max_{1 \leq j \leq N} |a_j| & p = \infty \end{cases}$$

is absolute normalized and the corresponding convex function ψ_p *is given by*

$$\begin{aligned} \psi_p(s_1, \ldots, s_{N-1}) \quad &:= \quad \|(1 - \sum_{i=1}^{N-1} s_i,\ s_1, \ldots, s_{N-1})\|_p \\ &= \begin{cases} \left\{\left(1 - \sum_{i=1}^{N-1} s_i\right)^p + s_1^p + \cdots + s_{n-1}^p\right\}^{1/p} & \text{if } p < \infty, \\ \max\left\{1 - \sum_{i=1}^{N-1} s_i, s_1, \ldots, s_{n-1}\right\} & \text{if } p = \infty. \end{cases} \end{aligned}$$

In particular $\psi_1(s_1, \ldots, s_{N-1}) = 1$.

Now, the following subclasses $\Psi_N^{(1)}$ and $\Psi_N^{(\infty)}$ of Ψ_N will play an important role in our later discussion. In the following let T be a nonempty subset of $\{1, \ldots, N\}$, χ_T the characteristic function of T. For $\boldsymbol{a} = (a_1, \ldots, a_N) \in \mathbb{R}_+^N$ let

$$\boldsymbol{a}_T = \sum_{j \in T} a_j \mathbf{e}_j = (\chi_T(1)a_1, \ldots, \chi_T(N)a_N),$$

where $\boldsymbol{e}_j = (0, \ldots, \overbrace{1}^{j}, \ldots, 0)$.

Definition 2.6 (Kato-Tamura [27, 30]) *(i) Let* $\psi \in \Psi_N$. *We say* $\psi \in \Psi_N^{(1)}$ *if there exists* $\boldsymbol{a} \in \mathbb{R}_+^N$ *and* $T \subsetneq \{1, \ldots, N\}$ $(T \neq \emptyset)$ *such that*

$$\|\mathbf{a}\|_\psi = \|\mathbf{a}_T\|_\psi + \|\mathbf{a}_{T^c}\|_\psi, \quad \text{where } \|\mathbf{a}_T\|_\psi,\ \|\mathbf{a}_{T^c}\|_\psi > 0.$$

(ii) We say $\psi \in \Psi_N^{(\infty)}$ *if there exists* $\boldsymbol{a} \in \mathbb{R}_+^N$ *and* $T \subsetneq \{1, \ldots, N\}$ $(T \neq \emptyset)$ *such that*

$$\|\mathbf{a}\|_\psi = \|\mathbf{a}_T\|_\psi = \|\mathbf{a}_{T^c}\|_\psi > 0.$$

The ℓ_1-norm $\|\cdot\|_1$ has the above property (i), and the ℓ_∞-norm $\|\cdot\|_\infty$ has the property (ii) (see the example below). These properties (i) and (ii) are much weaker than ℓ_1-norm's and ℓ_∞-norm's, respectively. We call, in general, a norm $\|\cdot\|$ on $\mathbb{R}^N$ with the properties (i) and (ii) a *partial ℓ_1-norm* and a *partial ℓ_∞-norm*, respectively.

Example 2.7 *$\psi_1 \in \Psi_N^{(1)}$ and $\psi_\infty \in \Psi_N^{(\infty)}$ since*

$$\left\|(1, \frac{1}{N-1}, \ldots, \frac{1}{N-1})\right\|_1 = \|(1,0,\ldots,0)\|_1 + \left\|(0, \frac{1}{N-1}, \ldots, \frac{1}{N-1})\right\|_1,$$
$$\|(1,1,\ldots,1)\|_\infty = \|(1,0,\ldots,0)\|_\infty = \|(0,1,\ldots,1)\|_\infty,$$

where $T = \{1\}$ in both cases.

On the other hand Dowling-Saejung [10] introduced the following notions.

Definition 2.8 *For $\boldsymbol{a} = (a_j) \in \mathbb{R}^N$ let* supp $\boldsymbol{a} = \{j : a_j \neq 0\}$.
(i) A norm $\|\cdot\|$ on $\mathbb{R}^N$ is said to have Property T_1^N if

$$\|\boldsymbol{a}\| = \|\boldsymbol{b}\| = \frac{1}{2}\|\boldsymbol{a}+\boldsymbol{b}\| = 1,\ \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^N \implies \text{supp } \boldsymbol{a} \cap \text{supp } \boldsymbol{b} \neq \emptyset.$$

(ii) A norm $\|\cdot\|$ on $\mathbb{R}^N$ is said to have Property T_∞^N if

$$\|\boldsymbol{a}\| = \|\boldsymbol{b}\| = \|\boldsymbol{a}+\boldsymbol{b}\| = 1 \implies \text{supp } \boldsymbol{a} \cap \text{supp } \boldsymbol{b} \neq \emptyset.$$

Note that ℓ_1-norm $\|\cdot\|_1$ and ℓ_∞-norm $\|\cdot\|_\infty$ do not have Property T_1^N and Property T_∞^N, respectively. We have the following.

Theorem 2.9 (Dhompomgsa-Kato-Tamura [8]) *Let $\psi \in \Psi_N$. Then*
(i) $\|\cdot\|_\psi$ *has Property T_1^N if and only if $\psi \notin \Psi_N^{(1)}$.*
(ii) $\|\cdot\|_\psi$ *has Property T_∞^N if and only if $\psi \notin \Psi_N^{(\infty)}$.*

3 Direct sums Let $\|\cdot\|_Z$ be an *absolute* norm on $\mathbb{R}^N$. The *Z-direct sum* $(X_1 \oplus \cdots \oplus X_N)_Z$ of Banach spaces $X_1, \ldots, X_N$ is their direct sum equipped with the norm

$$\|(x_1, \cdots, x_N)\|_Z := \|(\|x_1\|, \cdots, \|x_N\|)\|_Z, \quad (x_1, \ldots, x_N) \in X_1 \oplus \cdots \oplus X_N$$

(cf. Dowling-Saejung [10]).

Remark 3.1 *In the above definition the Z-norm $\|\cdot\|_Z$ on $\mathbb{R}^N$ is sometimes assumed to be absolute and monotone in $\mathbb{R}_+^N$ in [10]. But the latter condition can be dropped because of Lemma 2.1.*

A direct sum constructed in the same way as above from an *absolute normalized* norm $\|\cdot\|_{AN} = \|\cdot\|_\psi$ on $\mathbb{R}^N$ is called a *ψ-direct sum* and denoted by

$$(X_1 \oplus \cdots \oplus X_N)_\psi,$$

where ψ is the convex function corresponding to the norm $\|\cdot\|_{AN}$ (Kato-Saito-Tamura [21]; cf. [40]).

Let $\|\cdot\|_A$ be an *arbitrary* norm on $\mathbb{R}^N$. The *A-direct sum* $(X_1 \oplus \cdots \oplus X_N)_A$ is the direct sum of $X_1, \ldots, X_N$ equipped with the norm

$$\|(x_1, \ldots, x_N)\|_A = \|(\|x_1\|, \ldots, \|x_N\|)\|_A, \quad (x_1, \ldots, x_N) \in X_1 \oplus \cdots \oplus X_N$$

(Dhompongsa-Kato-Tamura [8]). Clearly, a ψ-direct sum is a Z-direct sum, which is an A-direct sum. These notions of direct sums are in fact all isometric.

Theorem 3.2 (Kato-Tamura [30]) *Let $\|\cdot\|_A$ be an arbitrary norm on $\mathbb{R}^N$. Then there exists $\psi \in \Psi_N$ such that $(X_1 \oplus \cdots \oplus X_N)_A$ is isometrically isomorphic to $(X_1 \oplus \cdots \oplus X_N)_\psi$. More precisely*

$$\|(x_1, \ldots, x_N)\|_A = \|(c_1 x_1, \ldots, c_N x_N)\|_\psi, \quad (x_1, \ldots, x_N) \in X_1 \oplus \cdots \oplus X_N,$$

where $c_k = \|(0, \ldots, 0, \overset{k}{\smile}{1}, 0, \ldots, 0)\|_A$ $(1 \leq k \leq N)$.

Sketch of proof Take $e_j \in X_j$ with $\|e_j\| = 1$ $(1 \leq j \leq N)$ and define a norm $\|\cdot\|_B$ on $\mathbb{R}^N$ by

$$\|(a_1, \ldots, a_N)\|_B = \|(a_1 e_1, \ldots, a_N e_N)\|_A.$$

Then $\|\cdot\|_B$ is absolute. Let

$$\|(x_1, \cdots, x_N)\|_B = \|(\|x_1\|, \cdots, \|x_N\|)\|_B$$

for $(x_1, \ldots, x_N) \in X_1 \oplus \cdots \oplus X_N$. Then

$$\|(x_1, \ldots, x_N)\|_A = \|(x_1, \ldots, x_N)\|_B,$$

Thus we may assume that, without loss of generality, the original norm $\|\cdot\|_A$ on $\mathbb{R}^N$ is absolute to construct the A-direct sum $(X_1 \oplus \cdots \oplus X_N)_A$. Next let $c_k = \|(0, \ldots, 0, \overset{k}{\smile}{1}, 0, \ldots, 0)\|_B$ and define a norm $\|\cdot\|_C$ on $\mathbb{R}^N$ by

$$\|(a_1, \ldots, a_N)\|_C = \|(a_1/c_1, \ldots, a_N/c_N)\|_B.$$

Then $\|\cdot\|_C$ is absolute and normalized, and

$$\|(a_1, \ldots, a_N)\|_B = \|(c_1 a_1, \ldots, c_N a_N)\|_C$$

Consequently we have

$$\|(x_1, \ldots, x_N)\|_A = \|(c_1 x_1, \ldots, c_N x_N)\|_C$$

for $(x_1, \ldots, x_N) \in X_1 \oplus \cdots \oplus X_N$. Thus $(X_1 \oplus \cdots \oplus X_N)_A$ is isometric to $(X_1 \oplus \cdots \oplus X_N)_C = (X_1 \oplus \cdots \oplus X_N)_\psi$ with some function $\psi \in \Psi_N$.

In particular any Z-direct sum is isometrically isomorphic to a ψ-direct sum. The advantage of the latter is to allow us to use a convex function $\psi \in \Psi_N$ in our discussion, especially to construct examples.

We shall see some basic properties for direct sums. A Banach space X is called *strictly convex* if

$$x, y \in S_X, x \neq y \implies \left\| \frac{x+y}{2} \right\| < 1.$$

X is called *uniformly convex* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|x - y\| \geq \varepsilon, x, y \in S_X \implies \left\| \frac{x+y}{2} \right\| < 1 - \delta.$$

Theorem 3.3 ([21, 40]) *A ψ-direct sum $(X_1 \oplus \cdots \oplus X_N)_\psi$ is strictly (uniformly) convex if and only if $X_1, \ldots, X_N$ are strictly (uniformly) convex and ψ is strictly convex.*

Now, ψ is strictly convex if and only if $\|\cdot\|_\psi$ is strictly convex ([38]), we have the general A-direct sum version of this theorem by Theorem 3.2.

Theorem 3.4 *An A-direct sum $(X_1 \oplus \cdots \oplus X_N)_A$ is strictly (uniformly) convex if and only if $X_1, \ldots, X_N$ are strictly (uniformly) convex and $\|\cdot\|_A$ is strictly convex.*

For similar results for the dual notions, smoothness and uniform smoothness we refer the reader to Mitani-Oshiro-Saito [33].

4 Weak nearly uniform smoothness A Banach space X is called *weakly nearly uniformly smooth* (WNUS in short) if X is reflexive and $R(X) < 2$, $R(X)$ is the *García-Falset coefficient*:

$$R(X) = \sup\{\liminf_{n\to\infty} \|x_n + x\|\},$$

where the supremum is taken over all weakly null sequences $\{x_n\}$ in B_X and all $x \in B_X$. (cf. García-Falset [14]; we refer the reader to Kutzarova et al. [31] for the original definition; cf. [27]). A Banach space X is said to have the *fixed point property* for nonexpansive mappings (FPP in short) provided that for any bounded closed convex subset C of X every nonexpansive self-mapping T on C has a fixed point, where T is called nonexpansive if

$$\|Tx - Ty\| \le \|x - y\| \quad \forall x, y \in C.$$

Uniformly convex resp., uniformly smooth spaces are WNUS ([35]). We also have

Theorem 4.1 (García-Falset [15, 14]) *Every weakly nearly uniformly smooth space has FPP.*

For WNUS-ness of direct sums we have the following.

Theorem 4.2 (Kato-Tamura [27]) *Let $X_1, \ldots, X_N$ be of infinite dimension. Let $\psi \in \Psi_N$. Then, the following are equivalent.*
(i) *$(X_1 \oplus \cdots \oplus X_N)_\psi$ is WNUS.*
(ii) *$X_1, \ldots, X_N$ are WNUS and $\psi \notin \Psi_N^{(1)}$.*

Remark 4.3 *(i) The implication (ii) $\Rightarrow$ (i) is valid without the assumption on the dimension of X_j's.*
(ii) For the case some of X_j's are of finite dimension we refer the reader to [30].

If $\psi \in \Psi_N$ is strictly convex, $\psi \notin \Psi_N^{(1)}$ ([27]). Therefore, taking Remark 4.3(i) into account, the next previous result is derived from Theorem 4.2.

Corollary 4.4 (Dhompongsa et al. [6]) *Let $X_1, \ldots, X_N$ be arbitrary Banach spaces. Let $\psi \in \Psi_N$ be strictly convex. Then the following are equivalent.*
(i) *$(X_1 \oplus \cdots \oplus X_N)_\psi$ is WNUS.*
(ii) *$X_1, \ldots, X_N$ are WNUS.*

Recall that $\psi \notin \Psi_N^{(1)}$ if and only if $\|\cdot\|_\psi$ has Property T_1^N (Theorem 2.9). Owing to Theorem 3.2, Theorem 4.2 is reformulated in the general A-direct sum setting as follows.

Theorem 4.5 *Let $X_1, \ldots, X_N$ be infinite dimensional. Let $\|\cdot\|_A$ be an arbitrary norm on $\mathbb{R}^N$. Then the following are equivalent.*
(i) *$(X_1 \oplus \cdots \oplus X_N)_A$ is WNUS.*
(ii) *$X_1, \ldots, X_N$ are WNUS and $\|\cdot\|_A$ has Property T_1^N.*

5 Uniform non-squareness A Banach space X is called *uniformly non-square* (R. C. James [17]) (UNSQ in short) if there exists a constant $\varepsilon > 0$ such that

$$\min\{\|x+y\|, \|x-y\|\} \leq 2(1-\varepsilon) \quad \text{for all } x, y \in S_X.$$

It is immediate to see that uniformly convex spaces are strictly convex and UNSQ, while there is no implication between the latter two notions. The UNSQ-ness has been playing an improtant role in the geometry of Banach spaces and FPP. One of the most remarkable recent results is the following.

Theorem 5.1 (García-Falset et al. [16]) *UNSQ spaces have FPP.*

An important classical result says that UNSQ spaces are reflexive ([17]); in fact, super-reflexive ([18]); we shall mention about super-reflexivity again in Section 7. Thus UNSQ-ness lies between uniform convexity and super-reflexivity, as well as FPP. It is worth stating that some geometric constants have close connections with these notions. In fact UNSQ-ness are characterized by $C_{NJ}(X) < 2$ and also by $J(X) < 2$, where $C_{NJ}(X)$ and $J(X)$ are the von Neumann-Jordan and the James constants of a Banach space X ([39, 13]; cf. [24, 20]). Therefore, if $C_{NJ}(X) < 2$ or $J(X) < 2$, X is reflexive and has FPP. These constants have been calculated for many concrete Banach spaces (we omit precise descriptions).

Theorem 5.2 (Kato-Saito-Tamura [22]) *A ψ-direct sum $X \oplus_\psi Y$ is uniformly non-square if and only if X, Y are uniformly non-square and $\psi \neq \psi_1, \psi_\infty$, that is, $\|\cdot\|_\psi \neq \|\cdot\|_1$, $\|\cdot\|_\infty$.*

In this paper they posed the following problem:

Problem 1. Characterize the uniform non-squareness for $(X_1 \oplus \cdots \oplus X_N)_\psi$.

This problem is quite complicated since in the case $N \geq 3$ we need to remove much more convex functions in Ψ_N (norms in AN_N). Dowling and Saejung [10] presented a partial answer for Z-direct sums, a fortiori, for ψ-direct sums.

Theorem 5.3 (Dowling-Saejung [10]) *Assume that Z-norm $\|\cdot\|_Z$ or the dual norm $\|\cdot\|_Z^*$ on $\mathbb{R}^N$ is strictly monotone. Then the following are equivalent.*

(i) *$(X_1 \oplus \cdots \oplus X_N)_Z$ is UNSQ.*

(ii) *$X_1, \ldots, X_N$ are UNSQ and $\|\cdot\|_Z$ has Properties T_1^N and T_∞^N.*

For the case $N = 3$ they dropped the assumption on strict monotonicity, which answers Problem 1 for $N = 3$:

Theorem 5.4 (Dowling-Saejung [10]) *The following are equivalent.*

(i) *$(X_1 \oplus X_2 \oplus X_3)_Z$ is UNSQ.*

(ii) *X_1, X_2, X_3 are UNSQ and $\|\cdot\|_Z$ has Properties T_1^3 and T_∞^3.*

Any A-direct sum is isometric to a Z-direct sum, whence we have the following.

Theorem 5.5 *Let $\|\cdot\|_A$ be an arbitrary norm on $\mathbb{R}^N$. Then the following are equivalent.*

(i) *$(X_1 \oplus X_2 \oplus X_3)_A$ is UNSQ.*

(ii) *X_1, X_2, X_3 are UNSQ and $\|\cdot\|_A$ has Properties T_1^3 and T_∞^3.*

Remark 5.6 *Why did they succeed in the case $N = 3$? Later we shall see the reason, which is a quite natural consequence of a recent result by Kato and Tamura [30].*

In 2015 Dhompongsa, Kato and Tamura [8] gave more precise results for Theorem 5.3 in the A-direct sum setting.

Theorem 5.7 *Let $\|\cdot\|_A$ be an arbitrary norm on $\mathbb{R}^N$. Assume that $\|\cdot\|_A$ is strictly monotone. Then the following are equivalent.*
(i) $(X_1 \oplus \cdots \oplus X_N)_A$ *is UNSQ.*
(ii) $X_1, \ldots, X_N$ *are UNSQ and the norm* $\|\cdot\|_A$ *has Property* T_1^N.

Theorem 5.8 *Let $\|\cdot\|_A$ be an arbitrary norm on $\mathbb{R}^N$. Assume that the dual norm $\|\cdot\|_A^*$ is strictly monotone. Then the following are equivalent.*
(i) $(X_1 \oplus \cdots \oplus X_N)_A$ *is UNSQ.*
(ii) $X_1, \ldots, X_N$ *are UNSQ and the norm* $\|\cdot\|_A$ *has Property* T_∞^N.

If $(X_1 \oplus \cdots \oplus X_N)_A$ is UNSQ, the norm $\|\cdot\|_A$ has Properties T_1^N and T_∞^N. (This is a corresponding result to Theorem 5.2; cf. [8, Cororally 4.5] and [30]). Therefore from Theorems 5.7 and 5.8 the following A-direct sum version of Theorem 5.3 is derived.

Corollary 5.9 *Let $\|\cdot\|_A$ be an arbitrary norm on $\mathbb{R}^N$. Assume that $\|\cdot\|_A$ or $\|\cdot\|_A^*$ is strictly monotone. Then the following are equivalent.*
(i) $(X_1 \oplus \cdots \oplus X_N)_A$ *is UNSQ.*
(ii) $X_1, \ldots, X_N$ *are UNSQ and* $\|\cdot\|_A$ *has Properties* T_1^N *and* T_∞^N.

Remark 5.10 *Dhompongsa-Kato-Tamura [8] first proved Theorems 5.7, 5.8, and Corollary 5.9 for ψ-direct sums by means of $\Psi_N^{(1)}$ and $\Psi_N^{(\infty)}$, and then derived these results for A-direct sums by Theorems 3.2 and 2.9. We shall see below the ψ-direct sum version of Theorem 5.5.*

Theorem 5.5′ ([8]) *Let $\psi \in \Psi_N$. Assume that $\|\cdot\|_\psi$ is strictly monotone. Then the following are equivalent.*
(i) $(X_1 \oplus \cdots \oplus X_N)_\psi$ *is UNSQ.*
(ii) $X_1, \ldots, X_N$ *are UNSQ and* $\psi \notin \Psi_N^{(1)}$.

Now we are in a position to explain why Dowling-Saejung [10] succeeded for the case $N = 3$. Very recently by introducing the class $\Psi_N^{(mix)}$ as the class which should be excluded, Kato-Tamura [30, in preparation] answered Problem 1 without the assumption on strict monotonicity:

A ψ-direct sum $(X_1 \oplus \cdots \oplus X_N)_\psi$ is UNSQ if and only if $X_1, \ldots, X_N$ are UNSQ and $\psi \notin \Psi_N^{(mix)}$.

(This will appear elsewhere.) They showed $\Psi_3^{(mix)} = \Psi_3^{(1)} \cup \Psi_3^{(\infty)}$ for $N = 3$ and obtained the following as a corollary.

Theorem 5.11 *Let $\psi \in \Psi_N$. Then the following are equivalent.*
(i) $(X_1 \oplus X_2 \oplus X_3)_\psi$ *is UNSQ.*
(ii) X_1, X_2, X_3 *are UNSQ and* $\psi \notin \Psi_3^{(1)} \cup \Psi_3^{(\infty)}$.

According to Theorem 2.9, $\psi \notin \Psi_3^{(1)} \cup \Psi_3^{(\infty)}$ if and only if $\|\cdot\|_\psi$ has Properties T_1^3 and T_∞^3. As any Z-direct sum is isometric to a ψ-direct sum, we have Dowling-Saejung's result for $(X_1 \oplus X_2 \oplus X_3)_Z$. Theorem 5.3 is also derived from the above-announced result by Kato-Tamura [30].

We shall conclude this section with the following result.

Theorem 5.12 (Betiuk-Pilarska and Prus [2]) *The following are equivalent.*
(i) $(X_1 \oplus \cdots \oplus X_N)_Z$ *is UNSQ.*
(ii) $X_1, \ldots, X_N$ *are UNSQ and* $(\mathbb{R}^N, \|\cdot\|_Z)$ *is UNSQ.*

Here it remains unknown when the space $(\mathbb{R}^N, \|\cdot\|_Z)$ is UNSQ. Kato-Tamura [30] answered to this question by introducing Property T^N_{mix} in the A-direct sum setting.

6 Uniform non-ℓ_1^n-ness A Banach space X is called *uniformly non-ℓ_1^n* if there exists ε $(0 < \varepsilon < 1)$ for which

$$\forall x_1, \cdots, x_n \in S_X,\ \exists \theta = (\theta_j)\ (\theta_j = \pm 1)\ s.t.\ \left\| \sum_{j=1}^{n} \theta_j x_j \right\| \leq n(1-\epsilon). \tag{6.1}$$

When $n = 2$ the uniform non-ℓ_1^2-ness coincides with the uniform non-squareness. For $n = 3$ uniformly non-ℓ_1^3 spaces are called *uniformly non-octahedral.* In the case $n = 1$ no Banach space is uniformly non-ℓ_1^1.

Proposition 6.1 *Uniformly non-ℓ_1^n spaces are uniformly non-ℓ_1^{n+1}.*

Theorem 5.2 for UNSQ-ness of $X \oplus_\psi Y$ is extended to uniform non-ℓ_1^n-ness.

Theorem 6.2 (Kato-Saito-Tamura [23]) *Assume that neither X nor Y is uniformly non-ℓ_1^{n-1}. Then the following are equivalent.*
(i) $X \oplus_\psi Y$ *is uniformly non-ℓ_1^n.*
(ii) X *and* Y *are uniformly non-ℓ_1^n and* $\psi \neq \psi_1, \psi_\infty$.

Remark 6.3 *(i) Theorem 6.2 covers Theorem 5.2 as the case $n = 2$, since no Banach space is uniformly non-ℓ_1^1.*
(ii) We cannot drop the assumption "neither X nor Y is uniformly non-ℓ_1^{n-1}".
(iii) For the N Banach spaces case some results were obtained under the assumption that the ψ-norm $\|\cdot\|_\psi$ is strictly monotone in Kato and Tamura [29].

As before we obtain the A-direct sum version of Theorem 6.2.

Theorem 6.4 *Let $\|\cdot\|_A$ be an arbitrary norm on $\mathbb{R}^N$. Assume that neither X nor Y is uniformly non-ℓ_1^{n-1}. Then the following are equivalent.*
(i) $X \oplus_A Y$ *is uniformly non-ℓ_1^n.*
(ii) X *and* Y *are uniformly non-ℓ_1^n and* $\|\cdot\|_A \neq \|\cdot\|_1, \|\cdot\|_\infty$.

Now, we shall look into the extreme cases, ℓ_1- and ℓ_∞-sums, which were excluded in Theorems 6.2 (and 6.4). According to this theorem, $X \oplus_1 Y$ and $X \oplus_\infty Y$ cannot be UNSQ for all X and Y, while $X \oplus_1 Y$ and $X \oplus_\infty Y$ can be uniformly non-ℓ_1^n $(n \geq 3)$ if either X or Y is uniformly non-ℓ_1^{n-1}. In fact the following are true.

Theorem 6.5 (Kato-Saito-Tamura [23]) *The following are equivalent.*
(i) $X \oplus_1 Y$ *is uniformly non-ℓ_1^n,* $n \geq 3$.
(ii) *There exist $n_1, n_2 \in \mathbb{N}$ with $n_1 + n_2 = n - 1$ such that X is uniformly non-$\ell_1^{n_1+1}$ and Y is uniformly non-$\ell_1^{n_2+1}$.*

As corollaries the following are obtained.

Corollary 6.6 *The following are equivalent.*
(i) $X \oplus_1 Y$ *is uniformly non-ℓ_1^3.*
(ii) X *and* Y *are UNSQ.*

Corollary 6.7 *The following are equivalent.*
(i) $X \oplus_1 Y$ *is uniformly non-*ℓ_1^4*.*
(ii) X *is UNSQ and* Y *is uniformly non-octahedral.*

Theorem 6.5 is extended as follows.

Theorem 6.8 *The following are equivalent.*
(i) $(X_1 \oplus \cdots \oplus X_N)_1$ *is uniformly non-*ℓ_1^{N+1}*.*
(ii) $X_1, \ldots, X_N$ *are UNSQ.*

This implies especially that the space ℓ_1^n is uniformly non-ℓ_1^{n+1}. For ℓ_∞-sums we have the following.

Theorem 6.9 (Kato-Tamura [26]) *Let* $n \geq 2$*. The following are equivalent.*
(i) $(X_1 \oplus \cdots \oplus X_{2^n-1})_\infty$ *is uniformly non-*ℓ_1^{n+1}*.*
(ii) $X_1, \ldots, X_{2^n-1}$ *are UNSQ.*

Corollary 6.10 *The following are equivalent.*
(1) $(X \oplus Y \oplus Z)_\infty$ *is uniformly non-*ℓ_1^3*.*
(2) X*,* Y *and* Z *are UNSQ.*

Remark 6.11 *Recall that the* ℓ_1*-sum* $X \oplus_1 Y$ *is uniformly non-*ℓ_1^3 *if and only if* X *and* Y *are UNSQ. Contrary to this, if* X *and* Y *are UNSQ, the* ℓ_∞*-sum* $X \oplus_\infty Y$ *is uniformly non-*ℓ_1^3 *([23, Corollary 5.3bis]), but the converse is not true ([23, Remark 5.5]). We added one more space* Z *to obtain the equivalence in Corollary 6.9. Compare also Theorems 6.7 and 6.8. These observations show one aspect of the defference between* ℓ_1*- and* ℓ_∞*-sums.*

7 Applications All UNSQ spaces have FPP. Thus it is natural to ask whether all uniformly non-octahedral spaces have FPP. We have the following.

Theorem 7.1 (Kato-Tamura [26]) *Let* X *be uniformly non-octahedral. If* X *is isometric to an* ℓ_∞*-sum of 3 Banach spaces,* X *has FPP, while* X *is not UNSQ.*

More generally we have

Theorem 7.2 *Let* X *be uniformly non-*ℓ_1^{n+1}*. If* X *is isometric to an* ℓ_∞*-sum of* $2^n - 1$ *Banach spaces,* X *has FPP, while* X *is not UNSQ.*

Example 7.3 *Let* $1 < p < \infty$*. Since* L_p *is uniformly convex, it is UNSQ. Therefore the space* $X = (L_p \oplus L_p \oplus L_p)_\infty$ *is uniformly non-octahedral by Theorem 6.9, and hence* X *has FPP by Theorem 7.1, while it is not UNSQ since* X *contains* ℓ_∞^3 *as a subspace.*

In the same way, the ℓ_∞*-sum of* $2^n - 1$ L_p*'s is uniformly non-*ℓ_1^{n+1}*, which is weaker than uniform non-octahedralness, has FPP but is not UNSQ.*

By using Theorem 4.2 a plenty of Banach spaces with FPP which fail to be UNSQ is constructed.

Example 7.4 (Kato-Tamura [27]) *Let* $N \geq 3$ *and let* $\varphi, \psi_1 \in \Psi_2$*,* $\varphi \neq \psi_1$*. Let*

$$\begin{aligned}&\psi(s_1, \ldots, s_{N-1})\\ &= \max\left\{ \|(1 - \sum_{i=1}^{N-1} s_i, s_1)\|_\varphi,\ \|(s_1, s_2)\|_\varphi,\ \|(s_2, s_3)\|_\varphi, \cdots, \|(s_{N-2}, s_{N-1})\|_\varphi \right\}\end{aligned}$$

$$\text{for } (s_1, \ldots, s_{N-1}) \in \Delta_N.$$

Then $\psi \in \Psi_N$ *and*

$$\|(a_1, a_2 \ldots, a_N)\|_\psi = \max\{\|(a_1, a_2)\|_\varphi, \|(a_2, a_3)\|_\varphi, \ldots, \|(a_{N-1}, a_N)\|_\varphi\} \quad \text{for } (a_1, \ldots, a_N) \in \mathbb{R}^N.$$

Further, $\psi \notin \Psi_N^{(1)}$ *and* $\|\cdot\|_\psi$ *is not UNSQ.*

Since WNUS spaces have FPP, we have the following.

Theorem 7.5 (Kato-Tamura [27]) *Let* $X_1, \ldots, X_N$ *be WNUS* ($N \geq 3$)*. Let* $\psi \in \Psi_N$ *be as in Example 7.4. Then* $(X_1 \oplus \cdots \oplus X_N)_\psi$ *has FPP, whereas it is not UNSQ.*

Indeed, since $\psi \notin \Psi_N^{(1)}$, $X = (X_1 \oplus \cdots \oplus X_N)_\psi$ is WNUS by Theorem 4.2 (with Remark 4.3 (i)) and hence X has FPP. On the other hand, X is not UNSQ as $(\mathbb{R}^N, \|\cdot\|_\psi)$ is not UNSQ.

Next, as $\psi_\infty \notin \Psi_N^{(1)}$, we have

Theorem 7.6 *Let* $X_1, \ldots, X_N$ *be WNUS. Then* $(X_1 \oplus \cdots \oplus X_N)_\infty$ *has FPP, whereas it is not UNSQ.*

Example 7.7 *Let* $1 < p_k < \infty,\ 1 \leq k \leq N$.

(i) Let $\psi \in \Psi_N$ *be as in Example 7.4. Since* L_{p_k} *are uniformly convex and hence WNUS, the space* $X = (L_{p_1} \oplus \cdots \oplus L_{p_N})_\psi$ *has FPP, while* X *is not UNSQ by Theorem 7.6.*

(ii) The ℓ_∞*-sum* $X = (L_{p_1} \oplus \cdots \oplus L_{p_N})_\infty$ *is WNUS and hence has FPP. On the other hand, the space* X *is not UNSQ.*

Finally we shall discuss super-reflexivity. A Banach space Y is said to be *finitely representable in* X provided for any $\epsilon > 0$ and for any finite dimensional subspace F of Y there is a finite dimensional subspace E of X with $\dim F = \dim E$ such that $d(F, E) := \inf\{\|T\|\|T^{-1}\| : T \text{ is an isomorphism of } F \text{ onto } E\} < 1 + \epsilon$. A Banach space X is called *super-reflexive* if every Banach space Y which is finitely representable in X is reflexive ([18]; cf. [1]). The next celebrated result states the connection between super-reflexivity and uniform convexity as well as UNSQ-ness.

Theorem 7.8 (cf. [12, 34, 18]) *The following are equivalent.*

(i) X *is super-reflexive.*
(ii) X *admits an equivalent uniformly convex norm.*
(iii) X *admits an equivalent uniformly non-square norm.*

UNSQ spaces are super-reflexive ([17]), whereas uniformly non-octahedral spaces are not always reflexive ([19]). For ℓ_1-sum spaces we have the following ([26]).

Theorem 7.9 *Let* X *be a uniformly non-octahedral Banach space which is isometric to the* ℓ_1*-sum of 2 Banach spaces. Then* X *is super-reflexive.*

Indeed, if X is isometric to $X_1 \oplus_1 X_2$, $X_1 \oplus_1 X_2$ is uniformly non-octahedral, from which it follows that X_1 and X_2 are UNSQ by Corollary 6.5, hence super-reflexive. Consequently the ℓ_1-sum, and hence X is super-reflexive.

By Theorem 6.9 we have the similar result for ℓ_∞-sum spaces.

Theorem 7.10 *Let* X *be a uniformly non-octahedral Banach space which is isometric to the* ℓ_∞*-sum of 3 Banach spaces. Then* X *is super-reflexive.*

Acknowledgement. The first author would like to thank the organizers of the conference FIM&ISME2017, especially, Professors J. Watada, Y. Yabuuchi and H. Sakai for their kind invitation and warm hopsitality.

References

[1] B. Beauzamy, Introduction to Banach spaces and their geometry, 2nd ed., North-Holland, 1985.

[2] A. Betiuk-Pilarska and S. Prus, Uniform nonsquareness of direct sums Banach spaces, Topol. Methods Nonlinear Anal. **34** (2009), 181–186.

[3] A. Betiuk-Pilarska and S. Prus, Moduli $RW(a, X)$ and $MW(X)$ of direct sums of Banach spaces, J. Nonlinear Convex Anal. **18** (2017), 309–315.

[4] R. Bhatia, Matrix Analysis, Springer-Verlag, New York, 1997.

[5] F. F. Bonsall and J. Duncan, *Numerical Ranges II*, London Math. Soc. Lecture Note Ser. **10** (1973).

[6] S. Dhompongsa, A. Kaewcharoen and A. Kaewkhao, Fixed point property of direct sums, Nonlinear Anal. **63**(2005), e2177–e2188.

[7] S. Dhompongsa, A. Kaewkhao and S. Saejung, Uniform smoothness and U-convexity of ψ-direct sums, J. Nonlinear Convex Anal. **6** (2005), 327–338.

[8] S. Dhompongsa, M. Kato and T. Tamura, Uniform non-squareness for A-direct sums of Banach spaces with a strictly monotone norm, Linear Nonlinear Anal. **1** (2015), 247–260.

[9] P. N. Dowling, On convexity properties of ψ-direct sums of Banach spaces, J. Math. Anal. Appl. **288** (2003), 540–543.

[10] P. N. Dowling and S. Saejung, Non-squareness and uniform non-squareness of Z-direct sums, J. Math. Anal. Appl. **369** (2010), 53–59.

[11] P. N. Dowling and B. Turett, Complex strict convexity of absolute norms on $\mathbb{C}^n$ and direct sums of Banach spaces, J. Math. Anal. Appl. **323** (2006), 930–937.

[12] P. Enflo, Banach spaces which can be given an equivalent uniformly convex norm, Israel J. Math. **13** (1972), 281–288.

[13] J. Gao and K. S. Lau, On two classes of Banach spaces with uniform normal structure, Studia Math. **99** (1991), 41–56.

[14] J. García-Falset, Stability and fixed points for nonexpansive mappings, Houston J. Math. **20** (1994), 495–506.

[15] J. García-Falset, The fixed point property in Banach spaces with the NUS property, J. Math. Anal. Appl. **215** (1997), 532–542.

[16] J. García-Falset, E. Llorens-Fuster and E. M. Mazcuñan-Navarro, Uniformly nonsquare Banach spaces have the fixed point property for nonexpansive mappings, J. Funct. Anal. **233** (2006), 494–514.

[17] R. C. James, Uniformly non-square Banach spaces, Ann. of Math. **80** (1964), 542–550.

[18] R. C. James, Super-reflexive Banach spaces, Canad. J. Math. **24** (1972), 896–904.

[19] R. C. James, A nonreflexive Banach space that is uniformly non-octahedral, Israel J. Math. **18** (1974), 145–155.

[20] M. Kato, L. Maligranda and Y. Takahashi, On James, Jordan-von Neumann constants and the normal structure coefficient of Banach spaces, Studia Math. **144** (2001), 275–295.

[21] M. Kato, K.-S. Saito and T. Tamura, On the ψ-direct sums of Banach spaces and convexity, J. Aust. Math. Soc. **75** (2003), 413–422.

[22] M. Kato, K.-S. Saito and T. Tamura, Uniform non-squareness of ψ-direct sums of Banach spaces $X \oplus_\psi Y$, Math. Inequal. Appl. **7** (2004), 429–437.

[23] M. Kato, K.-S. Saito and T. Tamura, Uniform non-ℓ_1^n-ness of ψ-direct sums of Banach spaces, J. Nonlinear Convex Anal. **11** (2010), 113–133.

[24] M. Kato and Y. Takahashi, On the von Neumann-Jordan constant for Banach spaces, Proc. Amer. Math. Soc. **125** (1997), 1055–1062.

[25] M. Kato and T. Tamura, Uniform non-ℓ_1^n-ness of ℓ_1-sums of Banach spaces, Comment. Math. Prace Mat. **47** (2007), 161–169.

[26] M. Kato and T. Tamura, Uniform non-ℓ_1^n-ness of ℓ_∞-sums of Banach spaces, Comment. Math. **49** (2009), 179–187.

[27] M. Kato and T. Tamura, Weak nearly uniform smoothness of the ψ-direct sums $(X_1 \oplus \cdots \oplus X_N)_\psi$, Comment. Math. **52** (2012), 171–198.

[28] M. Kato and T. Tamura, Direct sums of Banach spaces with FPP which fail to be uniformly non-square, J. Nonlinear Convex Anal. **16** (2015), 231–241.

[29] M. Kato and T. Tamura, On the uniform non-ℓ_1^n-ness and new classes of convex functions, J. Nonlinear Convex Anal. **16** (2015), 2225–2241.

[30] M. Kato and T. Tamura, On the uniform non-squareness of direct sums of Banach spaces, in preparation.

[31] D. Kutzarova, S. Prus and B. Sims, Remarks on orthogonal convexity of Banach spaces, Houston J. Math. **19** (1993), 603–614.

[32] J. Markowicz and S. Prus, James constant, García-Falset coefficient and uniform Opial property in direct sums of Banach spaces, J. Nonlinear Convex Anal. **17** (2016), 2237–2253.

[33] K-I. Mitani, S. Oshiro and K.-S. Saito, Smoothness of ψ-direct sums of Banach spaces, Math. Inequal. Appl. **8** (2005), 147–157.

[34] G. Pisier, Martingales with values in uniformly convex spaces, Israel J. Math. **20** (1975), 326–350.

[35] S. Prus, Nearly uniformly smooth Banach spaces, Boll. U. M. I.**(7)3-B** (1989), 507–521.

[36] K.-S. Saito and M. Kato, Uniform convexity of ψ-direct sums of Banach spaces, J. Math. Anal. Appl. **277** (2003), 1–11.

[37] K.-S. Saito, M. Kato and Y. Takahashi, Von Neumann-Jordan constant of absolute normalized norms on $\mathbb{C}^2$, J. Math. Anal. Appl. **244** (2000), 515–532.

[38] K.-S. Saito, M. Kato and Y. Takahashi, On absolute norms on $\mathbb{C}^n$, J. Math. Anal. Appl. **252** (2000), 879–905.

[39] Y. Takahashi and M. Kato, Von Neumann-Jordan constant and uniformly non-square Banach spaces, Nihonkai Math. J. **9** (1998), 155–169.

[40] Y. Takahashi, M. Kato and K.-S. Saito, Strict convexity of absolute norms on $\mathbb{C}^2$ and direct sums of Banach spaces, J. Inequal. Appl. **7** (2002), 179–186.

[41] T. Tamura, On Dominguez-Benavides coefficient of ψ-direct sums $(X_1 \oplus \cdots \oplus X_N)_\psi$ of Banach spaces, Linear Nonlinear Anal. **3** (2017), 87–99.

[42] T. Zachariades, On ℓ_ψ spaces and infinite ψ-direct sums of Banach spaces, Rocky Mount. J. Math. **41** (2011), 971-997.

[43] A. Wiśnicki, On the fixed points of nonexpansive mappings in direct sums of Banach spaces, Studia Math., **207** (2011), 75–84.

(M. KATO) Faculty of Engineering, Kyushu Institute of Technology, Kitakyushu 804-8550, Japan
Email: katom@mns.kyutech.ac.jp

(T. SOBUKAWA) Global education center, Waseda University, Tokyo 169-8050, Japan
Email: sobu@waseda.jp

(T. TAMURA) Graduate School of Social Sciences, Chiba University, Chiba 263-8522, Japan
Email: ttakayuki@faculty.chiba-u.jp