CAUCHY'S THEOREM FOR B-ALGEBRAS

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ABSTRACT. In this paper, we establish the Cauchy's Theorem for B-algebras. We also present some implications of Lagrange's Theorem and Cauchy's Theorem for B-algebras. In particular, the concept of B_p-algebras is introduced.

1 Introduction In [9], the notion of B-algebras was introduced by J. Neggers and H.S. Kim. A *B-algebra* is an algebra $(X; *, 0)$ of type (2, 0) (that is, a nonempty set X with a binary operation $*$ and a constant 0) satisfying the following axioms for all $x, y, z \in X$: (I) $x*x = 0$, (II) $x*0 = x$, (III) $(x*y)*z = x*(z*(0*y))$. A B-algebra $(X; *, 0)$ is *commutative* [9] if $x * (0 * y) = y * (0 * x)$ for all $x, y \in X$. In [10], J. Neggers and H.S. Kim introduced the notions of a subalgebra and normality of B-algebras and some of their properties are established. A nonempty subset N of X is called a *subalgebra* of X if $x * y \in N$ for any x, $y \in N$. It is called *normal* in X if for any $x * y, a * b \in N$ implies $(x * a) * (y * b) \in N$. A normal subset of X is a subalgebra of X. There are several properties of B-algebras as established by some authors [1−12]. The following properties are used in this paper, for any $x, y, z \in X$, we have (P1) $0 * (0 * x) = x$ [9], (P2) $x * y = 0 * (y * x)$ [11], (P3) $x*(y*z) = (x*(0*z))*y$ [9], (P4) $x*y = x*z$ implies $y = z$ [3], (P5) $(0*x)*(y*x) = 0*y$ [9]. In [2], J.S. Bantug and J.C. Endam established the Lagrange's Theorem for B-algebras. In this paper, we provide some partial results on the converse of this theorem. In particular, we establish the Cauchy's Theorem for B-algebras. As a consequence, we also introduce the concept of B_p-algebras. Throughout this paper, X means a B-algebra $(X; *, 0)$.

2 Preliminaries This section presents some concepts and results needed in this paper. We start with some examples of B-algebras.

Example 2.1. [9] Let $X = \{0, 1, 2\}$ be a set with the following table of operation:

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Example 2.2. [9] Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with the following table of operation:

$*$	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

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In [7], if S is a subset of X, then $\langle S\rangle_B$ is the intersection of all subalgebra H of X such that $S \subseteq H$, and the subalgebra $\langle S\rangle_B$ of X is called the *subalgebra generated by* S. If $X = \langle S\rangle_B$, then S is called a *set of generators* for X. Moreover, $\langle S\rangle_B$ is the smallest subalgebra of X containing S. If either $S = \varnothing$ or $S = \{0\}$, then $\langle S\rangle_B = \{0\}$. If S is a subalgebra of X, then $\langle S\rangle_B = S$. In particular, $\langle X\rangle_B = X$.

Let $x \in X$. In [9], J. Neggers and H.S. Kim defined $x^n = x^{n-1} * (0 * x)$ for $n \geq 1$ and $x^0 = 0$. Then $x^m * x^n = x^{m-n}$ if $m \geq n$ and $x^m * x^n = 0 * x^{n-m}$ otherwise. In [7], for each $x \in X$, N.C. Gonzaga and J.P. Vilela defined $-x = 0 * x$ and $x^{-n} = (-x)^n$ for each $n \geq 1$. In [5], J.C. Endam and R.C. Teves defined $x^m = 0 * x^{-m}$ for $m \leq -1$. If $m \geq 1$, then $x^m = 0 * (0 * x^m) = 0 * x^{-m}$. In effect, $x^m = 0 * x^{-m}$ for any $m \in \mathbb{Z}$. Furthermore, in [7], we have $x^m * x^n = x^{m-n}$, $(x^m)^n = x^{mn}$ for all $m, n \in \mathbb{Z}$, and $\langle x\rangle_B = \{x^n : n \in \mathbb{Z}\}$. If there exists a positive integer n such that $x^n = 0$, then the smallest such positive integer is denoted by $|x|_B$. If no such positive integer n exists, then we say that $|x|_B$ is infinite. If $A \subseteq X$, then we denote $|A|_B$ as the cardinality of A.

Let H and K be subalgebras of X. In [4], we define the subset HK of X to be the set $HK = \{x \in X : x = h * (0 * k) \text{ for some } h \in H, k \in K\}$. Clearly, we have $H \subseteq HK$, $H \subseteq KH$, $K \subseteq HK$, and $K \subseteq KH$. Moreover, if $H \subseteq K$, then $HK = KH = K$. Also, HK is a subalgebra of X if and only if $HK = KH$ if and only if $HK = \langle H \cup K\rangle_B$. A B-algebra X is called a *cyclic B-algebra* [7] if there exists $x \in X$ such that $X = \langle x\rangle_B$. Every cyclic B-algebra is commutative, but the converse need not be true. In [5], if $X = \langle x\rangle_B$ is a cyclic B-algebra with $|X|_B = m > 1$ and if H is a nontrivial subalgebra of X, then $H = \langle x^k\rangle_B$ for some integer $k > 1$ such that k divides m and $|H|_B$ divides m. Furthermore, for every positive divisor d of m, there exists a unique subalgebra H of X with $|H|_B = d$.

Let H be a subalgebra of X and $x \in X$. Let $xH = \{x * (0 * h) : h \in H\}$ and $Hx = \{h * (0 * x) : h \in H\}$, called the *left* and *right B-cosets* of H in X, respectively. If X is commutative, then $xH = Hx$ for all $x \in X$. Observe that $0H = H = H0$ and $x = x * (0 * 0) \in xH$ and $x = 0 * (0 * x) \in Hx$. It is easy to see that $xH = H$ if and only if $x \in H$.

Theorem 2.3. *[2] Let H be a subalgebra of X and $a, b \in X$. Then*
i. $aH = bH$ if and only if $(0 * b) * (0 * a) \in H$
ii. $Ha = Hb$ if and only if $a * b \in H$.

In [2], if H is a subalgebra of X, then $\{xH : x \in X\}$ forms a partition of X and there is a one-one correspondence of the set of all left B-cosets of H in X onto the set of all right B-cosets of H in X. Thus, we define the number of distinct left (or right) B-cosets, written $[X : H]_B$, of H in X as the *index* of H in X. If X is finite, then clearly $[X : H]_B$ is finite.

Theorem 2.4. *[2] (Lagrange's Theorem for B-algebras) Let H be a subalgebra of a finite B-algebra X. Then $|X|_B = [X : H]_B |H|_B$.*

Corollary 2.5. *[2] Let $|X|_B = p$, where p is prime. Then X is cyclic.*

Theorem 2.6. *[2] If H, K are finite subalgebras of X, then* $|HK|_B = \dfrac{|H|_B |K|_B}{|H \cap K|_B}$.

3 Some Implications of Lagrange's Theorem for B-algebras We now prove some results where Lagrange's Theorem plays a role.

Proposition 3.1. *Let X be a noncyclic B-algebra with $|X|_B = p^2$, where p is prime. Then $|x|_B = p$ for every nonzero $x \in X$.*

Proof. Let $x \in X$ and $x \neq 0$. By Lagrange's Theorem, $|x|_B$ divides $|X|_B = p^2$. Hence, $|x|_B$ is equal to 1, p, or p^2. If $|x|_B = p^2$, then $\langle x \rangle_B = X$ and so X is cyclic, a contradiction. Since $x \neq 0$, $|x|_B \neq 1$. Thus, $|x|_B = p$. □

Proposition 3.2. *If X is a B-algebra with prime order, then X has only the trivial subalgebras.*

Proof. Suppose that $|X|_B = p$, where p is prime. Let H be a subalgebra of X. By Lagrange's Theorem, $|H|_B$ is 1 or p. Thus, $H = \{0\}$ or $H = X$. □

Proposition 3.3. *Let $|X|_B = p^n$, where p is prime and $n \geq 1$. Then X contains an element of order p.*

Proof. Let $x \in X$ and $x \neq 0$. Then $H = \langle x \rangle_B$ is a cyclic subalgebra of X. By Lagrange's Theorem, $|H|_B$ divides $|X|_B = p^n$. Hence, $|H|_B = p^m$ for some $m \in \mathbb{Z}$, $0 < m \leq n$. It follows that for every divisor d of p^m, there exists a subalgebra of order d. In particular, for p, there exists a subalgebra K of H such that $|K|_B = p$. By Corollary 2.5, K is cyclic and so there exists $y \in K$ such that $K = \langle y \rangle_B$ and y is of order p. Hence, X contains an element of order p. □

Proposition 3.4. *Let X be a finite commutative B-algebra such that X contains two distinct elements of order 2. Then $|X|_B$ is a multiple of 4.*

Proof. Let x and y be two distinct elements of order 2. Let $H = \{0, x\}$ and $K = \{0, y\}$. Now, H and K are subalgebras of X. Since X is commutative, $HK = \{0, x, y, x * (0 * y)\}$ is a subalgebra of X of order 4. By Lagrange's Theorem, $|HK|_B = 4$ divides $|X|_B$. Thus, $|X|_B$ is a multiple of 4. □

The above result need not be true if X is not commutative. For instance, consider the B-algebra $X = \{0, 1, 2, 3, 4, 5\}$ in Example 2.2. Note that X is not commutative. Now, 3 and 4 are elements of X with $|3|_B = 2$ and $|4|_B = 2$. However, 4 does not divide $|X|_B = 6$.

Proposition 3.5. *Let X be a B-algebra with $|X|_B = pq$, where p and q are prime numbers. Then every proper subalgebra of X is cyclic.*

Proof. Let H be a proper subalgebra of X. By Lagrange's Theorem, $|H|_B$ is 1, p, q, or pq. Since H is proper, $|H|_B$ is p or q. By Corollary 2.5, H is cyclic. □

Proposition 3.6. *Let H and K be subalgebras of a finite B-algebra X such that $|H|_B > \sqrt{|X|_B}$ and $|K|_B > \sqrt{|X|_B}$. Then $|H \cap K|_B > 1$.*

Proof. Suppose that H and K are subalgebras of a finite B-algebra X such that $|H|_B > \sqrt{|X|_B}$ and $|K|_B > \sqrt{|X|_B}$. By Theorem 2.6, $|H \cap K|_B = \dfrac{|H|_B |K|_B}{|HK|_B}$. Since $|H|_B > \sqrt{|X|_B}$ and $|K|_B > \sqrt{|X|_B}$, it follows that $|H|_B |K|_B > |X|_B$. Since $|HK|_B \leq |X|_B$, it follows that $\dfrac{|X|_B}{|HK|_B} \geq 1$. Therefore, $|H \cap K|_B = \dfrac{|H|_B |K|_B}{|HK|_B} > \dfrac{|X|_B}{|HK|_B} \geq 1$. □

Proposition 3.7. *Let $|X|_B = pq$, where p and q are distinct primes with $p > q$. Then X has at most one subalgebra of order p.*

Proof. Suppose that H and K are subalgebras with $|H|_B = p = |K|_B$. Then $|H|_B > \sqrt{|X|_B}$ and $|K|_B > \sqrt{|X|_B}$. By Proposition 3.6, $|H \cap K|_B > 1$. Thus, $|H \cap K|_B = p$ and so $H = K$. □

4 Cauchy's Theorem for B-algebras This section establishes the Cauchy's Theorem for B-algebras and it also provides some implications of this theorem. We start with a simple observation given in the following lemma.

Lemma 4.1. *Let $a \in X$. Then $a \in Z(X)$ if and only if $[X : C(a)]_B = 1$ if and only if $C(a) = X$.*

Let $a \in X$. An element $b \in X$ is said to be a *conjugate of* a in X if there exists $c \in X$ such that $b = c * (c * a)$. Let $R = \{(a, b) \in X \times X\colon\ b$ is a conjugate of $a\}$.

Theorem 4.2. *Let $a \in X$. Then the relation R on X is an equivalence relation.*

Proof. Since $a = 0 * (0 * a)$, a is conjugate to a. Thus, R is reflexive. Let $(a, b) \in R$. Then there exists $c \in X$ such that $b = c * (c * a)$. Multiplying both sides by $0 * c$ twice, we have $(0 * c) * ((0 * c) * b) = (0 * c) * [(0 * c) * (c * (c * a))]$. By (P2), (P3), (I), and (P1), we obtain

$$\begin{aligned}(0 * c) * ((0 * c) * b) &= (0 * c) * [(0 * c) * (c * (c * a))] \\ &= (0 * c) * [((0 * c) * (0 * (c * a))) * c] \\ &= (0 * c) * [((0 * c) * (a * c)) * c] \\ &= (0 * c) * [(0 * a) * c)] \\ &= ((0 * c) * (0 * c)) * (0 * a) \\ &= 0 * (0 * a) \\ &= a.\end{aligned}$$

Hence, a is conjugate to b. Thus, R is symmetric. Let $(a, b), (b, c) \in R$. Then there exist $u, v \in X$ such that $b = u * (u * a)$ and $c = v * (v * b)$. Now, by (P2) and (P3), we obtain

$$\begin{aligned}c &= v * (v * b) \\ &= v * [v * (u * (u * a))] \\ &= v * [(v * (0 * (u * a))) * u] \\ &= v * [(v * (a * u)) * u] \\ &= (v * (0 * u)) * (v * (a * u)) \\ &= (v * (0 * u) * [(v * (0 * u)) * a]\end{aligned}$$

Hence, $(a, c) \in R$ and so R is transitive. Therefore, R is an equivalence relation on X. □

The equivalence relation R in Theorem 4.2 is called *conjugacy* on X. The equivalence class of $a \in X$, denoted by $[a]_c$, of the relation R is called the *conjugacy class of* a in X.

Example 4.3. Consider the B-algebra $X = \{0, 1, 2, 3, 4, 5\}$ in Example 2.2. Then there are three distinct conjugacy classes in X, namely, $[0]_c = \{0\}, [1]_c = [2]_c = \{1, 2\}, [3]_c = [4]_c = [5]_c = \{3, 4, 5\}$.

Remark 4.4. *Let $a \in X$. Then $a \in Z(X)$ if and only if $[a]_c = \{a\}$.*

The following theorem shows that the number of conjugates of a is equal to the index of $C(a)$ in X.

Theorem 4.5. *Let $a \in X$. Then $|[a]_c|_B = [X : C(a)]_B$.*

Proof. Let $a \in X$. Let $\mathcal{L}$ denote the set of all distinct left B-cosets of $C(a)$ in X. Then $|\mathcal{L}|_B = [X : C(a)]_B$. By definition, $b * (b * a) \in [a]_c$ for all $b \in X$. Define $f : \mathcal{L} \to [a]_c$ by $f(bC(a)) = b * (b * a)$. Suppose that $f(bC(a)) = f(cC(a))$. Then by (P2), (P3), (P5), (I), (III), and Theorem 2.3(i), we have

$$\begin{aligned}
f(bC(a)) = f(cC(a)) &\Rightarrow b * (b * a) = c * (c * a)\\
&\Rightarrow 0 * (b * (b * a)) = 0 * (c * (c * a))\\
&\Rightarrow (b * a) * b = (c * a) * c\\
&\Rightarrow (0 * c) * ((b * a) * b) = (0 * c) * ((c * a) * c)\\
&\Rightarrow (0 * c) * ((b * a) * b) = ((0 * c) * (0 * c)) * (c * a)\\
&\Rightarrow (0 * c) * ((b * a) * b) = 0 * (c * a)\\
&\Rightarrow (0 * c) * ((b * a) * b) = a * c\\
&\Rightarrow [(0 * c) * ((b * a) * b)] * (0 * b) = (a * c) * (0 * b)\\
&\Rightarrow [((0 * c) * (0 * b)) * (b * a)] * (0 * b) = a * ((0 * b) * (0 * c))\\
&\Rightarrow ((0 * c) * (0 * b)) * [(0 * b) * (0 * (b * a))] = a * [0 * ((0 * c) * (0 * b))]\\
&\Rightarrow ((0 * c) * (0 * b)) * ((0 * b) * (a * b)) = a * [0 * ((0 * c) * (0 * b))]\\
&\Rightarrow ((0 * c) * (0 * b)) * (0 * a) = a * [0 * ((0 * c) * (0 * b))]\\
&\Rightarrow (0 * c) * (0 * b) \in C(a)\\
&\Rightarrow bC(a) = cC(a).
\end{aligned}$$

Therefore, f is a one-one function. Let $y \in [a]_c$. Then there exists $x \in X$ such that $y = x * (x * a) = f(xC(a))$. Hence, f is onto. Therefore,f is a one-one function from $\mathcal{L}$ onto $[a]_c$. Consequently, $|[a]_c|_B = |\mathcal{L}|_B = [X : C(a)]_B$. □

Corollary 4.6. *Let X be a finite B-algebra. Then $|X|_B = \sum_a [X : C(a)]_B$, where the summation is over a complete set of distinct conjugacy class representatives.*

Proof. By Theorem 4.2, $X = \bigcup_a [a]_c$, where the union runs over a complete set of distinct conjugacy class representatives. Since the distinct conjugacy classes are mutually disjoint, we have $|X|_B = \left| \bigcup_a [a]_c \right|_B = \sum_a |[a]_c|_B$. By Theorem 4.5, it follows that $|X|_B = \sum_a [X : C(a)]_B$, where the summation is over a complete set of distinct conjugacy class representatives. □

Consider the B-algebra $X = \{0, 1, 2, 3, 4, 5\}$ in Example 2.2. Then $|X|_B = 6 = 1+2+3 = |[0]_c|_B + |[1]_c|_B + |[3]_c|_B = \sum_a |[a]_c|_B = \sum_a [X : C(a)]_B$.

Corollary 4.7. *If X is a finite B-algebra, then $|X|_B = |Z(X)|_B + \sum_{a \notin Z(X)} [X : C(a)]_B$, where the summation runs over a complete set of distinct conjugacy class representatives, which do not belong to $Z(X)$.*

Proof. By Corollary 4.6, $|X|_B = \sum_a [X : C(a)]_B$, where the summation is over a complete set of distinct conjugacy class representatives. Thus, we have $|X|_B = \sum_{a \in Z(X)} [X : C(a)]_B +$

$\sum_{a \notin Z(X)} [X : C(a)]_B$. By Lemma 4.1, we have $\sum_{a \in Z(X)} [X : C(a)]_B = |Z(X)|_B$. Hence, $|X|_B = |Z(X)|_B + \sum_{a \notin Z(X)} [X : C(a)]_B$, where the summation runs over a complete set of distinct conjugacy class representatives which do not belong to $Z(X)$. ☐

Example 4.8. Consider the B-algebra $X = \{0, 1, 2, 3, 4, 5\}$ in Example 2.2. Then $Z(X) = \{0\}$. Hence, $|Z(X)|_B + \sum_{a \notin Z(X)} [X : C(a)]_B = 1 + |[1]_c|_B + |[3]_c|_B = 1 + 2 + 3 = 6 = |X|_B$.

We now prove a partial converse of Lagrange's Theorem.

Lemma 4.9. *If X is a finite commutative B-algebra with $|X|_B = n$ such that n is divisible by a prime p, then X contains an element of order p and hence a subalgebra of order p.*

Proof. We proceed by induction on the order of X. If $|X|_B = p$ where p is prime, then every element of X (except 0) has order p. Thus, in particular, the lemma is true when $|X|_B = 2$. Suppose that the lemma is true for all B-algebras of order r, where $2 \leq r < n$. Suppose that X is a B-algebra of order n. Let $a \in X$ with $a \neq 0$ and let $|a|_B = m$. Then either $p|m$ or $p \nmid m$. If $p|m$, then $m = pk$ for some $k \in \mathbb{Z}^+$. In this case, $(a^k)^p = a^m = 0$. Hence, $a^k \neq 0$ and $|a^k|_B = p$. Suppose $p \nmid m$. Since X is commutative, the cyclic subalgebra $H = \langle a \rangle_B$ of X is a normal subalgebra of X. By Lagrange's Theorem, $|X|_B = m[X : H]_B$. Since $p \nmid m$, we have $p|[X : H]_B = |X/H|_B$. Since $|X/H|_B < n$, there exists $bH \in X/H$ s.t. $|bH|_B = p$. Now, $b^p H = (bH)^p = H$. Hence, $b^p \in H$. Thus, $(b^m)^p = (b^p)^m = 0$ and so either $b^m = 0$ or $|b^m|_B = p$. If $b^m = 0$, then $(bH)^m = H$ which implies $p|m$, a contradiction. Therefore, $|b^m|_B = p$ and so b^m is the desired element of X. ☐

Theorem 4.10. *(Cauchy's Theorem for B-algebras) Let X be a finite B-algebra with $|X|_B = n$ such that n is divisible by a prime p. Then X contains an element of order p and hence a subalgebra of order p.*

Proof. We proceed by induction on the order of X. If $n = 2$, then X is commutative and the result follows from Lemma 4.9. Suppose that the theorem is true for all B-algebras of order m s.t. $2 \leq m < n$. By Corollary 4.7, $|X|_B = |Z(X)|_B + \sum_{a \notin Z(x)} [X : C(a)]$. If $X = Z(X)$, then X is commutative and the result follows from Lemma 4.9. If $X \neq Z(X)$, then there exists $a \in X$ s.t. $a \notin Z(X)$. Then $X \neq C(a)$ and so $[X : C(a)]_B > 1$. By Lagrange's Theorem, $|X|_B = [X : C(a)]_B |C(a)|_B > |C(a)|_B$. If $p||C(a)|_B$, then $C(a)$ has an element of order p and so X has an element of order p. If $p \nmid |C(a)|_B$ for all $a \notin Z(X)$, then $p|[X : C(a)]_B$ for all $a \notin Z(X)$. Since p divides each term of the summation and also divides $|X|_B$, we have $p||Z(X)|$. By Lemma 4.9, X contains an element of order p and hence a subalgebra of order p ☐

The following theorem proves that the converse of Lagrange's Theorem for B-algebras hold for finite commutative B-algebras.

Theorem 4.11. *Let X be a finite commutative B-algebra with $|X|_B = n$. If $m \in \mathbb{Z}^+$ such that $m|n$, then X has a subalgebra of order m.*

Proof. If $m = 1$, then $\{0\}$ is the required subalgebra of order m. If $n = 1$, then $m = n = 1$ and the result follows easily. Assume that $m > 1$ and $n > 1$. We proceed by induction on n. If $n = 2$, then $m = 2$ and X is the required subalgebra of order m. Suppose that the theorem is true for all finite commutative B-algebras of order k s.t. $2 \leq k < n$. Let

p be a prime integer s.t. $p|m$. Then there exists $m_1 \in \mathbb{Z}^+$ s.t. $m = pm_1$. By Cauchy's Theorem, X has a subalgebra H of order p. Since X is commutative, H is normal and X/H is a B-algebra. Now, $1 \leq |X/H|_B = \frac{|X|_B}{|H|_B} < |X|_B$ and $|X/H|_B = \frac{n}{p}$. Now, $n = mm_2$ for some $m_2 \in \mathbb{Z}^+$. Thus, $|X/H|_B = \frac{pm_1m_2}{p} = m_1m_2$ and so m_1 divides $|X/H|_B$. Hence, X/H has a subalgebra K/H s.t. $|K/H|_B = m_1$, where K is a subalgebra of X. Now, $|K|_B = |K/H|_B|H|_B = m_1p = m$. Hence, K is a subalgebra of order m. □

As a consequence of Cauchy's Theorem, we now introduce the concept of B_p-algebras.

Definition 4.12. Let p be a prime number. A B-algebra X is called a *B_p-algebra* if the order of each element of X is a power of p. A subalgebra H of a B-algebra X is called *B_p-subalgebra* if H is a B_p-algebra.

The B-algebra in Example 2.1 is B_3-algebra. We now prove some results where Cauchy's Theorem plays a role. The following theorem provides a necessary and sufficient condition for a finite B-algebra to be a B_p-algebra.

Theorem 4.13. *Let X be a nontrivial B-algebra. Then X is a finite B_p-algebra if and only if $|X|_B = p^k$ for some $k \in \mathbb{Z}^+$.*

Proof. Suppose that X is a finite B_p-algebra. If $q||X|_B$ for some prime $q \neq p$, then by Cauchy's Theorem, X has an element of order q, a contradiction. Thus, p is the only prime divisor of $|X|_B$, that is, $|X|_B = p^k$ for some $k \in \mathbb{Z}^+$. Conversely, suppose that $|X|_B = p^k$ for some $k \in \mathbb{Z}^+$. Then by Lagrange's Theorem, the order of each element of X is a power of p. Therefore, X is a finite B_p-algebra. □

The following theorem shows that the center of a B_p-algebra is nontrivial.

Theorem 4.14. *If X is a finite B_p-algebra with $|X|_B > 1$, then $|Z(G)|_B > 1$.*

Proof. Suppose that X is a finite B_p-algebra with $|X|_B$ >1. If $X = Z(X)$, then $|Z(X)|_B = |X|_B > 1$. Suppose that $Z(X) \subset X$ and consider $a \in X$ such that $a \notin Z(X)$. Then $C(a)$ is a proper subalgebra of a B_p-algebra X. By Theorem 4.13, $p||X|_B$. It follows that $p|[X : C(a)]_B$ for all $a \notin Z(X)$. Thus, p divides $\sum_{a \notin Z(X)} [X : C(a)]_B$. By Corollary 4.7, $|X|_B = |Z(X)|_B + \sum_{a \notin Z(X)} [X : C(a)]_B$. Since $p||X|_B$ and $p|\sum_{a \notin Z(X)} [X : C(a)]_B$, it follows that $p||Z(X)|_B$. Therefore, $|Z(X)|_B > 1$. □

Corollary 4.15. *If $|X|_B = p^2$, where p is prime, then X is commutative.*

Proof. Suppose that $|X|_B = p^2$, where p is prime. By Theorem 4.14, $|Z(X)|_B > 1$. Since $Z(X)$ is a subalgebra, $|Z(X)|_B$ divides p^2 by Lagrange's Theorem. Hence, $|Z(X)|_B$ is p or p^2. If $|Z(X)|_B = p$. Then $Z(X) \neq X$ and so there exists $a \in X$ such that $a \notin Z(X)$. In [6], $C(a)$ is a subalgebra of X with $a \in C(a)$. Hence, $Z(X) \subset C(a)$. This implies that $|C(a)|_B = p^2$. Thus, $X = C(a)$ and so $a \in Z(X)$, a contradiction. Therefore, $|Z(X)|_B = p^2$ and so $X = Z(X)$. Consequently, X is commutative. □

Proposition 4.16. *Let H and K be subalgebras of a commutative B-algebra X. If $|H|_B = m$ and $|K|_B = n$, then X has a subalgebra of order lcm(m, n).*

Proof. Let H and K be subalgebras of a commutative B-algebra X with $|H|_B = m$ and $|K|_B = n$. Since $HK = KH$, HK is a subalgebra of X. Since H and K are finite, H and K are subalgebras of a finite B-algebra HK. By Lagrange's Theorem, $m||HK|_B$ and $n||HK|_B$. Hence, $lcm(m,n)||HK|_B$. By Theorem 4.11, HK has a subalgebra of order $lcm(m,n)$ and so X has a subalgebra of order $lcm(m,n)$. □

The version of Lagrange's Theorem for B-algebras in [2] is analogue to the Lagrange's Theorem for groups, and the version of Cauchy's Theorem for B-algebras in this paper is analogue to the Cauchy's Theorem for groups. It is then natural to seek an analogue results to the Sylow Theorems for groups.

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