HYPERGROUPS ARISING FROM CHARACTERS OF A COMPACT GROUP AND ITS SUBGROUP

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Abstract.

The purpose of the present paper is to investigate a hypergroup arising from irreducible characters of a compact group G and a closed subgroup G_0 of G with index $[G : G_0] < +\infty$. The convolution of this hypergroup is introduced by inducing irreducible representations of G_0 to G and by restricting irreducible representations of G to G_0. The method of proof relies on character formulae of induced representations of compact groups and of Frobenius' reciprocity theorem.

1. Introduction

One of the most challenging problems in the theory of hypergroups is a definite explanation of their algebraic structure. To solve this problem completely might be an utopian undertaking. But there are various ways to tackle parts of the problem. The approaches available are based on constructing new hypergroups from known ones. Much work has been done in the direction of extending hypergroups and of establishing new hypergroup structures defined by hypergroup actions ([HK1]). In succession of the authors' publications on semi-direct product hypergroups ([HK2]) and on hypergroup structures arising from certain dual objects of a hypergroup ([HK3]). The next step taken in the present paper is the supply of a hypergroup structure on the set of irreducible characters of a compact group G together with a closed subgroup G_0 of G. It turns out that the resulting hypergroup structure can be characterized in terms of an invariance condition on characters of irreducible representations of G_0.

The method chosen in order to establish this result depends on the application of a character formula ([H]), of Frobenius' reciprocity theorem ([F]) for compact groups, and on the recently developed character theory for induced representations of hypergroups ([HKY]).

It should be mentioned that further progress in the research on the structure of hypergroups is on its way to publication : an extension of the notion of hyperfields ([HKKK]) to not necessarily finite hypergroups ([HKTY1]), and a generalization of the present work to compact *hypergroups* and their closed *subhypergroups* ([HKTY2]).

A brief layout of the paper seems to be in order.

In Section 2 the preliminaries are restricted to the main notions of hypergroup theory ([BH], [J]) ; they can also be picked up from the introduction of [HK1]. In the latter reference semi-direct product hypergroups have been introduced.

Section 3 is devoted to defining admissible pairs (G, G_0) formed by a second countable compact group G and a closed subgroup G_0 of G of finite index, and to studying properties of

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these pairs (Lemma 3.5, 3.7, 3.9 and 3.10). In the Theorem of the section the set $\mathcal{K}(\hat{G}\cup\widehat{G_0})$ of characters of the union $\hat{G}\cup\widehat{G_0}$ of the duals of G and G_0 is characterized as a hypergroup by the fact that (G, G_0) is an admissible pair. Applications of the Theorem are given to symmetric groups (Corollary 3.14) and to semi-direct product groups (Corollary 3.17).

Section 4 contains a variety of examples comprising those given in [SW]. An extended list of examples can be visualized by Frobenius diagrams, a graph-theoretical illustration related to Dynkin diagrams and Coxeter graphs ([GHJ]). In all examples (except Example 4.6) the convolution of the hypergroup $\mathcal{K}(\hat{G}\cup\widehat{G_0})$ is described explicitly.

2. Preliminaries

For a locally compact space X we shall mainly consider the subspaces $C_c(X)$ and $C_0(X)$ of the space $C(X)$ of continuous functions on X which have compact support or vanish at infinity respectively. By $M(X)$, $M^b(X)$ and $M_c(X)$ we abbreviate the spaces of all (Radon) measures on X, the bounded measures and the measures with compact support on X respectively. Let $M^1(X)$ denote the set of probability measures on X and $M_c^1(X)$ its subset $M^1(X)\cap M_c(X)$. The symbol δ_x stands for the Dirac measures in $x \in X$.

A *hypergroup* $(K, *)$ is a locally compact space K together with a *convolution* $*$ in $M^b(K)$ such that $(M^b(K), *)$ becomes a Banach algebra and that the following properties are fulfilled.

(H1) The mapping
$$(\mu,\nu) \longmapsto \mu * \nu$$
from $M^b(K)\times M^b(K)$ into $M^b(K)$ is continuous with respect to the weak topology in $M^b(K)$.

(H2) For $x, y \in K$ the convolution $\delta_x * \delta_y$ belongs to $M_c^1(K)$.

(H3) There exists a *unit* element $e \in K$ with
$$\delta_e * \delta_x = \delta_x * \delta_e = \delta_x$$
for all $x \in K$, and an *involution*
$$x \longmapsto x^-$$
in K such that
$$\delta_{x^-} * \delta_{y^-} = (\delta_y * \delta_x)^-$$
and
$$e \in \operatorname{supp}(\delta_x * \delta_y) \ \text{ if and only if } \ x = y^-$$
whenever $x, y \in K$.

(H4) The mapping
$$(x, y) \longmapsto \operatorname{supp}(\delta_x * \delta_y)$$
from $K \times K$ into the space $\mathcal{C}(K)$ of all compact subsets of K furnished with Michael topology is continuous.

A hypergroup $(K, *)$ is said to be *commutative* if the convolution $*$ is commutative. In this case $(M^b(K), *, -)$ is a commutative Banach $*$-algebra with identity δ_e. There is an abundance of hypergroups and there are various constructions (polynomial, Sturm-Liouville) as the reader may learn from the pioneering papers on the subject and also from the monograph [BH].

Let $(K,*)$ and $(L,\circ)$ be two hypergroups with units e_K and e_L respectively. A continuous mapping $\varphi : K \to L$ is called a hypergroup *homomorphism* if $\varphi(e_K) = e_L$ and φ is the unique linear, weakly continuous extension from $M^b(K)$ to $M^b(L)$ such that

$$\varphi(\delta_x) = \delta_{\varphi(x)},\ \varphi(\delta_x^-) = \varphi(\delta_x)^- \text{ and } \varphi(\delta_x * \delta_y) = \varphi(\delta_x) \circ \varphi(\delta_y)$$

whenever $x, y \in K$. If $\varphi : K \to L$ is also a homeomorphism, it will be called an *isomorphism* from K onto L. An isomorphism from K onto K is called an *automorphism* of K. We denote by $\mathrm{Aut}(K)$ the set of all automorphisms of K. Then $\mathrm{Aut}(K)$ becomes a topological group equipped with the weak topology of $M^b(K)$. We call α an *action* of a locally compact group G on a hypergroup H if α is a continuous homomorphism from G into $\mathrm{Aut}(H)$. Associated with the action α of G on H one can define a semi-direct product hypergroup $K = H \rtimes_\alpha G$, see [HK1].

If the given hypergroup K is commutative, its dual $\widehat{K}$ can be introduced as the set of all bounded continuous functions $\chi \neq 0$ on K satisfying

$$\int_K \chi(z)(\delta_x^- * \delta_y)(dz) = \overline{\chi(x)}\chi(y)$$

for all $x, y \in K$. This set of characters $\widehat{K}$ of K becomes a locally compact space with respect to the topology of uniform convergence on compact sets, but generally fails to be a hypergroup. When $\widehat{K}$ is a hypergroup K is called a strong hypergroup. When the dual $\widehat{K}$ of a strong hypergroup K is also strong and $\widehat{\widehat{K}} \cong K$ holds K is called a Pontryagin hypergroup.

3. Hypergroups related to admissible pairs

Let G be a compact group which satisfies the second axiom of countability and let $\hat{G}$ be the set of all equivalence classes of irreducible representations of G. Then $\hat{G}$ is (at most countable) discrete space which we write explicitly as

$$\hat{G} = \{\pi_0, \pi_1, \cdots, \pi_n, \cdots\},$$

where π_0 is the trivial representation of G. We denote by $\mathrm{Rep}^{\mathrm{f}}(G)$ the set of equivalence classes of finite-dimensional representations of G. For $\pi \in \mathrm{Rep}^{\mathrm{f}}(G)$ we consider the normalized character of π given by

$$ch(\pi)(g) = \frac{1}{\dim \pi} \mathrm{tr}(\pi(g)),$$

for all $g \in G$.

Put

$$\mathcal{K}(\hat{G}) = \{ch(\pi) : \pi \in \hat{G}\}.$$

Then $\mathcal{K}(\hat{G})$ is known to be a discrete commutative hypergroup with unit $ch(\pi_0) = \pi_0$.

Let G_0 be a closed subgroup of G such that the index $[G : G_0]$ is finite. We write $\widehat{G_0} = \{\tau_0, \tau_1, \cdots, \tau_n, \cdots\}$, where τ_0 is the trivial representation of G_0.

The following is well-known fact.

Lemma 3.1

(1) $ch(\pi_i \otimes \pi_j) = ch(\pi_i)ch(\pi_j)$ for $\pi_i, \pi_j \in \mathrm{Rep}^{\mathrm{f}}(G)$.

(2) $ch(\mathrm{res}^G_{G_0}\pi) = \mathrm{res}^G_{G_0} ch(\pi)$ for $\pi \in \mathrm{Rep}^{\mathrm{f}}(G)$.

Lemma 3.2

(1) [Character formula] (see Hirai [H]) For $\tau \in \mathrm{Rep}^{\mathrm{f}}(G_0)$,

$$ch(\mathrm{ind}^G_{G_0}\tau)(g) = \int_G ch(\tau)(sgs^{-1})1_{G_0}(sgs^{-1})\omega_G(ds).$$

(2) [Frobenius' reciprocity theorem] (see Folland [F]) For $\tau \in \widehat{G_0}$ and for $\pi \in \hat{G}$,

$$[\mathrm{ind}^G_{G_0}\tau : \pi] = [\tau : \mathrm{res}^G_{G_0}\pi],$$

where [:] denotes the multiplicity of representations.

Remark $ch(\mathrm{ind}^G_{G_0}\tau) = \mathrm{ind}^G_{G_0} ch(\tau)$, see [HKY], where

$$\mathrm{ind}^G_{G_0} ch(\tau) := \int_G ch(\tau)(sgs^{-1})1_{G_0}(sgs^{-1})\omega_G(ds).$$

Definition On the set

$$\mathcal{K}(\hat{G} \cup \widehat{G_0}) := \{(ch(\pi), \circ), (ch(\tau), \bullet) : \pi \in \hat{G}, \tau \in \widehat{G_0}\}$$

we define a convolution $*$ as follows. For $\pi_i, \pi_j, \pi \in \hat{G}$ and $\tau_i, \tau_j, \tau \in \widehat{G_0}$,

$$\begin{aligned}
(ch(\pi_i), \circ) * (ch(\pi_j), \circ) &:= (ch(\pi_i)ch(\pi_j), \circ),\\
(ch(\pi), \circ) * (ch(\tau), \bullet) &:= (ch(\mathrm{res}^G_{G_0}\pi)ch(\tau), \bullet),\\
(ch(\tau), \bullet) * (ch(\pi), \circ) &:= (ch(\tau)ch(\mathrm{res}^G_{G_0}\pi), \bullet),\\
(ch(\tau_i), \bullet) * (ch(\tau_j), \bullet) &:= (ch(\mathrm{ind}^G_{G_0}(\tau_i \otimes \tau_j)), \circ).
\end{aligned}$$

We want to check the associativity relations of the convolution in the following cases. Whenever reference to a particular representation π is not needed, we abbreviate $(ch(\pi), \bullet)$ by $\bullet$ and $(ch(\pi), \circ)$ by $\circ$. Hence our task will be to verify the subsequent formulae :

$$\begin{aligned}
&(A1)\ (\circ * \circ) * \circ = \circ * (\circ * \circ),\\
&(A2)\ (\bullet * \circ) * \circ = \bullet * (\circ * \circ),\\
&(A3)\ (\bullet * \bullet) * \circ = \bullet * (\bullet * \circ) \text{ and}\\
&(A4)\ (\bullet * \bullet) * \bullet = \bullet * (\bullet * \bullet).
\end{aligned}$$

Lemma 3.3 The equalities $(A1)$, $(A2)$ and $(A3)$ hold without further assumptions. For $\pi_i, \pi_j, \pi_k, \pi \in \hat{G}$ and $\tau_i, \tau_j, \tau \in \widehat{G_0}$,

$(A1)$ $((ch(\pi_i), \circ) * (ch(\pi_j), \circ)) * (ch(\pi_k), \circ) = (ch(\pi_i), \circ) * ((ch(\pi_j), \circ) * (ch(\pi_k), \circ))$.

$(A2)$ $((ch(\tau), \bullet) * (ch(\pi_i), \circ)) * (ch(\pi_j), \circ) = (ch(\tau), \bullet) * ((ch(\pi_i), \circ) * (ch(\pi_j), \circ))$.

$(A3)$ $((ch(\tau_i), \bullet) * (ch(\tau_j), \bullet)) * (ch(\pi), \circ) = (ch(\tau_i), \bullet) * ((ch(\tau_j), \bullet) * (ch(\pi), \circ))$.

Proof $(A1)$ is clear because $\mathcal{K}(\hat{G})$ has a hypergroup structure.

$(A2)$ For $\tau \in \widehat{G_0}$ and $\pi_i, \pi_j \in \hat{G}$,

$$\begin{aligned} &((ch(\tau), \bullet) * (ch(\pi_i), \circ)) * (ch(\pi_j), \circ) \\ =&(ch(\tau)ch(\mathrm{res}^G_{G_0}\pi_i), \bullet) * (ch(\pi_j), \circ) \\ =&(ch(\tau)ch(\mathrm{res}^G_{G_0}\pi_i)ch(\mathrm{res}^G_{G_0}\pi_j), \bullet) \\ =&(ch(\tau)(\mathrm{res}^G_{G_0} ch(\pi_i))(\mathrm{res}^G_{G_0} ch(\pi_j)), \bullet). \end{aligned}$$

On the other hand,

$$\begin{aligned} &(ch(\tau), \bullet) * ((ch(\pi_i), \circ) * (ch(\pi_j), \circ)) \\ =&(ch(\tau), \bullet)) * ((ch(\pi_i)ch(\pi_j), \circ) \\ =&(ch(\tau)\mathrm{res}^G_{G_0}(ch(\pi_i)ch(\pi_j)), \bullet) \\ =&(ch(\tau)(\mathrm{res}^G_{G_0} ch(\pi_i))(\mathrm{res}^G_{G_0} ch(\pi_j)), \bullet). \end{aligned}$$

(A3) For $\tau_i, \tau_j \in \widehat{G_0}$ and $\pi \in \hat{G}$,

$$\begin{aligned} &((ch(\tau_i), \bullet) * (ch(\tau_j), \bullet)) * (ch(\pi), \circ) \\ =&(ch(\mathrm{ind}^G_{G_0}(\tau_i \otimes \tau_j)), \circ) * (ch(\pi), \circ) \\ =&(ch(\mathrm{ind}^G_{G_0}(\tau_i \otimes \tau_j))ch(\pi), \circ). \end{aligned}$$

For every $g \in G$,

$$\begin{aligned} &(ch(\mathrm{ind}^G_{G_0}(\tau_i \otimes \tau_j))ch(\pi))(g) \\ =&ch(\mathrm{ind}^G_{G_0}(\tau_i \otimes \tau_j)(g)ch(\pi)(g) \\ =&\int_G ch(\tau_i \otimes \tau_j)(sgs^{-1})1_{G_0}(sgs^{-1})\omega_G(ds)ch(\pi)(g) \\ =&\int_G (ch(\tau_i)ch(\tau_j))(sgs^{-1})ch(\pi)(g)1_{G_0}(sgs^{-1})\omega_G(ds) \\ =&\int_G ch(\tau_i)(sgs^{-1})ch(\tau_j)(sgs^{-1})ch(\pi)(sgs^{-1})1_{G_0}(sgs^{-1})\omega_G(ds). \end{aligned}$$

On the other hand,

$$\begin{aligned} &(ch(\tau_i), \bullet) * ((ch(\tau_j), \bullet) * (ch(\pi), \circ)) \\ =&(ch(\tau_i), \bullet) * (ch(\tau_j)ch(\mathrm{res}^G_{G_0}\pi), \bullet) \\ =&(ch(\mathrm{ind}^G_{G_0}(\tau_i \otimes \tau_j \otimes \mathrm{res}^G_{G_0}\pi)), \circ). \end{aligned}$$

For each $g \in G$,

$$\begin{aligned} &ch(\mathrm{ind}^G_{G_0}(\tau_i \otimes \tau_j \otimes \mathrm{res}^G_{G_0}\pi))(g) \\ =&\int_G ch(\tau_i \otimes \tau_j \otimes \mathrm{res}^G_{G_0}\pi)(sgs^{-1})1_{G_0}(sgs^{-1})\omega_G(ds) \\ =&\int_G (ch(\tau_i)ch(\tau_j)ch(\mathrm{res}^G_{G_0}\pi))(sgs^{-1})1_{G_0}(sgs^{-1})\omega_G(ds) \\ =&\int_G ch(\tau_i)(sgs^{-1})ch(\tau_j)(sgs^{-1})ch(\pi)(sgs^{-1})1_{G_0}(sgs^{-1})\omega_G(ds). \end{aligned}$$

[Q.E.D.]

Definition Let (G, G_0) be a pair of consisting of a compact group G and a closed subgroup G_0 of G. For $g \in G_0$

$$X(g) := \{s \in G : sgs^{-1} \in G_0\}.$$

We call (G, G_0) an *admissible pair* if for any $\tau \in \widehat{G_0}$, any $g \in G_0$ and any $s \in X(g)$,

$$ch(\tau)(sgs^{-1}) = ch(\tau)(g)$$

holds.

Lemma 3.4 If a compact group G together with a subgroup G_0 of G with $[G : G_0] < +\infty$ forms an admissible pair, then the associativity relation $(A4)$ holds.

Proof Assume that (G, G_0) is an admissible pair. For $\tau_i, \tau_j, \tau_k \in \widehat{G_0}$ and $g \in G_0$

$$\begin{aligned}
&((ch(\tau_i), \bullet) * (ch(\tau_j), \bullet)) * (ch(\tau_k), \bullet) \\
=&(ch(\mathrm{ind}_{G_0}^G(\tau_i \otimes \tau_j)), \circ) * (ch(\tau_k), \bullet) \\
=&(\mathrm{ind}_{G_0}^G ch(\tau_i \otimes \tau_j)), \circ) * (ch(\tau_k), \bullet) \\
=&(\mathrm{res}_{G_0}^G(\mathrm{ind}_{G_0}^G(ch(\tau_i)ch(\tau_j)))(ch(\tau_k), \bullet).
\end{aligned}$$

For $g \in G_0$,

$$\begin{aligned}
&(\mathrm{ind}_{G_0}^G(ch(\tau_i)ch(\tau_j)))(g)ch(\tau_k))(g) \\
=&\left(\int_G ch(\tau_i)(sgs^{-1})ch(\tau_j)(sgs^{-1})1_{G_0}(sgs^{-1})\omega_G(ds)\right) ch(\tau_k)(g) \\
=&\left(\int_G ch(\tau_i)(g)ch(\tau_j)(g)1_{G_0}(sgs^{-1})\omega_G(ds)\right) ch(\tau_k)(g) \\
=&\left(\int_G 1_{G_0}(sgs^{-1})\omega_G(ds)\right) ch(\tau_i)(g)ch(\tau_j)(g)ch(\tau_k)(g).
\end{aligned}$$

This implies the associativity relation $(A4)$. [Q.E.D.]

Lemma 3.5 If the associativity relation $(A4)$ holds for a compact group G and a subgroup G_0 of G with $[G : G_0] < +\infty$, then (G, G_0) is an admissible pair.

Proof Assume that the associativity relation $(A4)$ holds. Let τ_0 be the trivial representation of $\widehat{G_0}$. For $\tau \in \widehat{G_0}$ the associativity relation

$$((ch(\tau_0), \bullet) * (ch(\tau_0), \bullet)) * (ch(\tau), \bullet) = (ch(\tau_0), \bullet) * ((ch(\tau_0), \bullet) * (ch(\tau), \bullet))$$

holds.

$$((ch(\tau_0), \bullet) * (ch(\tau_0), \bullet)) * (ch(\tau), \bullet) = (ch(\mathrm{res}_{G_0}^G(\mathrm{ind}_{G_0}^G \tau_0))ch(\tau), \bullet)$$

and

$$(ch(\tau_0), \bullet) * ((ch(\tau_0), \bullet) * (ch(\tau), \bullet)) = (ch(\mathrm{res}_{G_0}^G(\mathrm{ind}_{G_0}^G \tau)), \bullet).$$

Then for $g \in G_0$

$$ch(\mathrm{ind}_{G_0}^G \tau_0)(g)ch(\tau)(g) = ch(\mathrm{ind}_{G_0}^G \tau)(g).$$

Now

$$ch(\mathrm{ind}_{G_0}^G \tau_0)(g) \geq \omega_G(G_0) > 0.$$

Indeed by the character formula

$$ch(\mathrm{ind}_{G_0}^{G}\tau_0)(g) = \int_G 1_{G_0}(sgs^{-1})\omega_G(ds) = \omega_G(X(g)).$$

Since $X(g) \supset G_0$, we see that $\omega_G(X(g)) \geq \omega_G(G_0)$. By the assumption $[G : G_0] < +\infty$ we obtain $\omega_G(G_0) = 1/[G : G_0] > 0$. Hence

$$ch(\tau)(g) = (ch(\mathrm{ind}_{G_0}^{G}\tau_0)(g))^{-1}ch(\mathrm{ind}_{G_0}^{G}\tau)(g).$$

For $s \in X(g)$

$$\begin{aligned} ch(\tau)(sgs^{-1}) &= (ch(\mathrm{ind}_{G_0}^{G}\tau_0)(sgs^{-1}))^{-1}ch(\mathrm{ind}_{G_0}^{G}\tau)(sgs^{-1}) \\ &= (ch(\mathrm{ind}_{G_0}^{G}\tau_0)(g))^{-1}ch(\mathrm{ind}_{G_0}^{G}\tau)(g) \\ &= ch(\tau)(g). \end{aligned}$$

Then (G, G_0) is an admissible pair. [Q.E.D.]

Theorem Let G_0 be a closed subgroup of a compact group G such that $[G : G_0] < +\infty$. Then $\mathcal{K}(\hat{G} \cup \widehat{G_0})$ is a hypergroup if and only if (G, G_0) is an admissible pair.

Proof The associativity relations $(A1)$, $(A2)$ and $(A3)$ are a consequence of Lemma 3.3, and $(A4)$ holds if and only if (G, G_0) is an admissible pair by Lemma 3.4 and Lemma 3.5. It is easy to check the remaining axioms of a hypergroup for $\mathcal{K}(\hat{G} \cup \widehat{G_0})$. The desired conclusion follows. [Q.E.D.]

Remark 3.6

(1) The above $\mathcal{K}(\hat{G} \cup \widehat{G_0})$ is a discrete commutative (at most countable) hypergroup such that the sequence :

$$1 \longrightarrow \mathcal{K}(\hat{G}) \longrightarrow \mathcal{K}(\hat{G} \cup \widehat{G_0}) \longrightarrow \mathbb{Z}_2 \longrightarrow 1$$

is exact.

(2) If $G_0 = G$, then $\mathcal{K}(\hat{G} \cup \widehat{G_0})$ is the hypergroup $\mathcal{K}(\hat{G}) \times \mathbb{Z}_2$.

(3) If G is a finite group and $G_0 = \{e\}$, then $\mathcal{K}(\hat{G} \cup \widehat{G_0})$ is the hypergroup join $\mathcal{K}(\hat{G}) \vee \mathbb{Z}_2$.

Lemma 3.7 If G is a compact Abelian group and G_0 a closed subgroup of G with $[G : G_0] < +\infty$. Then (G, G_0) is always an admissible pair.

Proof The desired assertion clearly follows from the fact that $sgs^{-1} = g$ for $g \in G_0$ and $s \in G$. [Q.E.D.]

Corollary 3.8 Let G be a compact Abelian group and G_0 a closed subgroup of G with $[G : G_0] < +\infty$. Then $\mathcal{K}(\hat{G} \cup \widehat{G_0})$ is a hypergroup.

Proof This assertion follows directly from the Theorem and Lemma 3.7. [Q.E.D.]

Lemma 3.9 If for each $\tau \in \widehat{G_0}$ there exists a representation $\tilde{\tau}$ of G such that $\mathrm{res}_{G_0}^{G}\tilde{\tau} = \tau$, then (G, G_0) is an admissible pair.

Proof For $\tau \in \widehat{G_0}$, $g \in G_0$ and $s \in X(g)$,

$$ch(\tau)(sgs^{-1}) = ch(\tilde{\tau})(sgs^{-1}) = ch(\tilde{\tau})(g) = ch(\tau)(g).$$

[Q.E.D.]

Corollary 3.10 Let G be a semi-direct product group $H \rtimes_\alpha G_0$, where H is a finite group and G_0 is a finite group. Then $\mathcal{K}(\hat{G} \cup \widehat{G_0})$ is a hypergroup.

Proof For $\tau \in \widehat{G_0}$, put

$$\tilde{\tau}((h,g)) = \tau(g)$$

for $(h,g) \in H \rtimes_\alpha G_0 = G$. Then $\tilde{\tau}$ is a finite dimensional representation of G and $\mathrm{res}^G_{G_0} \tilde{\tau} = \tau$. By the Theorem and Lemma 3.9 we arrive at the desired conclusion.

[Q.E.D.]

Lemma 3.11 If for $g \in G_0$ and $s \in X(g)$ there exists $t \in G_0$ such that $tgt^{-1} = sgs^{-1}$, then (G, G_0) is an admissible pair.

Proof For $\tau \in \widehat{G_0}$, $g \in G_0$ and $s \in X(g)$,

$$ch(\tau)(sgs^{-1}) = ch(\tau)(tgt^{-1}) = ch(\tau)(g).$$

[Q.E.D.]

Let S_n be the symmetric group of degree n.

Corollary 3.12 $\mathcal{K}(\widehat{S_n} \cup \widehat{S_{n-1}})$ $(n \geq 2)$ is a hypergroup.

Proof For $g \in S_{n-1}$, $s \in X(g)$ such that $s^{-1}(n) = a$,

$$sgs^{-1}(n) = s(g(s^{-1}(n))) = s(g(a)).$$

Since $sgs^{-1} \in S_{n-1}$, $sgs^{-1}(n) = n$ and $s(g(a)) = n$. Then $g(a) = s^{-1}(n) = a$.

Put $t = ss_1$ where s_1 is a transposed permutation (a, n). Then we see that

$$tgt^{-1} = sgs^{-1}$$

by the fact that for b such that $b \neq a$, $s^{-1}(b) \neq a$ and $g(s^{-1}(b)) \neq a$ hold. By the Theorem and Lemma 3.11 we get the desired conclusion. [Q.E.D.]

Lemma 3.13 Let G_0 and G_1 be closed subgroups of G such that $G_0 \subset G_1 \subset G$. If (G_1, G_0) and (G, G_1) are admissible pairs, then (G, G_0) is an admissible pair.

Proof Since (G_1, G_0) is an admissible pair, for $\tau \in \widehat{G_0}$ and $g \in G_0$,

$$\begin{aligned} ch(\mathrm{ind}^{G_1}_{G_0}\tau)(g) &= \int_{G_1} ch(\tau)(sgs^{-1}) 1_{G_0}(sgs^{-1}) d\omega_{G_1}(s) \\ &= \int_{G_1} ch(\tau)(g) 1_{G_0}(sgs^{-1}) d\omega_{G_1}(s) \\ &= \int_{G_1} 1_{G_0}(sgs^{-1}) d\omega_{G_1}(s) ch(\tau)(g) \\ &= ch(\mathrm{ind}^{G_1}_{G_0}\tau_0)(g) ch(\tau)(g), \end{aligned}$$

where τ_0 is the trivial representation of G_0. Since

$$ch(\mathrm{res}^{G_1}_{G_0}\tau_0)(g) \geq \omega_{G_1}(G_0) > 0,$$

we see that

$$ch(\tau)(g) = (ch(\mathrm{ind}_{G_0}^{G_1}\tau_0)(g))^{-1}ch(\mathrm{ind}_{G_0}^{G_1}\tau)(g).$$

Since (G, G_1) is an admissible pair and $\mathrm{ind}_{G_0}^{G_1}\tau_0 \in \mathrm{Rep^f}(G_1)$, $\mathrm{ind}_{G_0}^{G_1}\tau \in \mathrm{Rep^f}(G_1)$, we have for $g \in G_0 \subset G_1$ and $s \in X(g)$,

$$ch(\mathrm{ind}_{G_0}^{G_1}\tau_0)(sgs^{-1}) = ch(\mathrm{ind}_{G_0}^{G_1}\tau_0)(g)$$

and

$$ch(\mathrm{ind}_{G_0}^{G_1}\tau)(sgs^{-1}) = ch(\mathrm{ind}_{G_0}^{G_1}\tau)(g).$$

Then we obtain

$$\begin{aligned} ch(\tau)(sgs^{-1}) &= (ch(\mathrm{ind}_{G_0}^{G_1}\tau_0)(sgs^{-1}))^{-1}ch(\mathrm{ind}_{G_0}^{G}\tau)(sgs^{-1}) \\ &= (ch(\mathrm{ind}_{G_0}^{G_1}\tau_0)(g))^{-1}ch(\mathrm{ind}_{G_0}^{G_1}\tau)(g) \\ &= ch(\tau)(g) \end{aligned}$$

[Q.E.D.]

Corollary 3.14 For natural numbers m and n such that $m > n \geq 1$, $\mathcal{K}(\widehat{S_m} \cup \widehat{S_n})$ is a hypergroup.

Proof This statement follows from the Theorem and Lemma 3.13 combined with Corollary 3.12. [Q.E.D.]

Let G_0 be a closed normal subgroup of G. Then the coadjoint action $\hat{\alpha}$ of G on $\widehat{G_0}$ is defined by

$$\hat{\alpha}_s(\tau)(g) := \tau(sgs^{-1})$$

for $\tau \in \widehat{G_0}$, $g \in G_0$ and $s \in G$. If $\hat{\alpha}_s = id$ for all $s \in G$, we say that $\hat{\alpha}$ is trivial.

Lemma 3.15 Let G_0 be a closed normal subgroup of G. The pair (G, G_0) is an admissible pair if and only if the coadjoint action $\hat{\alpha}$ is trivial.

Proof Assume that (G, G_0) is an admissible pair. For $g \in G_0$ it is clear that $X(g) = G$. Then for $\tau \in \widehat{G_0}$

$$ch(\tau)(sgs^{-1}) = ch(\tau)(g)$$

for all $s \in G$. This implies that

$$ch(\hat{\alpha}_s(\tau))(g) = ch(\tau)(g)$$

for all $g \in G_0$. Hence we obtain

$$\hat{\alpha}_s(\tau) \cong \tau$$

for $\tau \in \widehat{G_0}$. In fact $\hat{\alpha}_s$ is the identity on $\widehat{G_0}$ which means that $\hat{\alpha}$ is trivial.

The converse is clear. [Q.E.D.]

Lemma 3.16 Let G_0 be a closed normal commutative subgroup of G. The pair (G, G_0) is an admissible pair if and only if $G \cong G_0 \times (G/G_0)$.

Proof Assume that (G, G_0) is an admissible pair and $sgs^{-1} \neq g$ for $g \in G_0$ and $s \in X(g) = G$. Since $\widehat{G_0}$ separates G_0, there exists $\tau \in \widehat{G_0}$ such that

$$\tau(sgs^{-1}) \neq \tau(g).$$

This contradicts the assumption that (G, G_0) is an admissible pair. But then for $g \in G_0$ and $s \in G$

$$sgs^{-1} = g,$$

namely

$$sg = gs$$

holds. For $g_1, g_2 \in G_0$ and $s_1, s_2 \in G$

$$(g_1 s_1)(g_2 s_2) = g_1(s_1 g_2)s_2 = g_1(g_2 s_1)s_2 = (g_1 g_2)(s_1 s_2).$$

This implies that $G \cong G_0 \times (G/G_0)$.

The converse is clear by Lemma 3.15. [Q.E.D.]

Corollary 3.17 Let G be a semi-direct product group $H \rtimes_\alpha G_0$ where H is a compact Abelian group and G_0 is a finite group. $\mathcal{K}(\hat{G} \cup \hat{H})$ is a hypergroup if and only if the action α is trivial, i.e. $G = H \times G_0$.

Proof We note that H is a closed normal subgroup of G and $G/H \cong G_0$. Then the assertion follows from the Theorem together with Lemma 3.16. [Q.E.D.]

4. Examples

Associated with a pair (G, G_0) of finite groups such that $G \supset G_0$, we obtain a certain finite graph $D(\hat{G} \cup \widehat{G_0})$ by Frobenius' reciprocity theorem. The set of vertices is

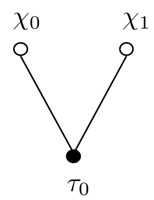
$$\{(\pi, \circ), (\tau, \bullet) : \pi \in \hat{G}, \tau \in \widehat{G_0}\}$$

and the edge between $(\pi, \circ)$ and $(\tau, \bullet)$ is given by the multiplicity

$$m_{\pi,\tau} := [\mathrm{ind}_{G_0}^G(\tau) : \pi] = [\tau : \mathrm{res}_{G_0}^G \pi] \neq 0.$$

We call this graph $D(\hat{G} \cup \widehat{G_0})$ a Frobenius diagram. Frobenius diagrams $D(\hat{G} \cup \widehat{G_0})$ sometimes appear as Dynkin diagrams and sometimes as Coxter graphs ([GHJ]). V. S. Sunder and N. J. Wildberger constructed in [SW] fusion rule algebras $\mathcal{F}(D)$ and hypergroups $\mathcal{K}(D)$ associated with certain Dynkin diagrams of type A_n, D_{2n} and so on. We give some examples of $\mathcal{K}(\hat{G} \cup \widehat{G_0})$ which are compatible with Frobenius diagrams $D(\hat{G} \cup \widehat{G_0})$.

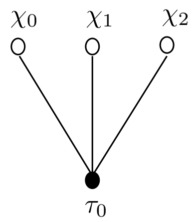
4.1 The case that $G = \mathbb{Z}_2 = \{e, g\}$ $(g^2 = e)$ and $G_0 = \{e\}$.



$\mathcal{K}(\hat{G} \cup \widehat{G_0}) = \{(ch(\chi_0), \circ), (ch(\chi_1), \circ), (ch(\tau_0), \bullet)\}$. Put $\gamma_0 = (ch(\chi_0), \circ)$, $\gamma_1 = (ch(\chi_1), \circ)$ and $\rho_0 = (ch(\tau_0), \bullet)$. Then the structure equations are

$$\gamma_1\gamma_1 = \gamma_0, \quad \rho_0\rho_0 = \frac{1}{2}\gamma_0 + \frac{1}{2}\gamma_1, \quad \gamma_1\rho_0 = \rho_0.$$

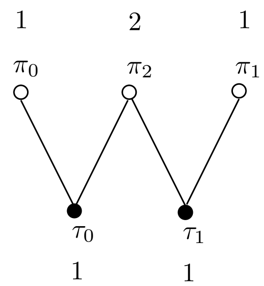
4.2 The case that $G = \mathbb{Z}_3 = \{e, g, g^2\}$ $(g^3 = e)$ and $G_0 = \{e\}$.



$\mathcal{K}(\hat{G} \cup \widehat{G_0}) = \{(ch(\chi_0), \circ), (ch(\chi_1), \circ), (ch(\chi_2), \circ), (ch(\tau_0), \bullet)\}$. Put $\gamma_0 = (ch(\chi_0), \circ)$, $\gamma_1 = (ch(\chi_1), \circ)$, $\gamma_2 = (ch(\chi_2), \circ)$ and $\rho_0 = (ch(\tau_0), \bullet)$. Then the structure equations are

$$\gamma_1\gamma_1 = \gamma_2, \quad \gamma_2\gamma_2 = \gamma_1, \quad \gamma_1\gamma_2 = \gamma_0,$$
$$\rho_0\rho_0 = \frac{1}{3}\gamma_0 + \frac{1}{3}\gamma_1 + \frac{1}{3}\gamma_2, \quad \gamma_1\rho_0 = \rho_0, \quad \gamma_2\rho_0 = \rho_0.$$

4.3 The case that G is the symmetric group $S_3 = \mathbb{Z}_3 \rtimes_\alpha \mathbb{Z}_2$ of degree 3 and $G_0 = \mathbb{Z}_2$.

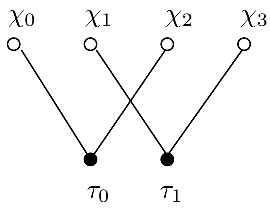


$\mathcal{K}(\hat{G} \cup \widehat{G_0}) = \{(ch(\pi_i), \circ), (ch(\tau_j), \bullet) : \pi_i \in \hat{G}, \tau_j \in \widehat{G_0}\}$. Put $\gamma_i = (ch(\pi_i), \circ)$ and $\rho_j = (ch(\tau_j), \bullet)$. Then the structure equations are

$$\gamma_1\gamma_1 = \gamma_0, \quad \gamma_2\gamma_2 = \frac{1}{4}\gamma_0 + \frac{1}{4}\gamma_1 + \frac{1}{2}\gamma_2, \quad \gamma_1\gamma_2 = \gamma_2, \quad \rho_0\rho_0 = \rho_1\rho_1 = \frac{1}{3}\gamma_0 + \frac{2}{3}\gamma_2,$$
$$\rho_0\rho_1 = \rho_1\rho_0 = \frac{1}{3}\gamma_1 + \frac{2}{3}\gamma_2, \quad \gamma_0\rho_0 = \rho_0, \quad \gamma_1\rho_0 = \rho_1, \quad \gamma_2\rho_0 = \frac{1}{2}\rho_0 + \frac{1}{2}\rho_1,$$
$$\gamma_0\rho_1 = \rho_1, \quad \gamma_1\rho_1 = \rho_0, \quad \gamma_2\rho_1 = \frac{1}{2}\rho_0 + \frac{1}{2}\rho_1.$$

Remark $\mathcal{K}(\widehat{\mathbb{Z}_2} \cup \widehat{\{e\}}) = \mathcal{K}(A_3)$, $\mathcal{K}(\widehat{\mathbb{Z}_3} \cup \widehat{\{e\}}) = \mathcal{K}(D_4)$ and $\mathcal{K}(\widehat{S_3} \cup \widehat{\mathbb{Z}_2}) = \mathcal{K}(A_5)$ where $\mathcal{K}(A_3)$, $\mathcal{K}(D_4)$ and $\mathcal{K}(A_5)$ are Sunder-Wildberger's hypergroups ([SW]) associated with Dynkin diagrams of type A_3, D_4 and A_5 respectively.

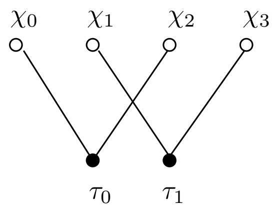
4.4 The case that $G = \mathbb{Z}_4 = \{e, g, g^2, g^3\}$ $(g^4 = e)$ and $G_0 = \mathbb{Z}_2$.



$\mathcal{K}(\hat{G} \cup \widehat{G_0}) = \{(ch(\chi_i), \circ), (ch(\tau_j), \bullet) : \chi_i \in \hat{G}, \tau_j \in \widehat{G_0}\}$. Put $\gamma_i = (ch(\chi_i), \circ)$ and $\rho_j = (ch(\tau_j), \bullet)$. Then the structure equations are

$$\gamma_1\gamma_1 = \gamma_2, \quad \gamma_2\gamma_2 = \gamma_0, \quad \gamma_3\gamma_3 = \gamma_1, \quad \gamma_1\gamma_2 = \gamma_3, \quad \gamma_1\gamma_3 = \gamma_0, \quad \gamma_2\gamma_3 = \gamma_1,$$
$$\rho_0\rho_0 = \rho_1\rho_1 = \frac{1}{2}\gamma_0 + \frac{1}{2}\gamma_2, \quad \rho_0\rho_1 = \rho_1\rho_0 = \frac{1}{2}\gamma_1 + \frac{1}{2}\gamma_3, \quad \gamma_0\rho_0 = \rho_0, \quad \gamma_1\rho_0 = \rho_1,$$
$$\gamma_2\rho_0 = \rho_0, \quad \gamma_3\rho_0 = \rho_1, \quad \gamma_0\rho_1 = \rho_1, \quad \gamma_1\rho_1 = \rho_0, \quad \gamma_2\rho_1 = \rho_1, \quad \gamma_3\rho_1 = \rho_0.$$

4.5 The case that $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(e,e), (e,g), (g,e), (g,g)\}$ $(g^2 = e)$ and $G_0 = \mathbb{Z}_2$.

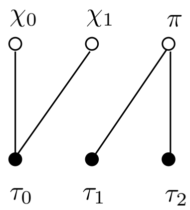


$\mathcal{K}(\hat{G} \cup \widehat{G_0}) = \{(ch(\chi_i), \circ), (ch(\tau_j), \bullet) : \chi_i \in \hat{G}, \tau_j \in \widehat{G_0}\}$. Put $\gamma_i = (ch(\chi_i), \circ)$ and $\rho_j = (ch(\tau_j), \bullet)$. Then the structure equations are

$$\gamma_1\gamma_1 = \gamma_0, \quad \gamma_2\gamma_2 = \gamma_0, \quad \gamma_3\gamma_3 = \gamma_0, \quad \gamma_1\gamma_2 = \gamma_3, \quad \gamma_1\gamma_3 = \gamma_2, \quad \gamma_2\gamma_3 = \gamma_1,$$
$$\rho_0\rho_0 = \rho_1\rho_1 = \frac{1}{2}\gamma_0 + \frac{1}{2}\gamma_2, \quad \rho_0\rho_1 = \rho_1\rho_0 = \frac{1}{2}\gamma_1 + \frac{1}{2}\gamma_3, \quad \gamma_0\rho_0 = \rho_0, \quad \gamma_1\rho_0 = \rho_1,$$
$$\gamma_2\rho_0 = \rho_0, \quad \gamma_3\rho_0 = \rho_1, \quad \gamma_0\rho_1 = \rho_1, \quad \gamma_1\rho_1 = \rho_0, \quad \gamma_2\rho_1 = \rho_1, \quad \gamma_3\rho_1 = \rho_0.$$

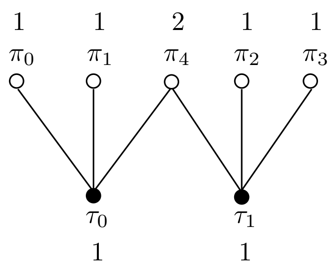
Remark We note that Frobenius diagrams of 4.4 and 4.5 are same but their hypergroup structures are different.

4.6 The case that $G = S_3 = \mathbb{Z}_3 \rtimes_\alpha \mathbb{Z}_2$ and $G_0 = \mathbb{Z}_3$.



$\mathcal{K}(\hat{G} \cup \widehat{G_0})$ is not a hypergroup by Corollary 3.17.

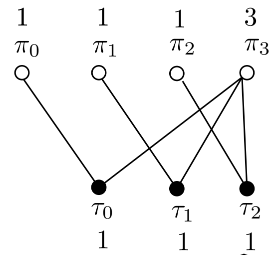
4.7 The case that G is the dihedral group $D_4 = \mathbb{Z}_4 \rtimes_\alpha \mathbb{Z}_2$ and $G_0 = \mathbb{Z}_2$.



$\mathcal{K}(\hat{G} \cup \widehat{G_0}) = \{(ch(\pi_i), \circ), (ch(\tau_j), \bullet) : \pi_i \in \hat{G}, \tau_j \in \widehat{G_0}\}$. Put $\gamma_i = (ch(\pi_i), \circ)$ and $\rho_j = (ch(\tau_j), \bullet)$. Then the structure equations are

$$\gamma_1\gamma_1 = \gamma_0, \quad \gamma_2\gamma_2 = \frac{1}{4}\gamma_0 + \frac{1}{4}\gamma_1 + \frac{1}{4}\gamma_3 + \frac{1}{4}\gamma_4, \quad \gamma_3\gamma_3 = \gamma_0, \quad \gamma_4\gamma_4 = \gamma_0,$$
$$\gamma_1\gamma_2 = \gamma_2, \quad \gamma_1\gamma_3 = \gamma_4, \quad \gamma_1\gamma_4 = \gamma_3, \quad \gamma_2\gamma_3 = \gamma_2, \quad \gamma_2\gamma_4 = \gamma_2, \quad \gamma_3\gamma_4 = \gamma_1,$$
$$\rho_0\rho_0 = \rho_1\rho_1 = \frac{1}{4}\gamma_0 + \frac{1}{4}\gamma_1 + \frac{1}{2}\gamma_2, \quad \rho_0\rho_1 = \frac{1}{2}\gamma_2 + \frac{1}{4}\gamma_3 + \frac{1}{4}\gamma_4,$$
$$\gamma_1\rho_0 = \rho_0, \quad \gamma_2\rho_0 = \frac{1}{2}\rho_0 + \frac{1}{2}\rho_1, \gamma_3\rho_0 = \rho_1, \quad \gamma_4\rho_0 = \rho_1,$$
$$\gamma_1\rho_1 = \rho_1, \quad \gamma_2\rho_1 = \frac{1}{2}\rho_0 + \frac{1}{2}\rho_1, \quad \gamma_3\rho_1 = \rho_0, \quad \gamma_4\rho_1 = \rho_0.$$

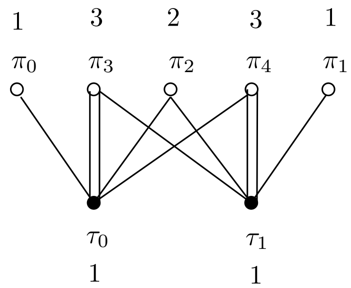
4.8 The case that G is the alternating group $A_4 = (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes_\alpha \mathbb{Z}_3$ of degree 4 and $G_0 = \mathbb{Z}_3$.



$\mathcal{K}(\hat{G} \cup \widehat{G_0}) = \{(ch(\pi_i), \circ), (ch(\tau_j), \bullet) : \pi_i \in \hat{G}, \tau_j \in \widehat{G_0}\}$. Put $\gamma_i = (ch(\pi_i), \circ)$ and $\rho_j = (ch(\tau_j), \bullet)$. Then the structure equations are

$$\gamma_1\gamma_1 = \gamma_2, \quad \gamma_2\gamma_2 = \gamma_1, \quad \gamma_3\gamma_3 = \frac{1}{9}\gamma_0 + \frac{1}{9}\gamma_1 + \frac{1}{9}\gamma_2 + \frac{2}{3}\gamma_3, \quad \gamma_1\gamma_2 = \gamma_0, \quad \gamma_1\gamma_3 = \gamma_3,$$
$$\gamma_2\gamma_3 = \gamma_3, \quad \rho_0\rho_0 = \rho_1\rho_2 = \frac{1}{4}\gamma_0 + \frac{3}{4}\gamma_3, \quad \rho_0\rho_1 = \frac{1}{4}\gamma_1 + \frac{3}{4}\gamma_3, \quad \rho_0\rho_2 = \frac{1}{4}\gamma_2 + \frac{3}{4}\gamma_3,$$
$$\gamma_1\rho_0 = \rho_1, \quad \gamma_2\rho_0 = \rho_2, \quad \gamma_1\rho_1 = \rho_2, \quad \gamma_2\rho_1 = \rho_0,$$
$$\gamma_3\rho_0 = \gamma_3\rho_1 = \gamma_3\rho_2 = \frac{1}{3}\rho_0 + \frac{1}{3}\rho_1 + \frac{1}{3}\rho_2.$$

4.9 The case that G is the symmetric group $S_4 = A_4 \rtimes_\alpha \mathbb{Z}_2$ of degree 4 and $G_0 = \mathbb{Z}_2$.

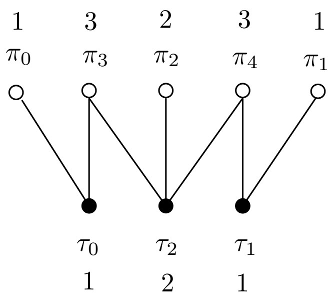


$\mathcal{K}(\hat{G} \cup \widehat{G_0}) = \{(ch(\pi_i), \circ), (ch(\tau_j), \bullet) : \pi_i \in \hat{G}, \tau_j \in \widehat{G_0}\}$. Put $\gamma_i = (ch(\pi_i), \circ)$ and

$\rho_j = (ch(\tau_j), \bullet)$. Then the structure equations are

$$\gamma_1\gamma_1 = \gamma_0, \quad \gamma_2\gamma_2 = \frac{1}{4}\gamma_0 + \frac{1}{4}\gamma_1 + \frac{1}{2}\gamma_2, \quad \gamma_3\gamma_3 = \gamma_4\gamma_4 = \frac{1}{9}\gamma_0 + \frac{2}{9}\gamma_2 + \frac{1}{3}\gamma_3 + \frac{1}{3}\gamma_4,$$
$$\gamma_1\gamma_2 = \gamma_2, \quad \gamma_1\gamma_3 = \gamma_4, \quad \gamma_1\gamma_4 = \gamma_3, \quad \gamma_2\gamma_3 = \gamma_2\gamma_4 = \frac{1}{2}\gamma_3 + \frac{1}{2}\gamma_4,$$
$$\gamma_3\gamma_4 = \frac{1}{9}\gamma_1 + \frac{2}{9}\gamma_2 + \frac{1}{3}\gamma_3 + \frac{1}{3}\gamma_4, \quad \rho_0\rho_0 = \rho_1\rho_1 = \frac{1}{12}\gamma_0 + \frac{1}{6}\gamma_2 + \frac{1}{2}\gamma_3 + \frac{1}{4}\gamma_4,$$
$$\rho_0\rho_1 = \rho_1\rho_0 = \frac{1}{12}\gamma_1 + \frac{1}{6}\gamma_2 + \frac{1}{4}\gamma_3 + \frac{1}{2}\gamma_4, \quad \gamma_0\rho_0 = \rho_0, \quad \gamma_1\rho_0 = \rho_1,$$
$$\gamma_2\rho_0 = \frac{1}{2}\rho_0 + \frac{1}{2}\rho_1, \quad \gamma_3\rho_0 = \frac{2}{3}\rho_0 + \frac{1}{3}\rho_1, \quad \gamma_4\rho_0 = \frac{1}{3}\rho_0 + \frac{2}{3}\rho_1, \quad \gamma_0\rho_1 = \rho_1,$$
$$\gamma_1\rho_1 = \rho_0, \quad \gamma_2\rho_1 = \frac{1}{2}\rho_0 + \frac{1}{2}\rho_1, \quad \gamma_3\rho_1 = \frac{1}{3}\rho_0 + \frac{2}{3}\rho_1, \quad \gamma_4\rho_1 = \frac{2}{3}\rho_0 + \frac{1}{3}\rho_1.$$

4.10 The case that G is the symmetric group S_4 of degree 4 and G_0 is the symmetric group S_3 of degree 3.



$\mathcal{K}(\hat{G} \cup \widehat{G_0}) = \{(ch(\pi_i), \circ), (ch(\tau_j), \bullet) : \pi_i \in \hat{G}, \tau_j \in \widehat{G_0}\}$. Put $\gamma_i = (ch(\pi_i), \circ)$ and $\rho_j = (ch(\tau_j), \bullet)$. Then the structure equations are

$$\gamma_1\gamma_1 = \gamma_0, \quad \gamma_2\gamma_2 = \frac{1}{4}\gamma_0 + \frac{1}{4}\gamma_1 + \frac{1}{2}\gamma_2, \quad \gamma_3\gamma_3 = \gamma_4\gamma_4 = \frac{1}{9}\gamma_0 + \frac{2}{9}\gamma_2 + \frac{1}{3}\gamma_3 + \frac{1}{3}\gamma_4,$$
$$\gamma_1\gamma_2 = \gamma_2, \quad \gamma_1\gamma_3 = \gamma_4, \quad \gamma_1\gamma_4 = \gamma_3, \quad \gamma_2\gamma_3 = \gamma_2\gamma_4 = \frac{1}{2}\gamma_3 + \frac{1}{2}\gamma_4,$$
$$\gamma_3\gamma_4 = \frac{1}{9}\gamma_1 + \frac{2}{9}\gamma_2 + \frac{1}{3}\gamma_3 + \frac{1}{3}\gamma_4, \quad \rho_0\rho_0 = \rho_1\rho_1 = \frac{1}{4}\gamma_0 + \frac{3}{4}\gamma_3,$$
$$\rho_2\rho_2 = \frac{1}{16}\gamma_0 + \frac{1}{16}\gamma_1 + \frac{1}{8}\gamma_2 + \frac{3}{8}\gamma_3 + \frac{3}{8}\gamma_4, \quad \rho_1\rho_2 = \frac{1}{4}\gamma_2 + \frac{3}{8}\gamma_3 + \frac{4}{8}\gamma_4,$$
$$\gamma_0\rho_0 = \rho_0, \quad \gamma_1\rho_0 = \rho_1, \quad \gamma_2\rho_0 = \rho_2, \quad \gamma_3\rho_0 = \frac{1}{3}\rho_0 + \frac{2}{3}\rho_2, \quad \gamma_4\rho_0 = \frac{1}{3}\rho_1 + \frac{2}{3}\rho_2,$$
$$\gamma_0\rho_1 = \rho_1, \quad \gamma_1\rho_1 = \rho_0, \quad \gamma_2\rho_1 = \rho_2, \quad \gamma_3\rho_1 = \frac{1}{3}\rho_1 + \frac{2}{3}\rho_2, \quad \gamma_4\rho_1 = \frac{1}{3}\rho_0 + \frac{2}{3}\rho_2,$$
$$\gamma_0\rho_2 = \rho_2, \quad \gamma_1\rho_2 = \rho_2, \quad \gamma_2\rho_2 = \frac{1}{4}\rho_0 + \frac{1}{4}\rho_1 + \frac{1}{2}\rho_2,$$
$$\gamma_3\rho_2 = \gamma_4\rho_0 = \frac{1}{6}\rho_0 + \frac{1}{6}\rho_1 + \frac{2}{3}\rho_2.$$

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