LEFT REGULAR AND INTRA-REGULAR ORDERED HYPERSEMIGROUPS IN TERMS OF SEMIPRIME AND FUZZY SEMIPRIME SUBSETS

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Received November 22, 2016 ; revised January 26, 2017

To the memory of Professor Kiyoshi Iséki

Abstract

We prove that an ordered hypersemigroup H is left (resp. right) regular if and only if every left (resp. right) ideal of H is semiprime and it is intra-regular if and only if every ideal of H is semiprime. Then we prove that an ordered hypersemigroup H is left (resp. right) regular if and only if every fuzzy left (resp. right) ideal of H is fuzzy semiprime and it is intra-regular if and only if every fuzzy ideal of H is fuzzy semiprime.

1 Introduction and prerequisites

A semigroup $(S, \cdot)$ is left (resp. right) regular if and only if every left (resp. right) ideal of S is semiprime, it is intra-regular if and only if every ideal of S is semiprime (cf. [1; Theorems 4.2, 4.4]). For an ordered semigroup $(S, \cdot, \leq)$ and a subset A of S, we denote by $(A]$ the subset of S defined by $(A] = \{t \in S \mid t \leq a \text{ for some } a \in A\}$. An ordered semigroup $(S, \cdot, \leq)$ is called *left regular* if for every $a \in S$ there exists $x \in S$ such that $a \leq xa^2$. This is equivalent to saying that $a \in (Sa^2]$ for every $a \in S$ or $A \subseteq (SA^2]$ for every $A \subseteq S$. It is called *right regular* if for every $a \in S$ there exists $x \in S$ such that $a \leq a^2x$, equivalently if $a \in (a^2S]$ for every $a \in S$ or $A \subseteq (A^2S]$ for every $A \subseteq S$. An ordered semigroup $(S, \cdot, \leq)$ is called *intra-regular* if for every $a \in S$ there exist $x, y \in S$ such that $a \leq xa^2y$. This is equivalent to saying that $a \in (Sa^2S]$ for every $a \in S$ or $A \subseteq (SA^2S]$ for every $A \subseteq S$. We have seen in [10] that an ordered semigroup

[0]2010 *Mathematics Subject Classification*. Primary: 06F99; 08A72.
Key words and Phrases. Hypergroupoid, ordered hypersemigroup, left (right) regular, intra-regular, left (right) ideal, fuzzy left (right) ideal, semiprime, fuzzy semiprime.

S is left (resp. right) regular if and only if the left (resp. right) ideals of S are semiprime and it is intra-regular if and only if the ideals of S are semiprime. We have also seen that an ordered semigroup S is left (resp. right) regular if and only if the fuzzy left (resp. fuzzy right) ideals of S are semiprime and it is intra-regular if and only if the fuzzy ideals of S are semiprime. In the present paper we examine these results for an hypersemigroup. For the sake of completeness, let us first give some definitions-remarks already given in [7, 8].

An *hypergroupoid* is a nonempty set H with an hyperoperation

$$\circ : H \times H \to \mathcal{P}^*(H) \mid (a, b) \to a \circ b$$

on H and an operation

$$* : \mathcal{P}^*(H) \times \mathcal{P}^*(H) \to \mathcal{P}^*(H) \mid (A, B) \to A * B$$

on $\mathcal{P}^*(H)$ (induced by the operation of H) such that

$$A * B = \bigcup_{(a,b)\in\, A\times B} (a \circ b)$$

for every $A, B \in \mathcal{P}^*(H)$ ($\mathcal{P}^*(H)$ is the set of nonempty subsets of H). As the operation "$*$" depends on the hyperoperation "$\circ$", an hypergroupoid can be denoted by $(H, \circ)$ (instead of $(H, \circ, *)$). If $(H, \circ)$ is an hypergroupoid and $A, B, C, D \in \mathcal{P}^*(H)$, then

$A \subseteq B$, implies $A * C \subseteq B * C$ and $C * A \subseteq C * B$. Equivalently,

$A \subseteq B$ and $C \subseteq D$ implies $A * C \subseteq B * D$ and $C * A \subseteq D * B$.

We also have $H * H \subseteq H$.

If H is an hypergroupoid then, for every $x, y \in H$, we have

$$\{x\} * \{y\} = x \circ y.$$

Indeed, $\{x\} * \{y\} = \bigcup\limits_{u\in\{x\}, v\in\{y\}} (u \circ v) = x \circ y$.

The following proposition, though clear, plays an essential role in the theory of hypergroupoids.

Proposition 1.1. *Let $(H, \circ)$ be an hypergroupoid, $x \in H$ and $A, B \in \mathcal{P}^*(H)$. Then we have the following:*

1. *$x \in A * B \Longleftrightarrow x \in a \circ b$ for some $a \in A$, $b \in B$.*
2. *If $a \in A$ and $b \in B$, then $a \circ b \subseteq A * B$.*

Lemma 1.2. [7] *Let $(H, \circ)$ be an hypergroupoid and $A_i, B \in \mathcal{P}^*(H)$, $i \in I$. Then we have the following:*

(1) $(\bigcup\limits_{i\in I} A_i) * B = \bigcup\limits_{i\in I} (A_i * B)$.

(2) $B * (\bigcup\limits_{i\in I} A_i) = \bigcup\limits_{i\in I} (B * A_i)$.

An hypergroupoid H is called *hypersemigroup* if, for every $x, y, z \in H$, we have

$$\{x\} * (y \circ z) = (x \circ y) * \{z\}$$

which is equivalent to saying that $\{x\} * \Big(\{y\} * \{z\}\Big) = \Big(\{x\} * \{y\}\Big) * \{z\}$ for every $x, y, z \in H$. If we like, we can identify the $\{x\}$ by x and the $\{z\}$ by z and write $x * (y \circ z)$ instead of $\{x\} * (y \circ z)$ and $(x \circ y) * z$ instead of $(x \circ y) * \{z\}$. So the associativity relation of an hypergroupoid can be also given, for short, as $x * (y \circ z) = (x \circ y) * z$.

Lemma 1.3 [7] *If $(H, \circ)$ is an hypersemigroup and $A, B, C \in \mathcal{P}^*(H)$, then we have*

$$\begin{aligned}(A * B) * C &= \bigcup_{(a,b,c) \in A \times B \times C} \Big((a \circ b) * \{c\}\Big) \\ &= \bigcup_{(a,b,c) \in A \times B \times C} \Big(\{a\} * (b \circ c)\Big) = A * (B * C) \\ &= \bigcup_{(a,b,c) \in A \times B \times C} \Big(\{a\} * \{b\} * \{c\}\Big).\end{aligned}$$

Thus we can write $(A * B) * C = A * (B * C) = A * B * C$. As a consequence, for any product $A_1 * A_2 * ..... * A_n$ of elements of $\mathcal{P}^*(H)$ we can put the parentheses in any place beginning with some A_i and ending in some A_j $(1 \le i, j \le n)$. In addition, using induction, we have the following which gives the form of the elements of the set $A_1 * A_2 * ..... * A_n$.

Lemma 1.4. For any finite family $A_1, A_2, ..., A_n$ of elements of $\mathcal{P}^*(H)$, we have

$$A_1 * A_2 * ..... * A_n = \bigcup_{(a_1, a_2 ... a_n) \in A_1 \times A_2 \times ... \times A_n} \Big(\{a_1\} * \{a_2\} * ... * \{a_n\}\Big).$$

For an hypergroupoid H, we denote by $(A]$ the subset of H defined by

$$(A] := \{t \in H \mid t \le a \text{ for some } a \in A\}.$$

Exactly as in ordered semigroups, we have $(H] = H$ and $((A]] = (A]$ for any nonempty subset A of H.
The results of the present paper hold not only for the elements but for the subsets of H as well which shows the pointless character of the results.

2 A characterization of left regular (resp. intra-regular) ordered hypersemigroups in terms of semiprime left ideals (resp. ideals)

Notation 2.1. Let $(H, \circ)$ be an hypergroupoid and "$\le$" an order relation on H. Denote by "$\preceq$" the relation on $\mathcal{P}^*(H)$ defined by

$$\preceq := \{(A, B) \mid \forall\, a \in A\ \exists\, b \in B \text{ such that } (a, b) \in \le\}.$$

So, for $A, B \in \mathcal{P}^*(H)$, we write $A \preceq B$ if for every $a \in A$ there exists $b \in B$ such that $a \leq b$. This is a reflexive and transitive relation on $\mathcal{P}^*(H)$, that is, a preorder on $\mathcal{P}^*(H)$.

A semigroup $(S, \cdot)$ is called an ordered semigroup if there exists an order relation "$\leq$" on S such that $(a, b) \in\leq$ implies $(ac, bc) \in\leq$ and $(ca, cb) \in\leq$ for every $c \in S$. Using the notation $a \leq b$ instead of $(a, b) \in\leq$, we write $a \leq b$ implies $ac \leq bc$ and $ca \leq cb$ for every $c \in S$. The definition of the ordered semigroup can be naturally transferred to hypersemigroups as follows:

Definition 2.2. (cf. also [13]) Let $(H, \circ)$ be an hypergroupoid and "$\leq$" an order relation on H. Then H is called an *ordered hypergroupoid*, denoted by $(H, \circ, \leq)$, if given an element $(x, y) \in\leq$, we have $(x \circ z, y \circ z) \in\preceq$ and $(z \circ x, z \circ y) \in\preceq$ for every $z \in H$. In other words,

$$x \leq y \text{ implies } x \circ z \preceq y \circ z \text{ and } z \circ x \preceq z \circ y \text{ for all } z \in H.$$

The concept of right regular ordered semigroups introduced by Kehayopulu in [4] is as follows: An ordered semigroup $(S, \cdot, \leq)$ is called right regular if for every $a \in S$ there exists $x \in S$ such that $a \leq a^2x$. Later, in an analogous manner she defined and studied the left regular ordered semigroups: An ordered semigroup S is called left regular if for every $a \in S$ there exists $x \in S$ such that $a \leq xa^2$. The concept of left regular ordered semigroups is naturally transferred to an ordered hypersemigroup H as follows: for every $a \in H$, there exists $x \in H$ such that $\{a\} \preceq \{x\} * (a \circ a)$. (Clearly $\{x\} * (a \circ a) = (x \circ a) * \{a\} = \{x\} * \{a\} * \{a\}$). This leads to the following definition.

Definition 2.3. An ordered hypersemigroup H is called *left regular* if for every $a \in H$ there exist $x, t \in H$ such that $t \in \{x\} * (a \circ a)$ and $a \leq t$.
It is called *right regular* if for every $a \in H$ there exist $x, t \in H$ such that $t \in (a \circ a) * \{x\}$ and $a \leq t$.

Proposition 2.4. *Let H be an ordered hypersemigroup. The following are equivalent:*

1. *H is left regular.*
2. *$a \in \Big(H * (a \circ a)\Big]$ for every $a \in H$.*
3. *$A \subseteq (H * A * A]$ for every $A \in \mathcal{P}^*(H)$.*

Proof. (1) $\Longrightarrow$ (2). Let $a \in H$. Since H is left regular, there exist $x, t \in H$ such that $t \in \{x\} * (a \circ a)$ and $a \leq t$. We have

$$a \leq t \in \{x\} * (a \circ a) \subseteq H * (a \circ a),$$

so $a \in \Big(H * (a \circ a)\Big]$.
(2) $\Longrightarrow$ (3). Let $A \in \mathcal{P}^*(H)$ and $a \in A$. Since $a \in H$, by (2), we have

$$a \in \Big(H * (a \circ a)\Big] = \Big(H * \{a\} * \{a\}\Big] \subseteq (H * A * A],$$

thus we get $a \in (H * A * A]$, and (3) holds.
(3) $\Longrightarrow$ (1). Let $a \in H$. Since $\{a\} \in \mathcal{P}^*(H)$, by (3), we have

$$a \in \{a\} \subseteq \Big(H * \{a\} * \{a\}\Big].$$

Then $a \leq t$ for some $t \in \Big(H * \{a\}\Big) * \{a\}$. By Proposition 1.1, there exists $y \in H * \{a\}$ such that $t \in y \circ a$. Since $y \in H * \{a\}$, again by Proposition 1.1, there exists $x \in H$ such that $y \in x \circ a$. We have

$$t \in y \circ a \subseteq (x \circ a) * \{a\} = \{x\} * (a \circ a).$$

Since $x, t \in H$ such that $t \in \{x\} * (a \circ a)$ and $a \leq t$, H is left regular. □
In a similar way we prove the following:

Proposition 2.5. *Let H be an ordered hypersemigroup. The following are equivalent:*

1. *H is right regular.*
2. *$a \in \Big((a \circ a) * H\Big]$ for every $a \in H$.*
3. *$A \subseteq (A * A * H]$ for every $A \in \mathcal{P}^*(H)$.*

A subset A of a groupoid or an ordered groupoid S is called semiprime if $x^2 \in A$ ($x \in S$) implies $x \in A$ [1, 2, 5]. This concept is naturally transferred in case of hypergroupoids in the definition below:

Definition 2.6. Let H be an hypergroupoid. A nonempty subset A of H is called *semiprime* if for every $t \in H$ such that $t \circ t \subseteq A$, we have $t \in A$.

Proposition 2.7. *Let $(H, \circ)$ be an hypergroupoid and $A \in \mathcal{P}^*(H)$. The following are equivalent:*

1. *A is semiprime.*
2. *For every $T \in \mathcal{P}^*(H)$ such that $T * T \subseteq A$, we have $T \subseteq A$.*

Proof. (1) $\Longrightarrow$ (2). Let $T \in \mathcal{P}^*(H)$, $T * T \subseteq A$ and $t \in T$. Since $t \in H$ and $t \circ t \subseteq T * T \subseteq A$, by (1), we have $t \in A$.
(2) $\Longrightarrow$ (1). Let $t \in H$ such that $t \circ t \subseteq A$. Since $\{t\} \in \mathcal{P}^*(H)$ and $\{t\} * \{t\} = t \circ t \subseteq A$, by (2), we have $\{t\} \subseteq A$, so $t \in A$, and T is semiprime. □

Lemma 2.8. *Let H be an ordered hypergroupoid and $A, B \in \mathcal{P}^*(H)$. Then we have*

$$(A] * (B] \subseteq (A * B].$$

Proof. Let $t \in (A] * (B]$. Then $t \in x \circ y$ for some $x \in (A]$, $y \in (B]$. Since $x \in (A]$, we have $x \leq a$ for some $a \in A$. Since $y \in (B]$, we get $y \leq b$ for some $b \in B$. Since $x \leq a$ and $y \leq b$, we have $t \in x \circ y \preceq a \circ b$. Then, there exists $z \in a \circ b$ such that $t \leq z$. We get $t \leq z \in a \circ b$, so $t \in (a \circ b]$. On the other hand, since $a \in A$ and $b \in B$, we have $a \circ b \subseteq A * B$. Then $(a \circ b] \subseteq (A * B]$, and $t \in (A * B]$. □

The concepts of left and right ideals of ordered groupoids introduced by Kehayopulu in [3] are naturally transferred in case of ordered hypergroupoids as follows: A nonempty subset A of an ordered hypergroupoid H is called a *left* (resp. *right*) *ideal* of H if

1. $H * A \subseteq A$ (resp. $A * H \subseteq A$) and
2. if $a \in A$ and $H \ni b \le a$, then $b \in A$, that is if $(A] = A$.

It is called an *ideal* of H if it is both a left and a right ideal of H.

Theorem 2.9. *An ordered hypersemigroup H is left regular if and only if every left ideal of H is semiprime.*

Proof. $\Longrightarrow$. Let A be a left ideal of H and $a \in H$ such that $a \circ a \subseteq A$. Since H is left regular and $a \in H$, there exist $x, t \in H$ such that $t \in \{x\} * (a \circ a)$ and $a \le t$. We have $t \in \{x\} * (a \circ a) \subseteq H * A \subseteq A$. Then $a \le t \in A$, and $a \in A$.
$\Longleftarrow$. Let $a \in H$. We have

$$(a \circ a) * (a \circ a) \subseteq H * (a \circ a) \subseteq \Big(H * (a \circ a)\Big].$$

The set $\Big(H * (a \circ a)\Big]$ is a left ideal of H. Indeed, it is a nonempty subset of H and we have

$$\begin{aligned} H * \Big(H * (a \circ a)\Big] &= (H] * \Big(H * (a \circ a)\Big] \subseteq \Bigg(H * \Big(H * (a \circ a)\Big)\Bigg] \\ &= \Big((H * H) * (a \circ a)\Big] \subseteq \Big(H * (a \circ a)\Big], \end{aligned}$$

and

$$\Bigg(\Big(H * (a \circ a)\Big]\Bigg] = \Big(H * (a \circ a)\Big]$$

(as $((A]] = (A]$ for any subset A of S). Since $\Big(H * (a \circ a)\Big]$ is semiprime, we have $(a \circ a) \subseteq \Big(H * (a \circ a)\Big]$, and $a \in \Big(H * (a \circ a)\Big]$. Then, by Proposition 2.4, H is left regular.

Now we will give a second proof of the Theorem using only sets: $\Longrightarrow$. Let A be a left ideal of H and $T \in \mathcal{P}^*(H)$ such that $T * T \subseteq A$. Since H is left regular, by Proposition 2.4, we have $T \subseteq (H * T * T] \subseteq (H * A] \subseteq (A] = A$. $\Longleftarrow$. Let $A \in \mathcal{P}^*(H)$. We have

$$(A * A) * (A * A) \subseteq (H * H) * A * A \subseteq H * A * A \subseteq (H * A * A].$$

Since $(H*A*A]$ is a left ideal of H, it is semiprime, and we have $A*A \subseteq (H*A*A]$ and $A \subseteq (H * A * A]$. Thus H is left regular. □

In a similar way we prove the following:

Theorem 2.10. *An ordered hypersemigroup H is right regular if and only if every right ideal of H is semiprime.*

Our aim now is to characterize the intra-regular ordered hypersemigroups in terms of semiprime ideals. The concept of an intra-regular ordered semigroup introduced by Kehayopulu in [6] is as follows: An ordered semigroup $(S, \cdot, \leq)$ is called intra-regular if for every $a \in S$ there exist $x, y \in S$ such that $a \leq xa^2y$. This concept is naturally transferred to an ordered hypersemigroup as follows: For every $a \in H$, $\{a\} \preceq \{x\} * (a \circ a) * \{y\}$. This leads to the following definition

Definition 2.11. An ordered hypersemigroup $(H, \circ, \leq)$ is called *intra-regular* if for every $a \in H$ there exist $x, y, t \in H$ such that $t \in \{x\} * (a \circ a) * \{y\}$ and $a \leq t$.
Clearly, $\{x\} * (a \circ a) * \{y\} = (x \circ a) * (a \circ y) = \{x\} * \{a\} * \{a\} * \{y\}$.

Proposition 2.12. *Let $(H, *, \leq)$ be an ordered hypersemigroup. The following are equivalent:*

1. *H is intra-regular.*
2. *$a \in \Big(H * (a \circ a) * H\Big]$ for every $a \in H$.*
3. *$A \subseteq (H * A * A * H]$ for every $A \in \mathcal{P}^*(H)$.*

Proof. (1) $\Longrightarrow$ (2). Let $a \in H$. Since H is intra-regular, there exist $x, y, t \in H$ such that $t \in \{x\} * (a \circ a) * \{y\}$ and $a \leq t$. We have

$$a \leq t \in \{x\} * (a \circ a) * \{y\} \subseteq H * (a \circ a) * H,$$

so $a \in \Big(H * (a \circ a) * H\Big]$.
(2) $\Longrightarrow$ (3). Let $A \in \mathcal{P}^*(H)$ and $a \in A$. By (2), we have

$$a \in \Big(H * (a \circ a) * H\Big] = \Big(H * \{a\} * \{a\} * H\Big] \subseteq (H * A * A * H],$$

so $a \in (H * A * A * H]$ and (3) is satisfied.
(3) $\Longrightarrow$ (1). Let $a \in H$. Since $\{a\} \in \mathcal{P}^*(H)$, by (3), we have

$$a \in \{a\} \subseteq \Big(H * \{a\} * \{a\} * H\Big].$$

Then $a \leq t$ for some $t \in H * \{a\} * \{a\} * H = \Big(H * (a \circ a)\Big) * H$. Then there exist $u \in H * (a \circ a)$ and $y \in H$ such that $t \in u \circ y$. Since $u \in H * (a \circ a)$, there exist $x \in H$ and $w \in (a \circ a)$ such that $u \in x \circ w$. We have

$$t \in u \circ y \subseteq (x \circ w) * \{y\} = \{x\} * \{w\} * \{y\} \subseteq \{x\} * (a \circ a) * \{y\}.$$

For the elements $x, y, t \in H$, we have $t \in \{x\} * (a \circ a) * \{y\}$ and $a \leq t$, so H is intra-regular. $\square$

Theorem 2.13. *An ordered hypersemigroup H is intra-regular if and only if every ideal of H is semiprime.*

$\Longrightarrow$. Let A be an ideal of H and $a \in H$ such that $a \circ a \subseteq A$. Since $a \in H$ and H is intra-regular, there exist $x, y, t \in H$ such that $t \in H * (a \circ a) * \{y\}$ and $a \leq t$. Then $t \in H * (a \circ a) * \{y\} \subseteq H * A * H \subseteq A$. Since $a \in H$ and $a \leq t \in A$, we have $a \in A$. Thus H is semiprime.
$\Longleftarrow$. Let $a \in H$. We have

$$(a \circ a) * (a \circ a) \subseteq H * \{a\} * \{a\} * H \subseteq \Big(H * \{a\} * \{a\} * H\Big].$$

The set $\Big(H * \{a\} * \{a\} * H\Big]$ is an ideal of H, so it is semiprime. Hence we have $a \circ a \subseteq \Big(H * \{a\} * \{a\} * H\Big]$, and $a \in \Big(H * \{a\} * \{a\} * H\Big] = \Big(H * (a \circ a) * H\Big]$. By Proposition 2.12, H is intra-regular.

A second proof of the theorem using only sets is as follows: $\Longrightarrow$. Let A be an ideal of H and $T \in \mathcal{P}^*(H)$ such that $T * T \subseteq A$. Since H is intra-regular, by Proposition 2.12, we have

$$T \subseteq (H * T * T * H] \subseteq (H * A * H] \subseteq (A] = A,$$

so A is semiprime. $\Longleftarrow$. Let A be a nonempty subset of H. Since $(A * A) * (A * A) \subseteq (H*A*A*H]$ and $(H*A*A*H]$ is semiprime, we have $A*A \subseteq (H*A*A*H]$, and $A \subseteq (H * A * A * H]$. $\square$

3 A characterization of left regular and intra-regular ordered hypersemigroups in terms of fuzzy semiprime subsets

Following Zadeh, any mapping $f : H \to [0,1]$ of a ordered hypergroupoid H into the closed interval $[0,1]$ of real numbers is called a *fuzzy subset* of H (or a *fuzzy set* in H) and f_A (: the characteristic function of A) is the mapping

$$f_A : H \to \{0,1\} \mid x \to f_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

The concepts of fuzzy right and fuzzy left ideals of an ordered groupoid due to Kehayopulu-Tsingelis [9] are naturally transferred to an ordered hypersemigroup as follows:

Definition 3.1. Let H be an ordered hypergroupoid. A fuzzy subset f of H is called a *fuzzy left ideal* of H if

1. $f(x \circ y) \geq f(y)$ for all $x, y \in H$, in the sense that if $x, y \in H$ and $u \in x \circ y$, then $f(u) \geq f(y)$ and
2. $x \leq y$ implies $f(x) \geq f(y)$.

A fuzzy subset f of H is called a *fuzzy right ideal* of H if

1. $f(x \circ y) \geq f(x)$ for all $x, y \in H$, meaning that if $x, y \in H$ and $u \in x \circ y$, then $f(u) \geq f(x)$ and
2. $x \leq y$ implies $f(x) \geq f(y)$.

A fuzzy subset of H is called a *fuzzy ideal* of H it is both a fuzzy left and a fuzzy right ideal of H. As one can easily see, a fuzzy subset f of H is a fuzzy ideal of H if and only if

1. if $f(x \circ y) \geq \max\{f(x), f(y)\}$ for all $x, y \in H$, in the sense that if $x, y \in H$ and $u \in x \circ y$, then $f(u) \geq \max\{f(x), f(y)\}$ and
2. if $x \leq y$, then $f(x) \geq f(y)$.

The concept of fuzzy semiprime subsets of groupoids introduced by Kuroki in [12] is as follows: A fuzzy subset f of a groupoid S is called semiprime if $f(a) \geq f(a^2)$ for every $a \in S$, and remains the same in case of ordered groupoids as well [11]. This concept is naturally transferred in case of an hypergroupoid as follows:

Definition 3.2. Let $(H, \circ)$ be an hypergroupoid. A fuzzy subset f of H is called *fuzzy semiprime* if

$$f(a) \geq f(a \circ a) \text{ for every } a \in H,$$

in the sense that if $u \in a \circ a$, then $f(a) \geq f(u)$.

Remark 3.3. Let $(H, \circ)$ be an hypergroupoid and f a semiprime fuzzy left ideal (or fuzzy right ideal) of H. Then, for every $a \in H$, we have $f(a) = f(a \circ a)$, meaning that if $u \in a \circ a$, then $f(a) = f(u)$. Indeed: Let $u \in a \circ a$. Since f is a fuzzy left (or right) ideal of H, we have $f(a \circ a) \geq f(a)$, then $f(u) \geq f(a)$. Since f is semiprime, we have $f(a) \geq f(a \circ a)$, then $f(a) \geq f(u)$. Thus we have $f(a) = f(u)$.

Lemma 3.4. *Let $(H, \circ, \leq)$ be an ordered hypergroupoid. If A is a left (resp. right) ideal of H, then the characteristic function f_A is a fuzzy left (resp. fuzzy right) ideal of H. "Conversely", if A is a nonempty subset of H such that f_A is a fuzzy left (resp. fuzzy right) ideal of H, then A is a left (resp. right) ideal of H.*

Proof. For the hypergroupoid $(H, \circ)$ the lemma is satisfied (cf. [8; Proposition 7]). It remains to prove that the following are equivalent:
(1) $y \in A$ and $H \ni x \leq y \Longrightarrow x \in A$ and
(2) $x \leq y \Longrightarrow f_A(x) \geq f_A(y)$.
(1) $\Longrightarrow$ (2). Let $x \leq y$. If $y \in A$ then, by (1), we have $x \in A$. Then $f_A(x) = 1 \geq f_A(y)$. If $y \notin A$, then $f_A(y) = 0 \leq f_A(x)$.
(2) $\Longrightarrow$ (1). Let $y \in A$ and $H \ni x \leq y$. Since $x \leq y$, by (2), we have $f_A(x) \geq f_A(y) = 1$. Then $f_A(x) = 1$, and $x \in A$. □

Lemma 3.5. *Let H be an ordered hypergroupoid. A nonempty subset A of H is an ideal of H if and only if the characteristic function f_A is a fuzzy ideal of H.*

Lemma 3.6. *Let $(H, \circ, \leq)$ be an ordered hypergroupoid. If I is a subset of H such that f_I is fuzzy semiprime, then I is semiprime. "Conversely", let I be a subset of H such that, for every $a \in H$, either $a \circ a \subseteq I$ or $(a \circ a) \cap I = \emptyset$. If I is semiprime, then f_I is fuzzy semiprime.*

Proof. $\Longrightarrow$. Let $a \in H$ such that $a \circ a \subseteq I$. Then $a \in I$. In fact: Since $a \circ a \subseteq I$, we have $f_I(a \circ a) = 1$. This is because if $u \in a \circ a$, then $u \in I$, so $f_I(u) = 1$. Since f_I is fuzzy semiprime, we have $f_I(a) \geq f_I(a \circ a) = 1$. Since f_I is a fuzzy subset of H, we have $f_I(a) \leq 1$. Thus we have $f_I(a) = 1$, and $a \in I$, so I is semiprime.

$\Longleftarrow$. Let I be semiprime. Then $f_I(a) \geq f_I(a \circ a)$. Indeed: If $a \circ a \subseteq I$ then, since I is semiprime, we have $a \in I$, then $f_I(a) = 1 \geq f_I(a \circ a)$. If $a \circ a \nsubseteq I$ then, by hypothesis, we have $(a \circ a) \cap I = \emptyset$, then $f_I(a \circ a) = 0$. This is because if $u \in a \circ a$, then $a \notin I$, so $f_I(a) = 0$. Hence we obtain $f_I(a \circ a) = 0 \leq f_I(a)$. $\square$

Lemma 3.7. *Let H be an ordered hypergroupoid. A nonempty subset A of H is a semiprime subset of H if and only if the fuzzy subset f_A of H is fuzzy semiprime.*

Theorem 3.8. *An ordered hypersemigroup $(H, \leq)$ is left regular if and only if the fuzzy left ideals of H are fuzzy semiprime.*

Proof. $\Longrightarrow$. Let f be a fuzzy left ideal of H and $a \in H$. Then $f(a) \geq f(a \circ a)$. In fact: Let $u \in a \circ a$. Then $f(a) \geq f(u)$. Indeed: Since $u \in H$ and H is left regular, there exist $x, t \in H$ such that $t \in (x \circ u) * \{u\}$ and $a \leq t$. Since $t \in (x \circ u) * \{u\}$, we have $t \in w \circ u$ for some $w \in x \circ u$. Since f is a fuzzy left ideal of H, we have $f(w \circ u) \geq f(u)$. Since $t \in w \circ u$, we have $f(t) \geq f(u)$. Since $a \leq t$, we have $f(a) \geq f(t)$. Thus we have $f(a) \geq f(u)$.
$\Longleftarrow$. By Theorem 2.9, it is enough to prove that every left ideal of H is semiprime. Let now A be a left ideal of H. By Lemma 3.4, f_A is a fuzzy left ideal of H. By hypothesis, f_A is semiprime. Then, by Lemma 3.6, A is semiprime. $\square$
The right analogue of the above theorem also holds, and we have

Theorem 3.9. *An ordered hypersemigroup H is right regular if and only if every fuzzy right ideal of H is fuzzy semiprime.*

Theorem 3.10. *An ordered hypersemigroup $(H, \circ, \leq)$ is intra-regular if and only if every fuzzy ideal of H is fuzzy semiprime.*

Proof. $\Longrightarrow$. Let f be a fuzzy ideal of H and $a \in H$. Then $f(a) \geq f(a \circ a)$. In fact: Let $u \in a \circ a$. Then $f(a) \geq f(u)$. Indeed: Since $u \in H$ and H is intra-regular, there exist $x, y, t \in H$ such that $t \in \{x\} * (u \circ u) * \{y\}$ and $a \leq t$. Since $t \in (x \circ u) * (u \circ y)$, there exist $v \in x \circ u$ and $w \in u \circ y$ such that $t \in v \circ w$. Since f is a fuzzy left ideal of H, we have $f(v \circ w) \geq f(w)$. Since $t \in v \circ w$, we have $f(t) \geq f(w)$. Since f is a fuzzy right ideal of H, we have $f(u \circ y) \geq f(u)$. Since $w \in u \circ y$, we have $f(w) \geq f(u)$. Since $a \leq t$, we have $f(a) \geq f(t)$. Thus we get $f(a) \geq f(u)$.

$\Longleftarrow$. By Theorem 2.13, it is enough to prove that every ideal of H is semiprime. Let now A be an ideal of H. By Lemma 3.5, the characteristic function f_A is a fuzzy ideal of H. By hypothesis, f_A is fuzzy semiprime. Then, by Lemma 3.6, A is semiprime. $\square$

I would like to thank Professor Klaus Denecke for his interst in my work and his prompt reply.

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