ON QUASI DROP PROPERTY OF UNBOUNDED SETS IN FRÉCHET SPACES

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Abstract. In reflexive Frechet spaces an unbounded closed convex set C has the quasi drop property if and only if i) $\text{int}(C) \neq \emptyset$ and ii) C has the Mackey (α)-property.

1 Introduction. Let $(X, \|\cdot\|)$ be a Banach space and B_X its closed unit ball. By the drop $D(x, B_X)$ defined by an element $x \in X \setminus B_X$ we mean the set *conv* $(\{x\} \cup B_X)$. Danes [2] proved that, for any Banach space $(X, \|\cdot\|)$ and every non-empty closed set $A \subset X$ at positive distance from B_X, there exists an $x_0 \in A$ such that $D(x_0, B_X) \cap A = \{x_0\}$. Motivated by Danes theorem, Rolewicz [23] introduced the notion of drop property for the norm of a Banach space: the norm $\|\cdot\|$ in X has the drop property if for every non-empty closed set A disjoint from B_X there exists $x_0 \in A$ such that $D(x_0, B_X) \cap A = \{x_0\}$. He proved that if the norm has the drop property then $(X, \|\cdot\|)$ is reflexive (see [23] Theorem 5). Later, Montesinos (see [15] Theorem 4) proved that a Banach space is reflexive if and only if it can be renormed to have the drop property.

Let B be a subset of a Banach space $(X, \|\cdot\|)$. The Kuratowski index of noncompactness of B, $\alpha(B)$, is the infimum of all positive numbers r such that B can be covered by a finite number of sets of diameter less than r. Given $f \in X^*$ such that $\|f\| = 1$ and $0 < \delta \leq 2$, consider the slice $S(f, B_X, \delta) = \{x \in B_X : f(x) \geq 1 - \delta\}$. The norm $\|\cdot\|$ in a Banach space X has property (α), if $\lim_{\delta \to 0} \alpha(S(f, B_X, \delta)) = 0$ for every $f \in X^*$, $\|f\| = 1$. Also, Rolewicz ([23] Theorem 4), proved that if the norm has the drop property then it has property (α), and Montesinos ([15] Theorem 3) established that these two properties are equivalent.

Giles, Sims and Yorke [4] said that the norm has the weak drop property if for every non-empty weakly sequentially closed set A disjoint from B_X, there exists an $x_0 \in A$ such that $D(x_0, B_X) \cap A = \{x_0\}$, and they proved that this property is equivalent to $(X, \|\cdot\|)$ being reflexive. Kutzarova [10] and Giles and Kutzarova [5] extended the discussion of these drop properties to closed bounded convex sets in Banach spaces. Cheng, Zhou and Zang [1], Zheng [26] and other authors studied those drop properties in locally convex spaces: a bounded, convex and closed subset B of a locally convex space (E, τ) is said to have the drop property if it is non-empty and for every non-empty sequentially closed subset $A \subset E$ disjoint from B there exists $a \in A$ such that $D(a, B) \cap A = \{a\}$.

Qiu in [19] and Monterde and Montesinos in [14], introduced another drop properties in locally convex spaces: a non-empty closed bounded convex subset B of a locally convex space (E, τ) is said to have the quasi weak drop (resp. quasi drop) property if for every non-empty weakly closed (resp. closed) subset $A \subset E$ disjoint from B, there exists an $x_0 \in A$ such that $D(x_0, B) \cap A = \{x_0\}$. In [19] and [20], Qiu established a number of equivalences for the quasi weak drop property in Frèchet spaces and in quasi-complete locally convex spaces. He characterized reflexivity of those spaces by the condition that every closed bounded convex subset of the space must satisfy the quasi weak drop property. Concerning drop properties and their applications, see for example [1]-[6], [10]-[16], [18]-[23] and [25]. In [13] can be

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found a extensive compilation of extensions and equivalent variational principles to drop properties.

In [11], Kutzarova and Rolewicz have dropped the boundedness assumption for the drop property in Banach spaces and proved

Theorem 1. *Let C be an unbounded closed convex set in a reflexive Banach space. The following conditions are equivalent:*

i) *C has the drop property;*

ii) $\mathrm{int}(C) \neq \emptyset$ *and C has the property (α).*

They asked if the existence of such a closed convex unbounded set C with the drop property forces the space to be reflexive. In [16], Montesinos proved that this is the case. Later, in [12], Lin and Yu proved that if C is an unbounded closed convex set with the weak drop property in a Banach space, then C has nonempty interior and the Banach spece is reflexive.

In [6], the author considered locally convex spaces with the strict Mackey convergence condition (sMc, see below) and studied the relation between the quasi drop property and the defined *Mackey (α)-property* (see below). Then he characterized quasi drop property for bounded disks in Frechet spaces. Now, based on techniques of Kutzarova-Rolewicz [11], Montesinos [16] and Lin-Yu [12] the Kutzarova-Rolewicz's Theorem is extended to the family of reflexive Fréchet spaces, i.e. in a reflexive Fréchet space an unbounded closed convex subset C has the quasi drop property if and only if $\mathrm{int}(C) \neq \emptyset$ and C has the Mackey (α)-property.

2 Preliminaries.

A closed, bounded and absolutely convex subset is called a disk. If D is a disk in the locally convex space (E, τ) then we let E_D denotes the linear span of D, equipped with the topology given by ρ_D the gauge (Minkowski's functional) of D. This topology has a base of zero neighborhoods of the form $\{aD : a > 0\}$, and makes E_D into a normed space such that $\tau|_{E_D} \leq \rho_D|_{E_D}$, for τ the original topology of E. And (E, τ) is said to be locally complete if every disk $D \subset E$, is a Banach disk, that is (E_D, ρ_D) is a Banach space. Note that for metrizable spaces, completeness and local completeness are equivalent. For local completeness, see [8] and [17].

According to Grothendieck (see [7]), we have that a space (E, τ) satisfies the strict Mackey convergence condition (sMc) if for every bounded subset $B \subset (E, \tau)$, there exists a disk $D \subset E$ containing B such that the topologies of E and E_D agree on B, i.e. $\tau|_B = \rho_D|_B$ and so on every subset of B. Note that every metrizable space satisfies the sMc (see [17], 5.1.27(ii)). So, every Fréchet space (E, τ) is locally complete and satisfies the sMc.

>From now, throughout this paper (E, τ) will be a Fréchet space over $\mathbb{R}$. Consider a family of τ-continuous seminorms $\{\rho_n : E \to \mathbb{R}\}_{n\in\mathbb{N}}$, where $\rho_n \leq \rho_{n+1}$ for every $n \in \mathbb{N}$, which define the topology τ. Following Rudin (see [24]), the function $d : E \times E \to \mathbb{R}$ given by $d(x, y) = \sum_{n=1}^{\infty} \frac{2^{-n}\rho_n(x-y)}{1+\rho_n(x-y)}$ defines an invariant metric compatible with the topology τ, i.e. $\tau = \tau_d$.

For a τ-closed convex set $C \subset E$, denote by $F(C)$ the set of all τ-continuous linear functionals $f \in (E, \tau)' \setminus \{0\}$ which are bounded above on C. For $f \in F(C)$ let $M_f = \sup\{f(x) : x \in C\}$, and for $\delta > 0$ consider the slice $S(f, C, \delta) = \{x \in C : f(x) \geq M_f - \delta\}$. The set C is said to have the (α)-property with respect to τ if for every $f \in F(C)$ and for every neighborhood U of 0 in τ, there exists $\delta > 0$ such that $S(f, C, \delta)$ can be covered by a finite number of translates of U.

If a slice $S(f, C, \delta_0)$ is τ-bounded, since the Fréchet space (E, τ) satisfies the sMc, there exists a disk $D \subset E$ containing $S(f, C, \delta_0)$ such that $\tau\left|_{S(f,C,\delta_0)} = \rho_D\right|_{S(f,C,\delta_0)}$. In this case, the Kuratowski index of noncompactness of $S(f, C, \delta_0)$ associated to the disk D is $\alpha_D(S(f, C, \delta_0))$ the infimum of all positive numbers r such that $S(f, C, \delta_0)$ is covered by a finite number of sets of ρ_D-diameter less than r. The τ-closed convex set $C \subset E$ is said to have the Mackey (α)-property if for every $f \in F(C)$ and D as above $\lim_{\delta \to 0} \alpha_D(S(f, C, \delta)) = 0$. In this case, due to the fact that ρ_D and τ induce the same topology on the slice, we get that C has the (α)-property with respect to τ. Obviously, if $(E, \|\cdot\|)$ is a normed space both (α)-properties coincide.

3 Results.

Proposition 1. *Let C be a non-empty closed convex (unbounded) subset of the Fréchet space (E, τ). Suppose that C has the quasi drop property. Then every C-stream in E has a τ-convergent subsequence.*

Proof. Suppose there exists a sequence $(x_n)_n \in E$ such that $x_{n+1} \in D(x_n, C) \setminus C$, for every $n \in \mathbb{N}$, but $(x_n)_n$ does not have any τ-convergent subsequence. So, for every subsequence $(x_{n_k})_k \subset (x_n)_n$ we have that $A = \{x_{n_k} : k \in \mathbb{N}\}$ is a closed set and C does not have the quasi drop property. □

Remark 1. *Note that if (E, τ) is a Fréchet space, $A \subset E$ is $\sigma(E, E')$-sequentially closed and $(x_n)_n \in A$ is τ-convergent to some $x_0 \in E$, then $x_n \to x_0$ respect to $\sigma(E, E')$. So, $x_0 \in A$ and A is τ-closed. Now, suppose that $C \subset E$ is τ-closed, convex and has the quasi drop property, then C has the weak drop property. Note also that in this case, as in Proposition 1, every C-stream has a weakly convergent subsequence.*

Proposition 2. *Let C be a non-empty closed convex (unbounded) subset of the Fréchet space (E, τ). Let $f \in F(C)$ and $M_f := \sup\{f(x) : x \in C\}$. Suppose that C has the quasi drop property. Then for every $\delta > 0$, the slice $S(f, C, \delta) = \{x \in C : f(x) \geq M_f - \delta\}$ is a bounded set.*

Proof. Suppose this is not true. Then there exist $f_0 \in F(C)$, $\delta_0 > 0$ and $U_0 \in \tau$ an open convex and simetric zero neighborhood such that for every $R > 0$ we have that $S(f_0, C, \delta_0)$ is not contained in RU_0; or equivalently, for every $R > 0$ there exists $x_R \in S(f_0, C, \delta_0)$ such that $\rho_{U_0}(x_R) > R$, where $\rho_{U_0}(\cdot)$ is the τ-continuous Minkowski's seminorm generated by U_0.

Let $M := \sup\{|f(x)| : x \in S(f_0, C, \delta_0)\} \geq M_{f_0}$. Let $x_1 \in E$ be such that $f_0(x_1) > M_{f_0}$. Find $0 < \lambda < 1$ such that $(1 - \lambda) f_0(x_1) - \lambda M > M_{f_0}$.

Take $\overline{x_2} \in S(f_0, C, \delta_0)$ satisfying $\rho_{U_0}(\lambda \overline{x_2} + (1 - \lambda)x_1 - x_1) \geq 1$. Let $x_2 := \lambda \overline{x_2} + (1 - \lambda)x_1$. So, $\rho_{U_0}(x_2 - x_1) \geq 1$ and

$$\begin{aligned} f_0(x_2) &= \lambda f_0(\overline{x_2}) + (1 - \lambda) f_0(x_1) \geq (1 - \lambda) f_0(x_1) - \lambda |f_0(\overline{x_2})| \\ &\geq (1 - \lambda) f_0(x_1) - \lambda M > M_{f_0} \end{aligned}$$

Suppose now, we have found $x_1, x_2, ..., x_n \in E$ such that $x_{i+1} \in D(x_i, C) \setminus C$, for $i = 1, 2, ..., (n-1)$, with $f_0(x_i) > M_{f_0}$, for $i = 1, 2, ..., n$ and $\rho_{U_0}(x_i - x_j) \geq 1$ for $i, j = 1, 2, ..., n$ and $i \neq j$. In order to find x_{n+1}, find $0 < \lambda < 1$ such that $(1 - \lambda) f(x_n) - \lambda M > M_{f_0}$ and take $\overline{x_{n+1}} \in S(f_0, C, \delta_0)$ satisfying $\rho_{U_0}(\lambda \overline{x_{n+1}} + (1 - \lambda)x_n - x_i) \geq 1$, for all $i = 1, 2, ..., n$. Let $x_{n+1} := \lambda \overline{x_{n+1}} + (1 - \lambda)x_n$. So, $\rho_{U_0}(x_{n+1} - x_i) \geq 1$, for all $i = 1, 2, ..., n$; and

$$\begin{aligned} f_0(x_{n+1}) &= \lambda f_0(\overline{x_{n+1}}) + (1 - \lambda) f_0(x_n) \geq (1 - \lambda) f_0(x_n) - \lambda |f_0(\overline{x_{n+1}})| \\ &\geq (1 - \lambda) f_0(x_n) - \lambda M > M_{f_0} \end{aligned}$$

Then the sequence $(x_n)_n$ is a C-stream in E with no convergent subsequences. This is a contradiction. □

Proposition 3. *Let C be a closed convex unbounded subset of the Fréchet space (E,τ). Suppose that C has the quasi-drop property. Then C has the Mackey (α)-property, and C has the $(\dot{\alpha})$-property respect to τ, too.*

Proof. Let $f \in F(C)$. Find $x_0 \in E$ such that $f(x_0) > M_f$. We may assume that $M_f > 1$, then by proposition 2, the slice $S_1 := S(f,C,1)$ is a τ-bounded closed set and $B := cvx\{S_1 \cup \{x_0\}\}$ is a bounded closed convex subset of E. Since (E,τ) is a Fréchet space, by the sMc, there exists $D \subset E$ a Banach disk such that $B \subset D$ and $\tau|_B = \rho_D|_B$. In particular, $\tau|_{S_1} = \rho_D|_{S_1}$. If $\inf\{\alpha_D(S(f,C,\varepsilon) : 1 > \varepsilon > 0\} > 2\delta_0$, for some $\delta_0 > 0$, then (see [21], Theorem 4) for every finite dimensional subspace $L \subset E_D$ we have:

$$(1) \qquad \sup_{x\in S(f,C,\varepsilon)} (\inf_{y\in L} \rho_D(x-y)) \geq \frac{1}{2}\inf_{\varepsilon>0} \alpha_D(S(f,C,\varepsilon)) > \delta_0$$

Take $\varepsilon_1 < f(x_0) - M_f$. And choose $\overline{x_1} \in S(f,C,\varepsilon_1)$ such that

$$\inf\{\rho_D(\overline{x_1} - z) : z \in span\{x_0\}\} > \delta_0.$$

Let $x_1 = \frac{x_0+\overline{x_1}}{2}$, then

$$f(x_1) = f(\frac{x_0+\overline{x_1}}{2}) = \frac{f(x_0)}{2} + \frac{f(\overline{x_1})}{2} > \frac{M_f+\varepsilon_1}{2} + \frac{M_f-\varepsilon_1}{2} = M_f.$$

Moreover

$$\inf\{\rho_D(x_1 - z) : z \in span\{x_0\}\} = \frac{1}{2}\inf\{\rho_D(\overline{x_1} - z) : z \in span\{x_0\}\} > \frac{\delta_0}{2}$$

Now, suppose we have $\{x_0, x_1, ..., x_n\}$, such that $x_i \neq x_j$ if $i \neq j \leq n$, and
i) $f(x_i) > M_f$
ii) $\inf\{\rho_D(x_i - z) : z \in span\{x_0, ..., x_{i-1}\}\} > \frac{\delta_0}{2}$
iii) $x_i \in D(x_{i-1}, C)$
for every $i \leq n$. Take $\varepsilon_{n+1} < f(x_n) - M_f$ and by (1) find $\overline{x_{n+1}} \in S(f,C,\varepsilon_{n+1})$ such that

$$\inf\{\rho_D(\overline{x_{n+1}} - z) : z \in span\{x_0, x_1, ..., x_n\}\} > \delta_0.$$

Let $x_{n+1} = \frac{x_n+\overline{x_{n+1}}}{2}$ then, in an analogous way to x_1, $f(x_{n+1}) > M_f$ and

$$\begin{aligned} & \inf\{\rho_D(x_{n+1} - z) : z \in span\{x_0, ..., x_n\}\} \\ = \ & \frac{1}{2}\inf\{\rho_D(\overline{x_{n+1}} - z) : z \in span\{x_0, ..., x_n\}\} > \frac{\delta_0}{2}. \end{aligned}$$

Then the sequence $(x_n)_n$ satisfies (i,ii,iii) and the set $A = \{x_0, x_1, ..., x_n, ...\} \subset B$ is ρ_D-closed. Since the topologies τ and ρ_D agree on B, A is τ-closed and $A \cap C = \emptyset$. Hence C, does not have the quasi drop property. This is a contradiction. □

Recall a generalization of Cantor's intersection theorem due to Kuratowski [9].

Lemma 1. *Given a complete metric space and a sequence of non-empty closed sets $\{F_n\}_{n\in\mathbb{N}}$, $F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n \supseteq \cdots$ with the property that, for α the Kuratowski index of noncompactness, $\lim_n \alpha(F_n) = 0$, then $\bigcap_{n=1}^{\infty} F_n$ is non-empty and compact.*

Remark 2. *a) Suppose that C has the Mackey (α)-property, let $f \in F(C)$ and $(\varepsilon_n)_n \in \mathbb{R}^+$ convergent to zero. Then for every sequence $(x_n)_n \in C$ such that $x_n \in S(f, C, \varepsilon_n)$ there is a subsequence $(x_{n_k})_k \subset (x_n)_n$ convergent in C.*

b) In particular, under the assumptions of Proposition 3, every $f \in F(C)$ attains its supremum on C.

Proposition 4. *Let C be a closed convex unbounded subset of the Fréchet space (E, τ). Suppose that C has the quasi-drop property. Then* $\operatorname{int}(C) \neq \emptyset$.

Proof. Since C is not bounded, there exists $(x_n)_n \in C$ such that does not have any convergent subsequence. For every $x \notin C$, define $y_n := \frac{1}{2^n}x + \sum_{i=1}^{n} \frac{1}{2^{n-i+1}}x_i$. So, $(y_n)_n$ is a sequence which has non-empty intersection with C. If this is not true, then we have two possibilities:

i) There exists a subsequence $(y_{n_k})_k \subset (y_n)_n$ which does not have weakly convergent subsequences. Then $A := \{y_{n_k} : k \in \mathbb{N}\}$ is closed disjoint to C, and contradicts the quasi drop property of C (see Remark 1).

ii) Every subsequence $(y_{n_k})_k \subset (y_n)_n$ has a convergent subsequence. Let $(y_{n_k})_k \subset (y_n)_n$ be convergent to $y_0 \in E$. This implies that $y_{n_k+1} \to y_0$, too. Since $y_{n_{k+1}} = \frac{1}{2}(y_{n_k} + x_{n_k+1})$ and $x_{n_k+1} = 2y_{n_k+1} - y_{n_k}$, then $(x_{n_k+1})_k$ converges to y_0. This is not possible.

Hence, $\{y_n : n \in \mathbb{N}\} \cap C \neq \emptyset$. Now, given $z \in E$ and $L \neq 0$, define the homeomorfism $T_{z,L} : E \to E$ given by $T_{z,L}(x) = z + L(x - z)$, and the application $T_{(x_1,\ldots,x_n)}(x) = T_{x_1,2} \circ T_{x_2,2} \circ \cdots \circ T_{x_n,2}(x)$. It is easy to verify that for the elements $\{x_n : n \in \mathbb{N}\}$ of the original sequence we have that $y_n \in C$ if and only if $x \in T_{(x_1,x_2,\ldots,x_n)}(C)$. Then $E \setminus C = \bigcup_{n\in\mathbb{N}} T_{(x_1,x_2,\ldots,x_n)}(C)$. Since (E, τ) is a Fréchet space, the Baire category theorem implies that $\operatorname{int}_\tau(C) \neq \emptyset$. □

Proposition 5. *Let (E, τ) be a Fréchet space and $C \subset E$ be a closed convex unbounded subset with $\operatorname{int}_\tau(C) \neq \emptyset$. Suppose that C has the Mackey (α)-property. Then for every $b \in E \setminus C$ the set $D(b, C)$ is closed.*

Proof. Suppose there is a point $b \in E\setminus C$ such that $D(b, C)$ is not τ-closed. Then there exists $a \in \overline{D(b, C)}^\tau$ such that $a \notin D(b, C)$. So, there are sequences $(y_n)_n \in C$ and $(\lambda_n)_n \in [0, 1]$ such that $x_n := \lambda_n b + (1 - \lambda_n)y_n \to a$ respect to τ. Then the sequence $\lambda_n \to 1$ and for every τ-continuous seminorm ρ such that distinguish a subsequence $(y_{n_k})_k \subset (y_n)_n$ we have that $\rho(y_{n_k}) \to \infty$. By a convexity argument the ray $r = \{b + \eta(a - b) : \eta \geq 1\}$ is contained in $\overline{D(b, C)}^\tau \setminus D(b, C)$. Note that $r \cap C = \emptyset$ and $\operatorname{int}_\tau(C) \neq \emptyset$ imply that there exists $f \in (E, \tau)' \setminus \{0\}$ such that $M_f := \sup\{f(x) : x \in C\} \leq \inf\{f(x) : x \in r\}$, evenmore $f(a) = f(b)$, so $r \subset H = \{x \in E : f(x) = f(a)\}$. Then $M_f \leq f(a)$. Since C has the Mackey (α)-property, for every $\delta > 0$ the set $S(f, C, \delta)$ is bounded. Consider the set $A := \{a, b\} \cup \{x_n : n \in \mathbb{N}\} \cup S(f, C, \delta)$. Since A is bounded and (E, τ) has the sMc, there exists a Banach disk $B \subset E$ such that $A \subset B$ and $\rho_B|_A = \tau|_A$, evenmore if we make $C_B := C \cap E_B$ then $\{y_n : n \in \mathbb{N}\} \subset C_B \subset E_B$ and $x_n \to a$ respect to ρ_B, so $\rho_B(y_n) \to \infty$. Then we have that $a \in \overline{D(b, C_B)}^{\rho_B}$ but $a \notin D(b, C_B)$. Note also that $\operatorname{int}_{\rho_B}(C_B) \neq \emptyset$ and $f_B := f|_{E_B} \in (E_B, \rho_B)' \setminus \{0\}$, so $f_B \in F(C_B)$ and f_B separates r and C_B. Hence all the previous construction and observations remains valid in the Banach space (E_B, ρ_B). If we prove that $a \in D(b, C_B)$, which clearly is contained in $D(b, C)$ we are done. But in these conditions the proof continues exactly as the rest of proof at this point of Proposition 5 in [11], where ρ_B substitutes $\|\cdot\|$. □

Note that Proposition 1 in [11] has been proved above for Fréchet spaces. Also, Lemma 2 and Lemma 12 in [11] follow directly being true in the frame of reflexive Fréchet spaces. Then Remark 2(iii) in [12] can be extended to

Remark 3. *Let (E,τ) be a reflexive Fréchet space and $C \subset E$ an unbounded closed convex subset. Suppose that C has the Mackey (α)-property and that* $\operatorname{int}(C) \neq \emptyset$ *then C contains a ray $\{c + \lambda b : \lambda \geq 0\}$. Moreover, for any $x \in E$ there is $\beta > 0$ such that C contains the ray $\{x + \lambda b : \lambda \geq \beta\}$*

Theorem 2. *Let (E,τ) be a reflexive Fréchet space and $C \subset E$ be an unbounded closed convex subset. Then the following conditions are equivalent:*

a) *C has the quasi-drop property*

b) $\operatorname{int}(C) \neq \emptyset$ *and C has the Mackey (α)-property.*

Proof. Assume that C does not have the quasi drop property. So, there is a closed set $A \subset E$ disjoint to C such that for every $x \in A$ there is $a \in A \setminus \{x\}$ satisfying $a \in A \cap D(x, C)$. Take any point $x_1 \in A$. Put $d_1' := \inf\{d(a, C) : a \in A \cap D(x_1, C)\}$ and find $x_2 \in A \cap D(x_1, C)$ such that $d_2 := d(x_2, C) < d_1' + 1$. Choose $\{x_1, x_2, ..., x_n\}$ such that $x_{i+1} \in A \cap D(x_i, C)$ and $x_{i+1} \neq x_i$ for $i = 1, ..., n-1$ and if we make $d_i' := \inf\{d(a, C) : a \in A \cap D(x_i, C)\}$ then $d_{i+1} := d(x_{i+1}, C) < d_i' + \frac{1}{i}$. Inductively construct, in this way, a C-stream $\{x_n : n \in \mathbb{N}\}$ with these characteristics, and note that $(d_n)_n \subset \mathbb{R}$ is a convergent sequence to some $\varepsilon_0 \geq 0$. Note that this C-stream $\{x_n : n \in \mathbb{N}\}$ does not have any convergent subsequence. In order to see this, suppose that the C-stream possess convergent subsequences and consider two cases:

i) $\varepsilon_0 = 0$. This means that there is a sequence $(y_n)_n \in C$ such that $d(x_n, y_n) \to 0$. Let $A_1 := cvx\{x_n : n \in \mathbb{N}\}$, by the lemma in [15], $A_1 \cap C = \emptyset$ and since $\operatorname{int}(C) \neq \emptyset$, there exists $f \in (E,\tau)' \setminus \{0\}$ which separates A_1 and C. We may assume that $f \in F(C)$. For $M_f := \{f(x) : x \in C\}$, we have that $f(y_n) \to M_f$. By the Mackey (α)-property, the Kuratwoski's lemma guarantees the existence of a subsequence $(y_{n_k})_k \subset (y_n)_{\mathbb{N}}$ which is convergent to some $y_0 \in C$ and $f(y_0) = M_f$. Then $d(x_n, y_o) \to 0$ and since A is closed we get that $y_0 \in A \cap C$. This is a contradiction.

ii) If $\varepsilon_0 > 0$ and $(x_n)_{\mathbb{N}}$ has a convergent subsequence to some $x_0 \in A \cap \bigcap_{i \in \mathbb{N}} D(x_i, C)$, i.e. $x_0 \in A \cap D(x_i, C)$, for every $i \in \mathbb{N}$. Then there exists $a \in A \setminus \{x_0\}$ satisfying $a \in A \cap D(x_0, C)$ and $d(a, C) < d(x_0, C)$. Find $n \in \mathbb{N}$ such that $\frac{1}{n} < d(x_0, C) - d(a, C)$, then $d(x_{n+1}, C) > d(x_0, C) > d(a, C) + \frac{1}{n} \geq d_n' + \frac{1}{n}$. Which is a contradiction. Then the C-stream does not have any convergent subsequence.

Now, by the Remark 3, there exists $b \in E \setminus \{0\}$ such that for every $x \in E$ there is $\beta > 0$ such that C contains the ray $\{x + \lambda b : \lambda \geq \beta\}$.

Let $\eta := \sup\{\beta : (\beta b + \{x_n\}_{\mathbb{N}}) \cap C = \emptyset\}$. Note that

i) $\eta b + C \subset C$

ii) if $\eta b + x_n \in C$, then $\eta b + x_m \in C$ for every $m > n$.

So, for every convex combination $\sum_{i=1}^{n} a_i x_i$ where each $a_i \geq 0$ and $\sum_{i=1}^{n} a_i = 1$, if $\eta b + \sum_{i=1}^{n} a_i x_i \in \operatorname{int}(C)$ then

$$\eta b + x_{n+1} \in cvx\left\{\left(\eta b + \sum_{i=1}^{n} a_i x_i\right) \cup (\eta b + C)\right\} \subset \operatorname{int}(C).$$

Which is not possible. Then $(\eta b + cvx\{x_n : n \in \mathbb{N}\}) + \operatorname{int}(C) = \emptyset$ and there exists $f \in (E,\tau)' \setminus \{0\}$ such that

$$\inf\{f(\eta b + x_n) : n \in \mathbb{N}\} = M_f = \sup\{f(x) : x \in C\}$$

By the definition of η, there exists a sequence $(y_{n_k})_k \in C$ such that

$$d(\eta b + x_{n_k}, y_{n_k}) \to 0 \text{ and } f(y_{n_k}) \to M_f$$

Since C has the Mackey (α)-property there exists a subsequence $(y_{n_l})_l \subset (y_{n_k})_k$ which is convergent to some $y_0 \in C$. Then the sequence $(x_n)_n$ has a convergent subsequence. This is a contradiction. □

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