CLASSIFICATION OF CONTRACTIBLE SPACES BY C^*-ALGEBRAS AND THEIR K-THEORY

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Received July 14, 2015

Abstract. We consider contractible spaces and the corresponding C^*-algebras to show that contractible spaces are classifiable or not (up to homeomorphisms) by the C^*-algebras and their K-theory.

1 Introduction We consider contractible spaces and the corresponding C^*-algebras to show that contractible spaces in some cases are classifiable or not (up to homeomorphism classes or manifold classes with some operations like jointed sums) by the C^*-algebras or their K-theory. Note that contractible spaces are homotopically identified with a point.

For the classification program in our sense, we introduce several notions for C^*-algebras and spaces and also do for several examples. As a summary, we obtain several tables as classification results as collections, and the overview obtained from these tables as maps would be useful for further study in this topic.

Refer to several textbooks [1], [2], [4], or [8] about C^*-algebras and their K-theory and in particular, contractible C^*-algebras. Beyond or extending several facts on them, we further go into studying targeted ones in details in a way this time.

See also [7] for another classification result for some topological manifolds by C^*-algebras and their K-theory, with the same sprit as in this paper.

Let us begin with **some notations** as follows.

For a compact Hausdorff sapce X, we denote by $C(X)$ the C^*-algebra of all continuous, complex-valued functions on X with the uniform (or supremum) norm and pointwise operations.

For a non-compact, locally compact Hausdorff sapce X, we denote by $C_0(X)$ the C^*-algebra of all continuous, complex-valued functions on X, vanishing at infinity. We denote by $X^+ = X \cup \{\infty\}$ the one-point compactification of X. We may say that a non-compact, locally compact Hausdorff space X^- is the one-point **un-compactification** of a compact Hausdorff space X if $X^- \cup \{\infty\} = X$.

We write $\mathfrak{A} \cong \mathfrak{B}$ if two C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ are $*$-isomorphic. We write $X \approx Y$ if two spaces X and Y are homeomorphic. Use the same symbol $\cong$ for (K-theory) group isomorphisms as well.

2 Contractible, spaces and C^*-algebras A topological space X is said to be **contractible** (in X) if there is a point p in X such that the identity map $\mathrm{id}_X : X \to X$ is homotopic to the constant map id_p on X, which sends elements of X to the point p. Namely, there is a continuous path of continuous maps (f_t) of X (to X) for $t \in [0,1] = I$ the interval such that $f_0 = \mathrm{id}_X$ and $f_1 = \mathrm{id}_p$ and the map $F(t,x) = f_t(x)$ is continuous on the product space $I \times X$. The map F is called a homotopy for X.

2000 *Mathematics Subject Classification.* Primary 46L05, 46L80, 19K14.
Key words and phrases. Contractible space, jointed sum, C*-algebra, K-theory.

Note that a contractible space may or may not be compact. For instance, the Euclidean space $\mathbb{R}^n$ as well as any convex subspace are all contractible by convexity. Note also that a contractible space is path-connected by definition.

We may say that a topological space X is **identically contractible** if X is contractible by a continuous path of homeomorphisms $(f_t)_{0\leq t<1}$ of X and $f_1 = \mathrm{id}_p$.

A C^*-algebra $\mathfrak{A}$ is said to be **contractible** (to zero) if the identity map $\mathrm{id}_{\mathfrak{A}} : \mathfrak{A} \to \mathfrak{A}$ is homotopic to the zero map $\mathrm{id}_0 = 0 : \mathfrak{A} \to \mathfrak{A}$ by a (norm or uniform) continuous path of $*$-homomorphisms (φ_t) of $\mathfrak{A}$ (to $\mathfrak{A}$) for $t \in [0,1] = I$ such that $\varphi_0 = \mathrm{id}_{\mathfrak{A}}$ and $\varphi_1 = \mathrm{id}_0$ and the map $\Phi(t,a) = \varphi_t(a)$ is continuous on the product space $I \times \mathfrak{A}$. We may call the map Φ a (norm or uniform) C^*-homotopy for $\mathfrak{A}$.

We also say that a C^*-algebra $\mathfrak{A}$ is **identically contractible** (to zero) if $\mathfrak{A}$ is contractible (to zero) by a continuous path of $*$-isomorphisms $(\varphi_t)_{0\leq t<1}$ of $\mathfrak{A}$ and $\varphi_1 = \mathrm{id}_0$.

We say that a C^*-algebra $\mathfrak{A}$ is **contractible** to $\mathbb{C}$ if the identity map $\mathrm{id}_{\mathfrak{A}} : \mathfrak{A} \to \mathfrak{A}$ is homotopic to a 1-dimensional representation (or character) $\chi : \mathfrak{A} \to \mathbb{C}1$ in $\mathfrak{A}$ by a continuous path of $*$-homomorphisms of $\mathfrak{A}$.

We also say that a C^*-algebra $\mathfrak{A}$ is **identically contractible** to $\mathbb{C}$ if $\mathfrak{A}$ is contractible to $\mathbb{C}$ by a continuous path of $*$-isomorphisms $(\varphi_t)_{0\leq t<1}$ of $\mathfrak{A}$ and $\varphi_1 = \chi$.

Note that a non-trivial C^*-algebra contractible to $\mathbb{C}$ is not simple.

Furthermore, we say that a C^*-algebra $\mathfrak{A}$ (especially when $\mathfrak{A} = C(X)$ or $C_0(X)$) is **weakly contractible** (to zero), weakly identically contractible (to zero), weakly contractible to $\mathbb{C}$, and weakly identically contractible to $\mathbb{C}$, respectively, if $\mathfrak{A}$ is contractible (to zero), identically contractible (to zero), contractible to $\mathbb{C}$, and identically contractible to $\mathbb{C}$, by a **pointwise** continuous C^*-homotopy Φ for $\mathfrak{A}$ (with respect to X), respectively. In these cases, we may call such a homotopy Φ either a **weak** homotopy, a weakly continuous path, or a pointwise continuous path for $\mathfrak{A}$.

A homotopy (f_t) for a space X induces directly a homotopy (φ_t) for $C(X)$ (or $C_0(X)$) as the composition as $\varphi_t(g) = g \circ f_t \in C(X)$, which we call the **induced** homotopy.

Indeed, as a summary, with (1) below certainly known ([2]),

Proposition 2.1. (1) *A unital C^*-algebra is not contractible. Equivalently, if a C^*-algebra is contractible, then it is non-unital.*

(2) *If a compact Hausdorff space X is contractible (in X) by a homotopy, then $C(X)$ is contractible to $\mathbb{C}$ by the induced homotopy.*

(3) *Similarly, if a compact Hausdorff space X is identically contractible by a homotopy, then $C(X)$ is identically contractible to $\mathbb{C}$ by the induced homotopy. The converse in this case also holds.*

(4) *Moreover, if a non-compact, locally compact Hausdorff space X is contractible (in X) by a homotopy, then $C_0(X)$ is weakly contractible to $\mathbb{C}$ by the induced homotopy.*

(5) *Similarly, if a non-compact, locally compact Hausdorff space X is identically contractible by a homotopy, then $C_0(X)$ is weakly identically contractible to $\mathbb{C}$ by the induced homotopy. The converse in this case also holds.*

Proof. For (1). Note that $*$-homomorphisms φ_t of a unital C^*-algebra $\mathfrak{A}$ (to $\mathfrak{A}$) are always unital, which can not be homotopic to the zero map on $\mathfrak{A}$. Because the constant map $1 = \varphi_t(1) \in \mathfrak{A}$ on $[0,1)$ converges continuously to $1 \in \mathfrak{A}$ at $1 \in [0,1]$.

For (2). Let (f_t) be a continuous path between id_X and id_p for some $p \in X$. Define a continuous path of $*$-homomorphisms of $C(X)$ by $\varphi_t(g) = g \circ f_t$ for $g \in C(X)$ and $t \in [0,1]$.

Indeed,

$$\begin{aligned}\varphi_t(\lambda g+h) &= (\lambda g+h)\circ f_t = (\lambda g\circ f_t)+(h\circ f_t) = \lambda\varphi_t(g)+\varphi_t(h),\\ \varphi_t(g\cdot h) &= (g\cdot h)\circ f_t = (g\circ f_t)\cdot(h\circ f_t) = \varphi_t(g)\cdot\varphi_t(h),\\ \varphi_t(g)^* &= (g\circ f_t)^* = \overline{g\circ f_t} = g^*\circ f_t = \varphi_t(g^*)\end{aligned}$$

for $g,h\in C(X)$ and $\lambda\in\mathbb{C}$, where the overline is the complex conjugate. Note that $\varphi_0(g) = g\circ \mathrm{id}_X = g$ and $\varphi_1(g)(x) = (g\circ \mathrm{id}_p)(x) = g(p)$ for any $x\in X$, so that $\varphi_1(g) = g(p)1 \equiv \chi_p(g)$ the character as the evaluation map at $p\in X$. Note also that the following norm estimate holds:

$$\begin{aligned}\|\Phi(t,g)-\Phi(s,h)\| &= \|\varphi_t(g)-\varphi_s(h)\|\\ &\le \|\varphi_t(g)-\varphi_s(g)\|+\|\varphi_s(g)-\varphi_s(h)\|\\ &\le \|g\circ f_t-g\circ f_s\|+\|g-h\|,\end{aligned}$$

which can be small enough when (t,g) and (s,h) are close enough on $[0,1]\times\mathfrak{A}$. Because X is compact, so that a continuous function $g\in C(X)$ is uniformly continuous on X. In particualr, when $s=1$, note that

$$\|g\circ f_t - g\circ f_1\| = \sup_{x\in X}|g(f_t(x))-g(p)|,$$

which goes to zero as $t\to 1$.

For (3). The same as above shows that if X is identically contractible, then $C(X)$ is identically contractible to $\mathbb{C}$.

Conversely, let $(\varphi_t)_{0\le t<1}$ be a continuous path of $*$-isomorphisms of $C(X)$ between $\varphi_0 = \mathrm{id}_{C(X)}$ and a character $\varphi_1 = \chi_p$ for some $p\in X$, by the Gelfand transform (see [4]). In fact, it is a well known fact that the space $C(X)^\wedge$ of all 1-dimensional represetations of $C(X)$ is identifed with the space X. Define a continuous path of homeomorphisms $f_t : X\to X$, induced from the following diagram to make it commutaive:

$$\begin{array}{ccc} C(X)\cong\varphi_t(C(X)) & \xrightarrow{\chi_x} & \mathbb{C}\\ {\scriptstyle\varphi_t}\uparrow & & \|\\ C(X) & \xrightarrow{\chi_{f_t(x)}} & \mathbb{C},\end{array}$$

since $\chi_x\circ\varphi_t$ for any x is written as χ_y for some $y\in X$, and set $y=f_t(x)$. Note that $\chi_{f_t(x)}\to\chi_{f_s(y)}$ as $(t,x)\to(s,y)\in I\times X$ in weak $*$-topology, if and only if for any $g\in C(X)$,

$$\begin{aligned}|\chi_{f_s(y)}(g)-\chi_{f_t(x)}(g)| &= |(\chi_y\circ\varphi_s)g-(\chi_x\circ\varphi_t)(g)|\\ &= |\varphi_s(g)(y)-\varphi_t(g)(x)|,\end{aligned}$$

which certainly goes to zero as $(t,x)\to(s,y)$, by continuity for the homotopy (φ_t). Note also that

$$(g\circ f_t)(x) = \chi_{f_t(x)}(g) = \varphi_t(g)(x).$$

For (4) and (5). Even if X is a non-compact, Hausforff space, the proof for this case is the similar as given above. Note that the space $C_0(X)^\wedge$ of all 1-dimensional representations of $C_0(X)$ is identified with X. Note also that for any $x\in X$,

$$\begin{aligned}|[\Phi(t,g)-\Phi(s,h)](x)| &= |[\varphi_t(g)-\varphi_s(h)](x)|\\ &\le |[\varphi_t(g)-\varphi_s(g)](x)|+|[\varphi_s(g)-\varphi_s(h)](x)|\\ &\le |(g(f_t(x))-g(f_s(x))|+\|g-h\|,\end{aligned}$$

which can be small enough when (t,g) and (s,h) are close enough on $[0,1]\times\mathfrak{A}$. In particualr, when $s=1$, note that

$$|[g\circ f_t - g\circ f_1](x)| = |g(f_t(x)) - g(p)|,$$

which goes to zero as $t\to 1$. Note that the uniform continuity for Φ is not expected from the assumption because the norm for the difference above can be non-zero constant, but the difference converges to zero pointwise (see the examples below). It is always assumed from the assumption in this case that only the pointwise continuity for Φ holds, which implies that the estimate above evaluated at $x\in X$ goes to zero, pointwise on X. $\square$

Remark. More generally, when (φ_t) is a continuous path of $*$-homomorphisms of $C(X)$ between $\mathrm{id}_{C(X)}$ and χ_p for some $p\in X$, each image $\varphi_t(C(X))$ as a quotient of $C(X)$ is a commutative C^*-subalgebra of $C(X)$, so that $\varphi_t(C(X))$ is isomorphic to $C(X_t)$ for some compact Hausdorff space X_t, which can be viewed as a closed subspace of X, from which, one can define a continuous path of continuous maps $f_t : X_t \to X_t$ in X, induced from the following diagram to make it commutative (only on X_t):

$$\begin{array}{ccc} \varphi_t(C(X)) \cong C(X_t) & \xrightarrow{\chi_x} & \mathbb{C} \\ {\scriptstyle \varphi_t}\uparrow & & \| \\ C(X) \supset \varphi_t(C(X)) & \xrightarrow{\chi_{f_t(x)}} & \mathbb{C}. \end{array}$$

If each f_t extends to X, then the extension of (f_t) to X gives a continuous path of continuous maps of X between id_X and id_p.

Furthermore, since a compact Hausdorff space X is normal, there is a continuous extension to X of a 1-dimensional closed interval valued, continuous function on a closed subset such as X_t of X, by Tietze-Urysohn extension theorem in general topology.

Example 2.2. • The C^*-algebra $C(I)$ on the closed interval $I=[0,1]$ is unital (so that not contractible) but weakly identically contractible to $\mathbb{C}$, by the C^*-homotopy induced by a homotopy in $[0,1]$.

If we define $\varphi_t(g)(x) = g((1-t)x)\in\mathbb{C}$ for $g\in C(I)$ and $t,x\in I$. Then (φ_t) is a continuous path of $*$-isomorphisms of $\mathfrak{A}=C(I)$ between $\mathrm{id}_{\mathfrak{A}}$ and χ_0, so that $\mathfrak{A}$ is weakly identically contractible to $\mathbb{C}$. Also, define $h_t(x)=(1-t)x\in[0,1-t]\approx[0,1]$ for $t,x\in I$, so that $\varphi_t(\mathfrak{A})\cong\mathfrak{A}$ for $t\in[0,1)$. Then (h_t) is a continuous path of homeomorphisms of $[0,1]$ such that $f_0=\mathrm{id}_X$ and id_0, so that $[0,1]$ is identically contractible (to 0).

• The (interval) C^*-algebra $I\mathfrak{A}=C(I,\mathfrak{A})$ over a C^*-algebra $\mathfrak{A}$, of all $\mathfrak{A}$-valued, continuous functions on I, viewed as the C^*-tensor product $C(I)\otimes\mathfrak{A}$, is weakly identicallty contractible to $\mathbb{C}$. In particular, $I\mathbb{C}=C(I)$. If $\mathfrak{A}$ is unital, then $C(I)\otimes\mathfrak{A}$ is unital and not contractible.

Note that $\|f\otimes a\| = \|f\|\|a\|$ for $f\otimes a\in I\mathbb{C}\otimes\mathfrak{A}$. Hence, the (norm) homotopy (φ_t) for $I\mathbb{C}$ to $\mathbb{C}$ is extended trivially as $\varphi_t(f\otimes a)=\varphi_t(f)\otimes a$.

• The C^*-algebra $C_0([0,1))$ on the half open interval $[0,1)$ (non-compact), viweded as the cone $C\mathbb{C}\cong C_0([0,1),\mathbb{C})\cong C_0([0,1))\otimes\mathbb{C}$ over $\mathbb{C}$, is non-unital and weakly identically contractible to $\mathbb{C}$ by the induced C^*-homotopy by a homotopy in $[0,1)$ (and is certainly contractible, but soon later discussed in the example given below).

Indeed, if we define $\psi_t(g)(x)=g(\frac{x}{1-t})$ for $g\in C_0([0,1))$, $t\in[0,1)$, and $x\in[0,1-t)$, and $\psi_1(g)(x)=g(0)$. Then (ψ_t) is a weakly continuous path of $*$-isomorphisms of $\mathfrak{A}=C_0([0,1))$ between $\mathrm{id}_{\mathfrak{A}}$ and χ_0, so that $\mathfrak{A}$ is weakly identically contractible to $\mathbb{C}$. Also, define $h_t(x)=\frac{x}{1-t}\in[0,1)$ for $t\in[0,1)$ and $x\in[0,1-t)\approx[0,1)$, and $h_1(x)=0$, so that $\psi_t(\mathfrak{A})\cong\mathfrak{A}$ for

$t \in [0,1)$. Then (h_t) is a continuous path of homeomorphisms of $[0,1)$ such that $f_0 = \mathrm{id}_X$ and $f_1 = \mathrm{id}_0$, so that $[0,1)$ is identically contractible (to 0).

Furthermore, now let $g(x) = x$ for $x \in [0, \frac{1}{2}]$ and $g(x) = 1 - x$ for $x \in [\frac{1}{2}, 1)$ and $g \in C_0([0,1))$. Then the norm $\|\psi_t(g)\| = \|g\| = 1$, but $\chi_0(g) = g(0) = 0$.

If a (compact or non-compact) space X is contractible to a point $p \in X$, then we define $\mathfrak{I}_p$ to be the closed ideal of all continuous functions of $(C(X)$ or $C_0(X))$ on X vanishing at the point p. Note that $\mathfrak{I}_p$ is isomorphic to $C_0(X \setminus \{p\})$.

As a generalization from the case of $C_0([0,1))$ as a closed ideal of $C([0,1])$,

Proposition 2.3. *If a compact Hausdorff space X is contractible to a point $p \in X$, then the closed ideal $\mathfrak{I}_p = C_0(X \setminus \{0\})$ is contractible to zero.*

As well, in this case, $\mathfrak{I}_p \otimes \mathfrak{A}$ for any C^*-algebra $\mathfrak{A}$ is contractible to zero.

Proof. As shown above, it follows that $C(X)$ is contractible to $\mathbb{C}$ (at $p \in X$). Therefore, $\mathfrak{I}_p$ is contractible to zero (at $p \in X$).

Since $\mathfrak{I}_p$ is contractible, so is $\mathfrak{I}_p \otimes \mathfrak{A}$ by the same reason as in the example above. □

Remark. Even if a non-compact, Hausdorff space X is contractible to a point $p \in X$, the closed ideal $\mathfrak{I}_p$ is not necessarily contractible. For instance, let $X = [0,1)$. Then X is contractible to $\{0\}$, but $\mathfrak{I}_0 = C_0((0,1))$ is not contractible. However, $\mathfrak{I}_0$ in this case is weakly contractible to $\mathbb{C}$ since $(0,1)$ is contractible and non compact. Note also that $(0,1)^+ \approx \mathbb{T}$ the one-dimensional torus, which is not contractible.

We now define that a non-compact topological space X is **extended contracible** (in the one-point compactification $X^+ = X \cup \{\infty\}$ of X) if the identity map $\mathrm{id}_{X^+} : X^+ \to X^+$ is homotopic to the constant map id_∞ on X^+, which sends elements of X^+ to the point ∞. We write F^+ for the corresponding homotopy on $I \times X^+$ and call it the extended homotopy for X^+.

Possibly, the most important thing to notice at this moment is that

Proposition 2.4. (1) *Let X be a non-compact, locally compact Hausdorff space. Then X is extended contractible in X^+ in our sense if and only if X^+ is contractible.*

(2) *If X is extended contractible in X^+ in our sense, in other words, if X is a one-point un-compactification of a contractible space, then $C_0(X)$ is contractible to zero.*

(3) *The direct product of finitely many, extended contractible, non-compact locally compact Hausdorff spaces is also extended contractible.*

Proof. By definition, the first statement (1) holds.

The second statement (2) follows from that $C_0(X) \cong \mathfrak{I}_\infty$ in $C(X^+)$.

For the third (3), if $X_1, \cdots, X_n$ are extended contractible, non-compact locally compact Hausdorff spaces, then $(\Pi_{i=1}^n X_i)^+$ is contractible because the coordinante homotopy in X_i^+ extends in $(\Pi_{i=1}^n X_i)^+$ as a product of the homotopies □

Example 2.5. • Let $\mathfrak{A} = C_0([0,1))$. Then $\mathfrak{A}$ is contractible (to zero) as in the references ([2], [4], and [8]).

Indeed, define $\varphi_t(g)(x) = g(t + x(1-t)) \in \mathbb{C}$ for $x \in [0,1)$ and $t \in [0,1]$. Then $\varphi_0(g)(x) = g(x)$ and $\varphi_1(g)(x) = g(1) = 0$, and φ_t for $t \in [0,1)$ are $*$-isomorphisms of $\mathfrak{A}$. Also the space $[0,1)$ is contractible (but to $1 \notin [0,1)$, however in $[0,1]$), because the maps on $[0,1)$ defined by $f_t(x) = t + x(1-t) \in [t,1) \approx [0,1)$ give a continuous path of homeomorphisms of $[0,1)$ such that $f_0 = \mathrm{id}_X$ and $f_1 = \mathrm{id}_1$.

Therefore, $[0,1)$ is extended contractible in $[0,1)^+ = [0,1]$ and $C_0([0,1))$ is identically contractible.

Remark. Note that a contractible space in the 1-dimensional closed interval $I = [0,1]$ is always identically contractible. Moreover, any 1-dimensional contractible space in I is homeomorphic to either $I = [0,1]$, $I_1 = [0,1)$, or $I_{0,1} = (0,1)$. Furthermore, $I = [0,1]$ is a 1-dimensional compact manifold with boundary $\partial I = \{0,1\}$, and $I_1 = [0,1)$ is a 1-dimensional non-compact manifold with boundary $\partial I_1 = \{0\}$, and $I_{0,1} = (0,1)$ is a 1-dimensional non-compact (or open) manifold without boundary.

On the other hand, an extended contractible space may or may not be connected.

Example 2.6. Let $X = [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$ be a union of half open intervals. Then X is non-connected and is viewed as the one-point un-compactification of $[0,1]$ a contractible space. Hence $C_0(X)$ is contractible to zero. Note that $C_0(X) \cong C_0([0, \frac{1}{2}]) \oplus C_0((\frac{1}{2}, 1])$ with both components contractible to zero.

Just as the 1-dimensional case of connected sums of topological manifolds, one can define (but) a **non-connected sum** of two contractible spaces X and Y in $[0,1]$, denoted as $X\#_pY$, for a point p viewed in the interiors X° and Y° of X and Y respectively, where X is viewed in the line of a Euclidean space and the boundary ∂X is $X \setminus X^\circ$. More precisely, $X\#_pY$ is defined by removing a point in the interiors X° and Y° of X and Y respectively, each identified with a point p, to make disjoint unions $X \setminus \{p\} = X_p^1 \sqcup X_p^2$ and $Y \setminus \{p\} = Y_p^1 \sqcup Y_p^2$ and by gluing X_p^1 and Y_p^1 together with p and gluing X_p^2 and Y_p^2 together with p to make two lines $X_p^1 \cup \{p\} \cup Y_p^1$ and $X_p^2 \cup \{p\} \cup Y_p^2$, where each p in these unions are assumed to be distinct. By definition, the non-connected sum $X\#_pY$ is a disjoint union of two contractible line segments L_j $(j = 1, 2)$ in $[0,1]$, so that $X\#_pY = L_1 \sqcup L_2$. Note that $X\#_pY$ is not contractible, and $C(X\#_pY) \cong C(L_1) \oplus C(L_2)$, and $C_0(X\#_pY) \cong C_0(L_1) \oplus C_0(L_2)$ where L_1 or L_2 may be compact and that $X\#_pY$ is compact if and only if X and Y are compact.

Example 2.7. We have $[0,1]\#_p[0,1] \approx [0,1] \sqcup [0,1] \equiv \sqcup^2[0,1]$, and $[0,1)\#_p[0,1) \approx [0,1] \sqcup (0,1)$, and $(0,1)\#_p(0,1) \approx \sqcup^2(0,1)$, and $[0,1]\#_p[0,1) \approx [0,1] \sqcup [0,1)$, and $[0,1]\#_p(0,1) \approx \sqcup^2[0,1)$, and $[0,1)\#_p(0,1) \approx [0,1)\#_p(0,1)$.

Note that only the case $X = [0,1]\#_p(0,1) \approx \sqcup^2[0,1)$ is extended contractible, with $X^+ \approx [0,1]$.

Moreover, we can define inductively a **successive** non-connected sum of n contractible spaces $X_1, \cdots, X_n$ in $[0,1]$ as

$$\#_{p_i}^n X_i \equiv (\cdots((X_1\#_{p_1}X_2)\#_{p_2}X_3)\cdots\#_{p_{n-1}}X_n,$$

where each point p_k is identified with both a point of the interior of $\#_{p_i}^{k-1}X_i$ and a point of the interior of X_{k+1}. The operation taking a non-connected sum is associative. Namely, for example, $(X_1\#_{p_1}X_2)\#_{p_2}X_3 \approx X_1\#_{p_1}(X_2\#_{p_2}X_3)$, where for this we may assume that $p_2 \in X_2$. Note that the points p_1 and p_2 and the points p_k in more general may or may not be the same. Even if $p_i = p_j$ in $[0,1]$ with $i \neq j$, the attached points corresponding to p_i and p_j are assumed to be distinct. Therefore, we always have

$$\#_{p_i}^n X_i \approx L_1 \sqcup L_2 \sqcup \cdots \sqcup L_n \equiv \sqcup_i^n L_i,$$

where each L_i is a contractible space in $[0,1]$.

Proposition 2.8. *Let $X_1, \cdots, X_n$ be contractible spaces in $[0,1]$. Then a disconnected sum $\#_{p_i}^n X_i$ is a non-contractible, locally compact Hausforff space, and is compact if and only if each X_i is compact. We have $\partial(\#_{p_i}^n X_i) = \cup_i \partial X_i$.*

A non-compact $\#_{p_i}^n X_i$ is extended contractible if and only if $\#_{p_i}^n X_i$ is homeomorphic to the disjoint union $\sqcup^n[0,1)$. Hence, $C_0(\sqcup^n[0,1)) \cong \oplus^n C_0([0,1))$ is contractible to zero.

Proof. The first part is clear.

For the second, note that if a non-compact $\#_{p_i}^n X_i$ contains a X_i, homeomorphic to $(0,1)$, then the one-point compactification $(\#_{p_i}^n X_i)^+$ contains a circle embedded as a subset, so that it can not be contractible. □

Recall that the connected sum $M\#N$ of two topological manifolds M and N of dimension $d \geq 2$ is obtained by removing the d-dimensional closed unit ball B viewed in M and N and attaching $M \setminus B$ and $N \setminus B$ together with the boundary ∂B of B along. Note that ∂B is not contractible. Hence $M\#N$ is always not contractible even when M and N are contractible.

On the other hand, one can also define a **pointed jointed sum** of two spaces X and Y, denoted as $X \sqcup_p Y$, for a point p viewed in X and Y. More precisely, $X\#_p Y$ is defined by joining X and Y at p in the disjoint union $X \sqcup Y$. By definition, if X and Y are contractible, then the pointed jointed sum $X \sqcup_p Y$ is contractible.

Moreover, we can define inductively a **successive** pointed jointed sum of n spaces $X_1, \cdots, X_n$ as

$$\sqcup_{p_i}^n X_i \equiv (\cdots((X_1 \sqcup_{p_1} X_2) \sqcup_{p_2} X_3) \cdots \sqcup_{p_{n-1}} X_n,$$

where each point p_k is identified with both a point of $\sqcup_{p_i}^{k-1} X_i$ and a point of X_{k+1}. By definition, if $X_1, \cdots, X_n$ are contractible, then a successive pointed jointed sum $\sqcup_{p_i}^n X_i$ is contractible. To have associativity for successive pointed jointed sums, such as

$$(X_1 \sqcup_{p_1} X_2) \sqcup_{p_2} X_3 \approx X_1 \sqcup_{p_1} (X_2 \sqcup_{p_2} X_3),$$

we may assume that each p_i is in X_i. We **assume** this associativity in what follows. Note that homeomorphism classes of pointed jointed sums do depend on both the way of arrangement (or permutation with respect to i) of X_i and the choice (distinct or not) of the points p_i in general. For instance,

$$([0,1] \sqcup_{\frac{1}{3}} (0,1)) \sqcup_{\frac{1}{2}} [0,1] \not\approx ((0,1) \sqcup_{\frac{1}{3}} [0,1]) \sqcup_{\frac{1}{2}} [0,1].$$

Proof. Indeed, consider the interval $[\frac{1}{3}, \frac{1}{2}]$ viewed in the middle intervals. The jointed points $\frac{1}{3}$ and $\frac{1}{2}$ emit three intervals closed or open at the other end points respectively (2 closed and 1 open at $\frac{1}{3}$ and $\frac{1}{2}$ and 2 open and 1 closed at $\frac{1}{3}$ and 3 closed at $\frac{1}{2}$), whose respective parts in the jointed sums are not homeomorphic respectively. □

Proposition 2.9. *Let $X_1, \cdots, X_n$ be contractible spaces. A pointed jointed sum $\sqcup_{p_i}^n X_i$ is a contractible, locally compact Hausforff space, and is compact if and only if each X_i is compact, and $\partial(\sqcup_{p_i}^n X_i) = \cup_i \partial X_i$.*

A non-compact $\sqcup_{p_i}^n X_i$ is extended contractible if and only if its boundary has only one point.

Moreover, if each X_i is identically contractible, then $\sqcup_{p_i}^n X_i$ is identically contractible.

Proof. Note that for a non-compact $\sqcup_{p_i}^n X_i$, if $\partial(\sqcup_{p_i}^n X_i)$ has more than one point, then the one-point compactfication $(\sqcup_{p_i}^n X_i)^+$ contains a circle embedded as a subset and thus the compactification is not contractible.

Since each X_i is identically contractible by a homotopy, so is the jointed sum $\sqcup_{p_i}^n X_i$ by taking the (simultaneous) homotopy induced by the homotopies of X_i □

Let M and N be topological manifolds of dimension $d \geq 1$ (or greater than d). We define a d-dimensional **balled jointed sum** of M and N to be obtained by identifying the d-dimensional closed unit balls B viewed in M and N, and to be denoted by $M \sqcup_B N$.

Note that the 1-dimensional closed unit ball is the closed interval $[-1,1]$. Also, a pointed jointed sum may be defined to be a **zero**-dimensional jointed sum. Moreover, one can define inductively a **successive** d-dimensional (or at most) balled jointed sum of topological manifolds $M_1, \cdots, M_n$ of dimension d (or greater than d) by

$$\sqcup_{B_i}^n M_i \equiv (\cdots((M_1 \sqcup_{B_1} M_2) \sqcup_{B_2} M_3) \cdots) \sqcup_{B_{n-1}} M_n,$$

where each B_i is a d-dimensional (or at most) closed unit ball viewed in M_i and M_{i+1}. Note that the dimension d may not be constant as $\dim B_i = d_i$ for i. By definition, if $M_1, \cdots, M_n$ are contractible, then $\sqcup_{B_i}^n M_i$ is also contractible, but only a space, not a manifold in general. To have associativity for successive balled jointed sums, such as

$$(M_1 \sqcup_{B_1} M_2) \sqcup_{B_2} M_3 \approx M_1 \sqcup_{B_1} (M_2 \sqcup_{B_2} M_3),$$

we may assume that each B_i is in M_i. We **assume** this associativity in what follows. Note that homeomorphism classes of balled jointed sums do depend on both the way of arrangement (or permutation with respect to i) of M_i and the choice (distinct or not) of the balls B_i in general.

As a collection, we obtain

Table 1: Classification for contractible spaces and examples by C^*-algebras

C^*-algebras $\setminus$ Spaces	Compact	Non-compact, contractible
Contractible to zero (non-unital)	No	Extended contractible: $I_1 = [0,1)$, $I_1^d = \Pi^d I_1$, $(\sqcup_{p_i}^{n-1} I) \sqcup_{p_{n-1}} I_1$, $(\sqcup_{B_i}^{n-1} I^d) \sqcup_{B_{n-1}} I_1^d$
Non-contractible to zero (unital or non-unital)	Contractible: $I = [0,1]$, I^d $\sqcup_{p_i}^n I$, $\sqcup_{B_i}^n I^d$	Non-extended contractible: $I_{0,1} = (0,1)$, $I_{0,1}^d \approx \mathbb{R}^d$, $\sqcup_{p_i}^{n+m+l} X_i$, $\sqcup_{B_i}^{n+m+l} X_i^d$ $(m+l \geq 2)$ $(X_i = I, I_1, I_{0,1}$ n, m, l copies)

Remark. There are non-contractible spaces whose C^*-algebras are contractible to zero, such as disjoint unions of extended contractible, non-compact locally compact Hausdorff spaces like $\sqcup^n [0,1)$.

It follows from the Table 1 that

Corollary 2.10. *The being or not being contractible to zero for C^*-algebras (together with unitalness or non-unitalness for C^*-algebras) classifies contractible spaces to be compact or non-compact and to be extended contractible or not.*

Remark. Note that compactness and non-compactness for spaces just correspond to unitalness and non-unitalness for C^*-algebras, respectively.

Now let X be a topological space. Denote by $\partial\overline{X}$ the boundary of $\overline{X}$, which is equal to $\overline{X} \setminus (\overline{X})^\circ$, where $\overline{X}$ is the closure of X in a suitable topology (or a suitable compactification of X along $\partial\overline{X}$) and $(\overline{X})^\circ$ is the interior of $\overline{X}$, where note that we mostly deal with topological spaces X viewed as (homeomorphically bounded) subsets with relative topology in Euclidean spaces and take their closures $\overline{X}$ in there. We may say that $\partial\overline{X} \setminus X = \overline{X} \setminus X$ is the **attached boundary** of X and $\overline{X}$ is the **flat** compactification of X.

Example 2.11. Let $I = [0,1]$. Then $\partial I = \{0,1\}$ and $\partial I \setminus I = \emptyset$, and also $\partial(I^d) \setminus I^d = \emptyset$.

Let $I_1 = [0,1)$. Then $\overline{I_1} = [0,1]$, $\partial \overline{I_1} = \{0,1\}$ and $\partial \overline{I_1} \setminus I_1 = \overline{I_1} \setminus I_1 = \{1\}$.

Let $I_{0,1} = (0,1)$. Then $\overline{I_{0,1}} = [0,1]$, $\partial \overline{I_{0,1}} = \{0,1\}$ and $\partial \overline{I_{0,1}} \setminus I_{0,1} = \{0,1\}$.

We have $\overline{I_1^2} = [0,1]^2$, and $\partial(\overline{I_1^2}) \setminus I_1^2 = (\{1\} \times [0,1]) \cup ([0,1] \times \{1\}) \approx [0,1]$, which is contractible and has covering dimension one.

We have $\overline{I_{0,1}^2} = [0,1]^2$, and $\partial(\overline{I_{0,1}^2}) \setminus I_{0,1}^2 \approx S^1$ the 1-dimensional sphere, which is not contractible and has covering dimension one.

Table 2: Classification for examples of contractible spaces by boundaries

Attached boundaries	Contractible spaces
No	Compact: $I = [0,1]$, I^d, $\sqcup_{p_i}^n I$, $\sqcup_{B_i}^n I^d$
One point	Non-compact: $I_1 = [0,1)$, $(\sqcup_{p_i}^{n-1} I) \sqcup_{p_{n-1}} I_1$ $(n \geq 2)$
Contractible, dimension $d-1$	Non-compact: I_1^d, $(\sqcup_{B_i}^{n-1} I^d) \sqcup_{B_{n-1}} I_1^d$
Two points $m + 2l$ points	Non-compact: $I_{0,1} = (0,1)$, $\sqcup_{p_i}^{n+m+l} X_i$, $\sqcup_{B_i}^{n+m+l} X_i$ $(m + 2l \geq 2)$ $(X_i = I, I_1, I_{0,1}$ n, m, l copies, resp)
Non-contractible, dim $d-1$	$I_{0,1}^d \approx \mathbb{R}^d$ $(d \geq 2)$
Non-contractible, dim $d-1$, $m + l$ components	$\sqcup_{B_i}^{n+m+l} X_i^d$ $(m + l \geq 2, d \geq 2)$ $(X_i = I, I_1, I_{0,1}$ n, m, l copies, resp)

It follows from the Tables 1 and 2 that

Corollary 2.12. *The being contractible and being unital or not for C^*-algebras, together with attached boundaries for spaces as similar invariants, and with dimension and pointed or balled jointedness for spaces or manifolds classify* (*up to homeomorphisms in part*) *1-dimensional, contractible manifolds and d-dimensional, jointed sums of d-dimensional contractible, their product manifolds, as in the collection lists above.*

Remark. The homeomorphism classes of the spaces $\sqcup_{p_i}^{n+m+l} X_i$ with $X_i = I$, I_1, or $I_{0,1}$ n, m, l copies respectively do depend on how to take the points p_j. For instance, all p_j may be the unique point, like $p_j = \frac{1}{2}$. Namely, the homeomorphism classes depend on that p_j are mutually, the same or different and as well their positions, in general. The similar things hold for $\sqcup_{B_i}^{n+m+l} X_i^d$.

It follows from the Table 3 (at the top of the next page) that

Corollary 2.13. *The being either unital and identically contractible to $\mathbb{C}$ or being non-unital and weakly identically contractible to $\mathbb{C}$ for C^*-algebras classifies contractible spaces to be compact or not to be.*

3 K-theory

We now consider K-theory (abelian) groups for C^*-algebras.

It is known that if a C^*-algebra $\mathfrak{A}$ is contractible to zero, then the K-theory groups $K_0(\mathfrak{A})$ and $K_1(\mathfrak{A})$ both are zero, Note that the K-theory groups are homotopy invariant. In fact, the zero C^*-algebra $\{0\}$ has K_0 zero and the unitization $\{0\}^+ = \mathbb{C}$ has K_1 zero, so that the zero C^*-algebra has K_1 zero.

In particular,

Table 3: Classification for identically contractible spaces and examples by C^*-algebras

C^*-algebras \ Spaces	Compact	Non-compact, contractible
Unital, identically contractible to $\mathbb{C}$	Contractible: I^d, $\sqcup_{p_i}^n I^d$, $\sqcup_{B_i}^n I^d$	No
Non-unital, weakly identically contractible to $\mathbb{C}$	No	Extended: I_1^d, $(\sqcup_{p_i}^{n-1} I^d) \sqcup_{p_{n-1}} I_1^d$, $(\sqcup_{B_i}^{n-1} I^d) \sqcup_{B_{n-1}} I_1^d$ Non-extended: $I_{0,1}^d$, $\sqcup_{p_i}^{n+m+l} X_i$, $\sqcup_{B_i}^{n+m+l} X_i^d$ $(m+2l \geq 2)$ $(X_i = I, I_0, I_{0,1}$ n, m, l copies)

Example 3.1. Since $C_0([0,1)) = C\mathbb{C}$ the cone over $\mathbb{C}$ is contractible, it follows that $K_0(C_0([0,1))) \cong 0$ and $K_1(C_0([0,1))) \cong 0$. The same holds by replacing $[0,1)$ with $(\sqcup_{p_i}^{n-1} I) \sqcup_{p_{n-1}} I_1$ and also by $C\mathbb{C}$ with $C\mathfrak{A} \cong C_0([0,1)) \otimes \mathfrak{A}$ for any C^*-algebra $\mathfrak{A}$.

As a contrast, with (1) below certainly known ([8]),

Proposition 3.2. (1) *Let X be a contractible, compact space. Then*

$$K_0(C(X)) \cong \mathbb{Z} \quad \text{and} \quad K_1(C(X)) \cong 0.$$

(2) *For a non-comapct space X, we have*

$$K_0(C_0(X)) \cong K_0(C(X^+))/\mathbb{Z} \quad \text{and} \quad K_1(C_0(X)) \cong K_1(C(X^+)).$$

(3) *If a non-compact space X is extended contractible, then we have*

$$K_0(C_0(X)) \cong 0 \quad \text{and} \quad K_1(C_0(X)) \cong 0.$$

Proof. The first statement (1) holds because $K_j(C(X)) \cong K_j(\mathbb{C})$ for $j = 0, 1$.

For the second (2), there is the short exact sequence of C^*-algebras:

$$0 \to C_0(X) \to C(X^+) \to \mathbb{C} \to 0$$

that splits, where the section from $\mathbb{C}$ to $C(X^+)$ is given by sending $1 \in \mathbb{C}$ to $1 \in C(X^+)$. The associated six-term exact sequence of K-theory groups implies that

$$K_j(C(X^+)) \cong K_j(C_0(X)) \oplus K_j(\mathbb{C})$$

for $j = 0, 1$, with $K_0(\mathbb{C}) \cong \mathbb{C}$ and $K_1(\mathbb{C}) = 0$.

The third (3) follows from (1) and (2) above. $\square$

Example 3.3. We have $K_0(C([0,1])) \cong \mathbb{Z}$ and $K_0(C([0,1])) \cong 0$. Since a compact, pointed or balled, jointed sums $J = \sqcup_{p_i}^n I$ or $J = \sqcup_{B_i}^n I^d$ contractible, thus $K_0(C(J)) \cong \mathbb{Z}$ and $K_1(C(J)) \cong 0$.

There is the following short exact sequence of C^*-algebras:

$$0 \to C_0((0,1)) \to C_0([0,1)) \to \mathbb{C} \to 0,$$

which is not splitting, but the six-term exact sequence of K-theory groups, associated, becomes:

$$\begin{array}{ccccc} K_0(C_0((0,1))) & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} \\ \partial \uparrow & & & & \downarrow \partial \\ 0 & \longleftarrow & 0 & \longleftarrow & K_1(C_0((0,1))) \end{array}$$

with the maps ∂ as the up and down arrows in the left and right, respectively, the index map and the exponential map (as a dual of the index map), and hence $K_0(C_0((0,1))) \cong 0$ and $K_1(C_0((0,1))) \cong \mathbb{Z}$.

The converses of (1) and (3) in the proposition above do not hold for contractible spaces.

Example 3.4. Let $X = \mathbb{R}^{2n}$ be the $2n$-dimensional Euclidean space, for $n \geq 1$, which is contractible but non-compact. Then

$$K_0(C_0(\mathbb{R}^{2n})) \cong K_0(\mathbb{C}) \cong \mathbb{Z} \quad \text{and} \quad K_1(C_0(\mathbb{R}^{2n})) \cong K_1(\mathbb{C}) \cong 0$$

by Bott periodicity of K-theory groups. Also, X^+ is homeomorphic to S^{2n} the $2n$-dimensional sphere, which is not contractible, because $K_0(C(S^{2n})) \cong \mathbb{Z}^2$ and $K_1(C(S^{2n})) \cong 0$, so that X is not extended contractible.

Let $X = \mathbb{R}^{2n} \times [0,1)$ the product space. Then

$$K_j(C_0(X)) \cong K_j(C_0(\mathbb{R}^{2n}) \otimes C_0([0,1))) \cong K_j(C_0([0,1))) \cong 0$$

for $j = 0,1$. Also, X^+ is homeomophic to $S^{2n} \sqcup_1 I_1$, which is not contractible, because $S^{2n} \sqcup_1 I_1$ is homotopic to S^{2n}, so that X is not extended contractible.

Proposition 3.5. *Let $\#_{p_i} X_i$ be the successive non-connected sum of n contractible spaces $X_1, \cdots, X_n$ in $[0,1]$, with $\#_{p_i} X_i \approx \sqcup_{i=1}^n L_i$. Then*

$$K_j(C_0(\#_{p_i} X_i)) \cong \oplus_{i=1}^n K_j(C_0(L_i))$$

for $j = 0,1$.

Proposition 3.6. *Let $X \sqcup_p Y$ be the (pointed) jointed sum of two spaces X, Y. If $X \sqcup_p Y$ is compact, then*

$$\begin{aligned} K_0(C(X \sqcup_p Y)) &\cong K_0(C_0(X \setminus \{p\})) \oplus K_0(C_0(Y \setminus \{p\})) \oplus \mathbb{Z}, \\ K_1(C(X \sqcup_p Y)) &\cong K_1(C_0(X \setminus \{p\})) \oplus K_1(C_0(Y \setminus \{p\})), \end{aligned}$$

and if $X \sqcup_p Y$ is not compact, then

$$\begin{aligned} K_0(C_0(X \sqcup_p Y)) &\cong K_0(C_0(X \setminus \{p\})) \oplus K_0(C_0(Y \setminus \{p\})), \\ K_1(C_0(X \sqcup_p Y)) &\cong [K_1(C_0(X \setminus \{p\})) \oplus K_1(C_0(Y \setminus \{p\}))]/\mathbb{Z}. \end{aligned}$$

Proof. There is the following short exact sequence of C^*-algebras:

$$0 \to C_0(X \setminus \{p\}) \oplus C_0(Y \setminus \{p\}) \to C_0(X \sqcup_p Y) \to \mathbb{C} \to 0,$$

which splits only when $X \sqcup_p Y$ is compact, where the quotient map is the evaluation map at p. It follows that if $X \sqcup_p Y$ is compact, then

$$K_j(C(X \sqcup_p Y)) \cong K_j(C_0(X \setminus \{p\})) \oplus K_j(C_0(Y \setminus \{p\})) \oplus K_j(\mathbb{C})$$

for $j = 0,1$. If $X \sqcup_p Y$ is not compact, then the induced quotient map from $K_0(C_0(X \sqcup_p Y))$ to $K_0(\mathbb{C})$ is zero, so that it follows from exactness of the six-term exact sequences of K-theory groups that

$$K_0(C_0(X \sqcup_p Y)) \cong K_0(C_0(X \setminus \{p\})) \oplus K_0(C_0(Y \setminus \{p\}))$$

and

$$K_1(C_0(X \sqcup_p Y)) \cong [K_1(C_0(X \setminus \{p\})) \oplus K_1(C_0(Y \setminus \{p\}))]/K_1(\mathbb{C}).$$

□

Moreover

Proposition 3.7. *Let $\sqcup_{p_i}^n X_i$ be the successive (pointed) jointed sum of n path-connected spaces $X_1, \cdots, X_n$. If $\sqcup_{p_i}^n X_i$ is compact, then*

$$K_0(C(\sqcup_{p_i}^n X_i)) \cong \oplus_{i=1}^n K_0(C_0(X_i \setminus \{p_{i-1}\})) \oplus \mathbb{Z},$$
$$K_1(C(\sqcup_{p_i}^n X_i)) \cong \oplus_{i=1}^n K_1(C_0(X_i \setminus \{p_{i-1}\})).$$

If $\sqcup_{p_i}^n X_i$ is not compact, then

$$K_0(C_0(\sqcup_{p_i}^n X_i)) \cong \oplus_{i=1}^n K_0(C_0(X_i \setminus \{p_{i-1}\})),$$
$$K_1(C_0(\sqcup_{p_i}^n X_i)) \cong [\oplus_{i=1}^n K_1(C_0(X_i \setminus \{p_{i-1}\}))]/\mathbb{Z}.$$

Proof. There is a homotopy between $X = \sqcup_{p_i}^n X_i$ and the jointed sum $Y = \sqcup_p^n X_i$ with the common point p as in the case where $p_i = p_{i+1}$ (identified) for $1 \leq i \leq n-2$. Then there is the following short exact sequence of C^*-algebras:

$$0 \to \oplus_{i=1}^n C_0(X_i \setminus \{p_{i-1}\}) \to C_0(Y) \to \mathbb{C} \to 0,$$

which splits only when Y is compact, where the quotient map is the evaluation map at the common point p and $X_i \setminus \{p\} \approx X_i \setminus \{p_{i-1}\}$. It follows that if Y is compact (if and only if X is compact), then

$$K_j(C(Y)) \cong [\oplus_{i=1}^n K_j(C_0(X_i \setminus \{p_{i-1}\}))] \oplus K_j(\mathbb{C})$$

for $j = 0, 1$. If Y is not compact, then the induced quotient map from $K_0(C_0(Y))$ to $K_0(\mathbb{C})$ is zero, so that it follows from exactness of the six-term exact sequences of K-theory groups that

$$K_0(C_0(Y)) \cong \oplus_{i=1}^n K_0(C_0(X_i \setminus \{p_{i-1}\}))$$

and $K_1(C(Y)) \cong [\oplus_{i=1}^n K_1(C_0(X_i \setminus \{p_{i-1}\}))]/K_1(\mathbb{C})$. □

As examples,

Example 3.8. Let $X = \sqcup_{p_i}^n I_1$ be a (pointed) jointed sum of n copies of $I_1 = [0, 1)$ $(n \geq 2)$. Then

$$K_0(C_0(\sqcup_{p_i}^n I_1)) \cong 0 \quad \text{and} \quad K_1(C_0(\sqcup_{p_i}^n I_1)) \cong \mathbb{Z}^{n-1}.$$

This also holds for $n = 1$, with $\sqcup^1 I_1 = I_1$ and $\mathbb{Z}^0 = 0$.

Proof. There is a homotopy between X and $\sqcup_0^n I_1$ the jointed sum of n copies of I_1 at the common zero point 0. Because if $I_1 = [0, p_i) \cup [p_i, 1)$ and $[0, p_i)$ does not contain other p_j, then it is homotopic to $[p_i, 1)$ in X. We continue this process inductively and finitely to obtain the required homotopy.

When $n = 2$, X is homotopic to $(0, 1) \approx \sqcup_0^2 I_1$.

When $n = 3$, there is the following short exact sequence:

$$0 \to C_0((0, 1)) \to C_0(\sqcup_0^3 I_1) \to C_0(\sqcup_0^2 I_1) \to 0,$$

where $\sqcup_0^2 I_1$ in the quotient is homeomoprhic to $(0, 1)$ and closed in $\sqcup_0^3 I_1$, and its complement is $(0, 1)$ in the ideal. The six-term exact sequence of K-theory groups, associated, becomes:

$$\begin{array}{ccccc} 0 & \longrightarrow & K_0(C_0(\sqcup_0^3 I_1)) & \longrightarrow & 0 \\ \partial \uparrow & & & & \downarrow \partial \\ \mathbb{Z} & \longleftarrow & K_1(C_0(\sqcup_0^3 I_1)) & \longleftarrow & \mathbb{Z}. \end{array}$$

It follows that $K_0(C_0(\sqcup_0^3 I_1)) \cong 0$ and $K_1(C_0(\sqcup_0^3 I_1)) \cong \mathbb{Z}^2$.

By induction, we assume that $K_0(C_0(\sqcup_0^n I_1)) \cong 0$ and $K_1(C_0(\sqcup_0^n I_1)) \cong \mathbb{Z}^{n-1}$. Then there is the following short exact sequence:

$$0 \to C_0((0,1)) \to C_0(\sqcup_0^{n+1} I_1) \to C_0(\sqcup_0^n I_1) \to 0$$

since $\sqcup_0^n I_1$ is closed in $\sqcup_0^{n+1} I_1$ and its complement is $(0,1)$. The six-term exact sequence of K-theory groups, associated, becomes:

$$\begin{array}{ccccc} 0 & \longrightarrow & K_0(C_0(\sqcup_0^{n+1} I_1)) & \longrightarrow & 0 \\ \partial \uparrow & & & & \downarrow \partial \\ \mathbb{Z}^{n-1} & \longleftarrow & K_1(C_0(\sqcup_0^{n+1} I_1)) & \longleftarrow & \mathbb{Z}. \end{array}$$

It follows that $K_0(C_0(\sqcup_0^{n+1} I_1)) \cong 0$ and $K_1(C_0(\sqcup_0^{n+1} I_1)) \cong \mathbb{Z}^n$.

There is also the following short exact sequence:

$$0 \to C_0(\sqcup^n I_{0,1}) \to C_0(\sqcup_0^n I_1) \to \mathbb{C} \to 0,$$

which is not splitting, with $C_0(\sqcup_0^n I_1) \cong \oplus^n C_0((0,1))$. The six-term exact sequence of K-theory groups, associated, becomes:

$$\begin{array}{ccccc} \oplus^n 0 & \longrightarrow & K_0(C_0(Z)) & \longrightarrow & \mathbb{Z} \\ \partial \uparrow & & & & \downarrow \partial \\ 0 & \longleftarrow & K_1(C_0(Z)) & \longleftarrow & \oplus^n \mathbb{Z} \end{array}$$

and $K_0(C_0(Z)) \cong 0$ and $K_1(C_0(Z))) \cong \mathbb{Z}^{n-1}$. □

Example 3.9. Let $X = \sqcup_{p_i}^n I_{0,1}$ be a (pointed) jointed sum of n copies of $I_{0,1} = (0,1) \approx \mathbb{R}$ $(n \geq 2)$. Then

$$K_0(C_0(\sqcup_{p_i}^n I_{0,1})) \cong 0 \quad \text{and} \quad K_1(C_0(\sqcup_{p_i}^n I_{0,1})) \cong \mathbb{Z}^{2n-1}.$$

This also holds for $n = 1$, with $\sqcup^1 I_{0,1} = I_{0,1}$.

Proof. There is a homotopy between X and $\sqcup_0^{2n} I_1$ the jointed sum at the common zero point 0. By Proposition 3.7 above, we obtain the conclusion. □

Example 3.10. Let $X = \sqcup_{p_i}^{n+m+l} X_i$ be a (pointed) jointed sum of $X_i = I$, I_1, or $I_{0,1}$, with n copies of I, m copies of I_1, and l copies of $I_{0,1}$. Then

$$K_0(C_0(\sqcup_{p_i}^{n+m+l} X_i)) \cong 0 \quad \text{and} \quad K_1(C_0(\sqcup_{p_i}^{n+m+l} X_i)) \cong \mathbb{Z}^{m+2l-1}.$$

Proof. There is a homotopy between X and $\sqcup_0^{m+2l} I_1$ the jointed sum at the common zero point 0, as considered above. By Proposition 3.7 above, we obtain the conclusion. □

As 2-dimensional analogues as examples,

Example 3.11. Let $X = \sqcup_{p_i}^n (I^2)^-$ be a (pointed) jointed sum of n copies of $(I^2)^-$ the one-potint uncompactification of the 2-direct product of $I = [0,1]$. Then

$$K_0(C_0(X)) \cong 0 \quad \text{and} \quad K_1(C_0(X)) \cong \mathbb{Z}^{n-1}.$$

Proof. To determine $K_j(C_0(X))$, it is enough to compute $K_j(C_0((I^2)^- \setminus \{p_i\}))$. Then one can show that the space $(I^2)^- \setminus \{p_i\}$ is homotopic to $(0,1)$. Because p_i is different from the removed point (say q_i) of each I^2 to make $(I^2)^-$, and that I^2 is homotopic to a 1-dimensional closed interval with end points identified with p_i and q_i, so that $(I^2)^- \setminus \{p_i\}$ is homotopic to the interior of the interval. □

Quite similarly, as higher-dimensional analogues as examples,

Example 3.12. Let m be a positive integer with $m \geq 2$. Let $X = \sqcup_{p_i}^n (I^m)^-$ be a (pointed) jointed sum of n copies of $(I^m)^-$ the one-potint uncompactification of the m-direct product of $I = [0,1]$. Then

$$K_0(C_0(X)) \cong 0 \quad \text{and} \quad K_1(C_0(X)) \cong \mathbb{Z}^{n-1}.$$

Moreover,

Example 3.13. Let $X = \sqcup_{p_i}^n \mathbb{R}^2$ be a (pointed) jointed sum of n copies of $\mathbb{R}^2$. Then

$$K_0(C_0(X)) \cong \mathbb{Z}^n \quad \text{and} \quad K_1(C_0(X)) \cong \mathbb{Z}^{n-1}.$$

Proof. Note that $\mathbb{R}^2$ is viewed as $(S^2)^-$, so that $(S^2)^- \setminus \{p_i\}$ is homeomorphic to $S^1 \times \mathbb{R}$, where the removed two points from S^2 may be assumed to be north and south poles in S^2. Then we have $K_j(C_0(S^1 \times \mathbb{R})) \cong K_{j+1}(C(S^1)) \cong \mathbb{Z}$ for $j = 0, 1 \pmod 2$. □

Similarly,

Example 3.14. Let $X = \sqcup_{p_i}^n \mathbb{R}^{2m}$ be a (pointed) jointed sum of n copies of $\mathbb{R}^{2m}$. Then

$$K_0(C_0(X)) \cong \mathbb{Z}^n \quad \text{and} \quad K_1(C_0(X)) \cong \mathbb{Z}^{n-1}.$$

Proof. Note that $\mathbb{R}^{2m}$ is viewed as $(S^{2m})^-$, so that $(S^{2m})^- \setminus \{p_i\}$ is homeomorphic to $S^{2m-1} \times \mathbb{R}$, where we may assume that the removed two points from S^{2m} are north and south poles in S^{2m}. Then we have $K_j(C_0(S^{2m-1} \times \mathbb{R})) \cong K_{j+1}(C(S^{2m-1})) \cong \mathbb{Z}$ for $j = 0, 1 \pmod 2$. □

On the other hand,

Example 3.15. Let $X = \sqcup_{p_i}^n \mathbb{R}^{2m+1}$ be a (pointed) jointed sum of n copies of $\mathbb{R}^{2m+1}$. Then

$$K_0(C_0(X)) \cong 0 \quad \text{and} \quad K_1(C_0(X)) \cong \mathbb{Z}^{2n-1}.$$

Proof. Note that $\mathbb{R}^{2m+1}$ is viewed as $(S^{2m+1})^-$, so that $(S^{2m+1})^- \setminus \{p_i\}$ is homeomorphic to $S^{2m} \times \mathbb{R}$, where we may assume that the removed two points from S^{2m+1} are north and south poles in S^{2m+1}. Then we have $K_j(C_0(S^{2m} \times \mathbb{R})) \cong K_{j+1}(C(S^{2m}))$ for $j = 0, 1 \pmod 2$ and $K_0(C(S^{2m})) \cong \mathbb{Z}^2$ and $K_1(C(S^{2m})) \cong 0$. □

Furthermore,

Example 3.16. Let $X = \sqcup_{p_i}^{n+m} X_i$ be a (pointed) jointed sum of X_i of n Euclidean spaces with dimensions even and m Euclidean spaces with dimensions odd. Then

$$K_0(C_0(X)) \cong \mathbb{Z}^n \quad \text{and} \quad K_1(C_0(X)) \cong \mathbb{Z}^{n+2m-1}.$$

Next, we consider the balled case.

Proposition 3.17. *Let $M \sqcup_B N$ be the d-dimensional (balled) jointed sum of two topological manifolds M, N of dimension d (or greater than d). If $M \sqcup_B N$ is compact, then*

$$K_0(C(M \sqcup_B N)) \cong K_0(C_0(M \setminus B)) \oplus K_0(C_0(N \setminus B))) \oplus \mathbb{Z},$$
$$K_1(C(M \sqcup_B N)) \cong K_1(C_0(M \setminus B))) \oplus K_1(C_0(N \setminus B))),$$

and if $M \sqcup_B N$ is not compact, then

$$K_0(C_0(M \sqcup_B N)) \cong K_0(C_0(M \setminus B))) \oplus K_0(C_0(N \setminus B))),$$
$$K_1(C_0(M \sqcup_B N)) \cong [K_1(C_0(M \setminus B)) \oplus K_1(C_0(N \setminus B))]/\mathbb{Z}.$$

Proof. The proof is exactly the same as that for Proposition 3.6. Note that $K_j(C(B)) \cong K_j(\mathbb{C})$ for $j = 0, 1$ and the d-dimensional closed ball B is contractible. □

Moreover,

Proposition 3.18. *Let $\sqcup_{B_i}^n M_i$ be the successive d-dimensional (balled) jointed sum of path-connected, topological manifolds $M_1, \cdots, M_n$ of dimension d (or greater than d). If $\sqcup_{B_i}^n M_i$ is compact, then*

$$K_0(C(\sqcup_{B_i}^n M_i)) \cong \oplus_{i=1}^n K_0(C_0(M_i \setminus B_{i-1})) \oplus \mathbb{Z},$$
$$K_1(C(\sqcup_{B_i}^n M_i)) \cong \oplus_{i=1}^n K_1(C_0(M_i \setminus B_{i-1})).$$

If $\sqcup_{B_i}^n M_i$ is not compact, then

$$K_0(C_0(\sqcup_{B_i}^n M_i)) \cong \oplus_{i=1}^n K_0(C_0(M_i \setminus B_{i-1})),$$
$$K_1(C_0(\sqcup_{B_i}^n M_i)) \cong [\oplus_{i=1}^n K_1(C_0(M_i \setminus B_{i-1}))]/\mathbb{Z}.$$

Proof. The proof is exactly the same as that for Proposition 3.7. □

Example 3.19. Let $M = \sqcup_{B_i}^n I_1^d$, with $I_1 = [0, 1)$ and $n \geq 2$. Then

$$K_0(C_0(M)) \cong 0 \quad \text{and} \quad K_1(C_0(M)) \cong \mathbb{Z}^{n-1}.$$

If $M = I_1^d$, then $K_0(C_0(M)) \cong 0 \cong K_1(C_0(M))$.

Proof. We compute $K_j(C_0(I_1^d \setminus B_i))$. Since each ball B_i is contractible, there is the following short exact sequence of C^*-algebras:

$$0 \to C_0(I_1^d \setminus B_i) \to C_0(I_1^d) \to \mathbb{C} \to 0.$$

Since $C_0(I_1^d) \cong \otimes^d C_0(I_1)$ is a contractible C^*-algebra, hence $K_j(C_0(I_1^d)) \cong 0$ for $j = 0, 1$. Note also that the space I_1^d is extended contractible since $(I_1^d)^+$ is contractible. It follows from the six-term exact sequence of K-theory groups that

$$K_0(C_0(I_1^d \setminus B_i)) \cong 0 \quad \text{and} \quad K_1(C_0(I_1^d \setminus B_i)) \cong \mathbb{Z}.$$

□

Example 3.20. Let $M = \sqcup_{B_i}^n I_{0,1}^d$, with $I_{0,1} = (0, 1)$. If d is even, then

$$K_0(C_0(M)) \cong \mathbb{Z}^n \quad \text{and} \quad K_1(C_0(M)) \cong \mathbb{Z}^{n-1},$$

and if d is odd, then $K_0(C_0(M)) \cong 0$ $K_1(C_0(M)) \cong \mathbb{Z}^{2n-1}$.

Proof. We compute $K_j(C_0(I_{0,1}^d \setminus B_i))$. Since each ball B_i is contractible, there is the following short exact sequence of C^*-algebras:

$$0 \to C_0(I_{0,1}^d \setminus B_i) \to C_0(I_{0,1}^d) \to \mathbb{C} \to 0.$$

Since $C_0(I_{0,1}^d) \cong \otimes^d C_0(\mathbb{R}) = S^d\mathbb{C}$, we have $K_0(S^d\mathbb{C}) \cong \mathbb{Z}$ and $K_0(S^d\mathbb{C}) \cong 0$ if d is even and $K_0(S^d\mathbb{C}) \cong 0$ and $K_0(S^d\mathbb{C}) \cong \mathbb{Z}$ if d is odd. It follows from the six-term exact sequence of K-theory groups that if d is even, then

$$K_0(C_0(I_{0,1}^d \setminus B_i)) \cong \mathbb{Z} \quad \text{and} \quad K_1(C_0(I_{0,1}^d \setminus B_i)) \cong \mathbb{Z},$$

and if d is odd, then $K_0(C_0(I_{0,1}^d \setminus B_i)) \cong 0$ and $K_1(C_0(I_{0,1}^d \setminus B_i)) \cong \mathbb{Z}^2$. □

Furthermore, combining Examples 3.19 and 3.20 with Proposition 3.18 we obtain

Example 3.21. Let $M = \sqcup_{B_i}^{n+m+l} X_i^d$, where X_i are n, m, l copies of $I, I_1, I_{0,1}$ respectively. If $m + l \geq 1$, then M is non-compact, and if d is even, then

$$K_0(C_0(M)) \cong \mathbb{Z}^l \quad \text{and} \quad K_1(C_0(M)) \cong \mathbb{Z}^{m+l-1}$$

and if d is odd, then $K_0(C_0(M)) \cong 0$ and $K_1(C_0(M)) \cong \mathbb{Z}^{m+2l-1}$.

Table 4: Classification for contractible spaces by K-theory of C^*-algebras

K-theory of C^*-algebras	Contractible spaces
$K_0 = 0$, $K_1 = 0$	Non-compact, extended contractible: I_1, $(I^n)^- \approx I_1^n$ $(n \geq 2)$,
$K_0 = \mathbb{Z}$, $K_1 = 0$	Compact: I^n Noncompact, non-extended: $I_{0,1}^{2n} \approx \mathbb{R}^{2n}$
$K_0 = \mathbb{Z}^n$, $K_1 = \mathbb{Z}^{n-1}$	$\sqcup_{p_i}^n \mathbb{R}^{2m}$ (pointed), $\sqcup_{B_i}^n I_{0,1}^{2m}$ (balled)
$K_0 = 0$, $K_1 = \mathbb{Z}$	Noncompact, non-extended: $I_{0,1}^{2n+1} \approx \mathbb{R}^{2n+1}$, $\sqcup_p^2 I_1$, $\sqcup_p^2 (I^m)^-$ (pointed), $\sqcup_B^2 I_1^d$ (balled)
$K_0 = 0$, $K_1 = \mathbb{Z}^{n-1}$ $K_0 = 0$, $K_1 = \mathbb{Z}^{m+2l-1}$, $K_0 = 0$, $K_1 = \mathbb{Z}^{2n-1}$	$\sqcup_{p_i}^n I_1$, $\sqcup_{p_i}^n (I^m)^-$ (pointed), $\sqcup_{B_i}^n I_1^d$ (balled) $\sqcup_{p_i}^{n+m+l} X_i$, $\sqcup_{B_i}^{n+m+l} X_i^{2d+1}$, $(X_i = I, I_1, I_{0,1}$ n, m, l copies, $m, l \geq 1)$, $\sqcup_{p_i}^n \mathbb{R}$, $\sqcup_{p_i}^n \mathbb{R}^{2m+1}$ (pointed), $\sqcup_{B_i}^n I_{0,1}^{2m+1}$ (balled)
$K_0 = \mathbb{Z}^l, K_1 = \mathbb{Z}^{m+2l-1}$,	$\sqcup_{B_i}^{n+m+l} X_i^{2d}$, $(X_i = I, I_1, I_{0,1}$ n, m, l copies, $m, l \geq 1)$,
$K_0 = \mathbb{Z}^n, K_1 = \mathbb{Z}^{n+2m-1}$,	$\sqcup_{p_i}^{n+m} X_i$, $\sqcup_{B_i}^{n+m} X_i$ (dim mixed), with $X_i = \mathbb{R}^{2n_i}$ $(1 \leq i \leq n)$, $X_i = \mathbb{R}^{2m_i+1}$ $(n+1 \leq i \leq n+m)$

It follows from the Table 4 that

Corollary 3.22. *The ranks of K-theory groups for C^*-algebras (together with compactness of spaces and dimension of spaces and that of balls in (generic) jointed sums and with jointedness (jointed or not) and with arrangement (or permutation) in jointed sums) classify contractible spaces as in the table (up to homeomorphisms) and to be compact, non-compact and extended, or non-compact and non-extended.*

Remark. Similarly, one can obtain almost the same table for identically contractible spaces.

In the statements above and below, to obtain classification results up to homeomorphisms we may **assume** that pointed or balled jointed sums are generic, i.e., points or balls involved are mutually distinct.

Recall ([5] or [6]) that the Euler characteristic $\chi(\mathfrak{A})$ of a C^*-algebra $\mathfrak{A}$ is defined to be the (formal) difference:

$$\chi(\mathfrak{A}) = \mathrm{rank}_{\mathbb{Z}} K_0(\mathfrak{A}) - \mathrm{rank}_{\mathbb{Z}} K_1(\mathfrak{A}) \in \mathbb{Z} \cup \{\pm\infty\} \cup \{\infty - \infty\}$$

of the $\mathbb{Z}$-ranks of the free abelian direct summands of the K-theory groups of $\mathfrak{A}$. In particular, it is shown that $\chi(C(X)) = \chi(X)$, where $\chi(X)$ is the Euler characteristic of a compact space (or a finite cell complex) X in homology (or cohomology) for spaces.

What's more, it is deduced from the table 4 above that

Table 5: Classification for contractible spaces by the Euler characteristic

Euler numbers of C^*-algebras	Contractible spaces
Zero: $\chi = 0 - 0 = 0$	Non-compact, extended contractible: I_1, $(I^n)^- \approx I_1^d$ $(n \geq 2)$
Positive: $\chi = 1 - 0 = 1 > 0$ $\chi = n - (n-1) = 1 > 0$	Compact: I^n Noncompact, non-extended: (even dim): $I_{0,1}^{2n} \approx \mathbb{R}^{2n}$, $\sqcup_{p_i}^n \mathbb{R}^{2m}$ (pointed), $\sqcup_{B_i}^n I_{0,1}^{2m}$ (balled)
Negative: $\chi = 0 - 1 = -1 < 0$	Noncompact, non-extended: (odd dim): $I_{0,1}^{2n+1} \approx \mathbb{R}^{2n+1}$, 2-fold: $\sqcup_p^2 I_1$, $\sqcup_p^2 (I^m)^-$ (pointed), $\sqcup_B^2 I_1^d$ (balled)
$\chi = 0 - (n-1) = 1 - n < 0$ $\chi = 0 - (m + 2l - 1)$ $= 1 - m - 2l < 0$ $\chi = 0 - (2n-1)$ $1 - 2n < 0$	n-fold: $\sqcup_{p_i}^n I_1$, $\sqcup_{p_i}^n (I^m)^-$ (pointed), $\sqcup_{B_i}^n I_1^d$ (balled) $\sqcup_{p_i}^{n+m+l} X_i$, $\sqcup_{B_i}^{n+m+l} X_i^{2d+1}$, $(X_i = I, I_1, I_{0,1}$ n, m, l copies, $m, l \geq 1)$, n-fold (odd dim): $\sqcup_{p_i}^n \mathbb{R}$, $\sqcup_{p_i}^n \mathbb{R}^{2m+1}$ (pointed), $\sqcup_{B_i}^n I_{0,1}^{2m+1}$ (balled)
$\chi = l - (m + 2l - 1)$ $= 1 - m - l < 0$	$\sqcup_{B_i}^{n+m+l} X_i^{2d}$, $(X_i = I, I_1, I_{0,1}$ n, m, l copies, $m, l \geq 1)$
$\chi = n - (n + 2m - 1)$ $= 1 - 2m < 0$	$\sqcup_{p_i}^{n+m} X_i$, $\sqcup_{B_i}^{n+m} X_i$ (dim mixed), with $X_i = \mathbb{R}^{2n_i}$ $(1 \leq i \leq n)$, $X_i = \mathbb{R}^{2m_i+1}$ $(n+1 \leq i \leq n+m)$

It follows from the Table 5 that

Corollary 3.23. *The numbers or signs* (*being positive, zero, or negative*) *of the Euler characteristic for C^*-algebras* (*together with compactness, dimension, jointedness of spaces, and arrangement* (*or permutation*) *in* (*generic*) *jointed sums*) *classify contractible spaces as in the table* (*up to homeomorphisms*) *and to be compact, non-compact and extended, or non-compact and non-extended.*

Remark. Our classification tables obtained as collections in this paper would be useful for further classification of contractible spaces in more general, with more examples as representatives to be added.

Once more,

Corollary 3.24. *Our classfication tables say that contractible spaces restricted to examples viewed as representatives of equivalence classes by homeomorphisms are classifiable by their corresponding C^*-algebras and K-theory data, plus, compactness, dimension, pointed or balled jointedness for spaces, and arrangement* (*or permutation*) *in* (*generic*) *jointed sums, as complete invariants.*

Remark. The covering dimension for spaces as an invariant can be replaced by the real rank for C^*-algebras ([3]). Being compact for spaces corresponds to being unital for their corresponding C^*-algebras. Also, being jointed for spaces corresponds to being jointed for their corresponding C^*-algebras, and arrangement (or permutation) in jointed sums for spaces corresponds to that in jointed sums for their corresponding C^*-algebras.

Corollary 3.25. *Both the ranks of K-theory groups for C^*-algebras and the Euler characteristic for C^*-algebras can not classify jointedness for spaces, and as well, can not do pointed or balled jointed sums of contractible spaces, up to arrangement* (*or permutation*), *in general, except that all the components in jointed sums are the same.*

However, if restricted to this exceptional case, and further restricted with dimension fixed in spaces and balls in (*generic*) *jointed sums, the ranks and the Euler characteristic together with compactness and jointedness for spaces can be complete invariants to classify the contractible spaces as in the lists above, up to homeomorphisms.*

Consequently, we obtain

Corollary 3.26. *Let M, N be product manifolds of finitely many 1-dimensional contractible manifolds. Then the d and d'-dimensional* (*with $d, d' \geq 0$*), *jointed sums $\sqcup_{B_i}^n M$ and $\sqcup_{B_i'}^m N$ are homeomorphic,* (*which is equivalent to that*

$$C(\sqcup_{B_i}^n M) \cong C(\sqcup_{B_i'}^m N) \quad \textit{or} \quad C_0(\sqcup_{B_i}^n M) \cong C_0(\sqcup_{B_i'}^m N),$$

where both M and N are compact or not), *if and only if*

$$K_j(C(\sqcup_{B_i}^n M)) \cong K_j(C(\sqcup_{B_i'}^m N)) \quad \textit{or} \quad K_j(C_0(\sqcup_{B_i}^n M)) \cong K_j(C_0(\sqcup_{B_i'}^m N))$$

for $j = 0, 1$, and $n = m$ (*jointedness*), *and $\dim M = \dim N$ and $\dim B_i = d = d' = \dim B_i'$ for every i.*

Furthermore, the K-theory group isomorphisms can be replaced by

$$\chi(C(\sqcup_{B_i}^n M)) = \chi(C(\sqcup_{B_i'}^m N)) \quad \textit{or} \quad \chi(C_0(\sqcup_{B_i}^n M)) = \chi(C_0(\sqcup_{B_i'}^m N)),$$

with the same other conditions.

Proof. As a note, suppose that there is a homeomorphism $\varphi : X \to Y$ of locally compact Hausdorff spaces. Then there is a $*$-isomorphism $\psi : C_0(Y) \to C_0(X)$ defined by $\psi(f) = f \circ \varphi$ for $f \in C_0(Y)$. The converse also holds by that X is the spectrum of $C_0(X)$ by the Gelfand transform. $\square$

4 Noncommutative jointed sums We may say that a **jointed sum** of two C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ with a common quotient $\mathfrak{D}$ is defined to be the pull back C^*-algebra $\mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B}$ as

$$\begin{array}{ccc} \mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B} = \{(a,b) \in \mathfrak{A} \oplus \mathfrak{B} \,|\, \varphi(a) = \psi(b)\} & \xrightarrow{\rho} & \mathfrak{B} \\ \pi \downarrow & & \downarrow \psi \\ \mathfrak{A} & \xrightarrow{\varphi} & \mathfrak{D} \end{array}$$

where $\varphi : \mathfrak{A} \to \mathfrak{D}$ and $\psi : \mathfrak{B} \to \mathfrak{D}$ are quotient maps and $\pi : \mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B} \to \mathfrak{A}$ and $\rho : \mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B} \to \mathfrak{B}$ are natural projections.

The Mayer-Vietoris sequence for K-theory of C^*-algebras (see [1]) is the following six-term diagram:

$$\begin{array}{ccccc} K_0(\mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B}) & \xrightarrow{(\pi_*, \rho_*)} & K_0(\mathfrak{A}) \oplus K_0(\mathfrak{B}) & \xrightarrow{\psi_* - \varphi_*} & K_0(\mathfrak{D}) \\ \uparrow & & & & \downarrow \\ K_1(\mathfrak{D}) & \xleftarrow{\psi_* - \varphi_*} & K_1(\mathfrak{A}) \oplus K_1(\mathfrak{B}) & \xleftarrow{(\pi_*, \rho_*)} & K_1(\mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B}) \end{array}$$

In particular, it follows that

Proposition 4.1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be contractible C^*-algebras with a common quotient $\mathfrak{D}$ that is contractible to $\mathbb{C}$. Then*

$$K_0(\mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B}) \cong 0 \quad \textit{and} \quad K_1(\mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B}) \cong \mathbb{Z}.$$

Proof. Indeed, the Mayer-Vietoris sequence becomes in this case:

$$\begin{array}{ccccc} K_0(\mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B}) & \xrightarrow{(\pi_*, \rho_*)} & 0 \oplus 0 & \xrightarrow{\psi_* - \varphi_*} & \mathbb{Z} \\ \uparrow & & & & \downarrow \\ 0 & \xleftarrow{\psi_* - \varphi_*} & 0 \oplus 0 & \xleftarrow{(\pi_*, \rho_*)} & K_1(\mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B}). \end{array}$$

□

Now suppose that the jointed sum C^*-algebra $\mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B}$ and a C^*-algebra $\mathfrak{C}$ have a common quotient E. Then one can define a **successive** jointed sum of three C^*-algebras

$$(\mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B}) \oplus_E \mathfrak{C}$$

as the successive pull back C^*-algebra. Note that the associativity for successive jointed sums may not hold or not be defined in general. To have the associativity as

$$(\mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B}) \oplus_E \mathfrak{C} \cong \mathfrak{A} \oplus_{\mathfrak{D}} (\mathfrak{B} \oplus_E \mathfrak{C})$$

we further need to assume that E is a common quotient of $\mathfrak{B}$, $\mathfrak{C}$, and $\mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B}$.

Proposition 4.2. *Let $(\mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B}) \oplus_E \mathfrak{C}$ be a successive jointed sum C^*-algebra of contractible C^*-algebras $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ with quotients $\mathfrak{D}$ and E that are contractible to $\mathbb{C}$. Then*

$$K_0((\mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B}) \oplus_E \mathfrak{C}) \cong 0 \quad \textit{and} \quad K_1((\mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B}) \oplus_E \mathfrak{C}) \cong \mathbb{Z}^2.$$

Proof. Indeed, the Mayer-Vietoris sequence becomes in this case:

$$\begin{array}{ccccc} K_0((\mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B}) \oplus_E \mathfrak{C}) & \xrightarrow{(\pi_*, \rho_*)} & 0 \oplus 0 & \xrightarrow{\psi_* - \varphi_*} & \mathbb{Z} \\ \uparrow & & & & \downarrow \\ 0 & \xleftarrow{\psi_* - \varphi_*} & \mathbb{Z} \oplus 0 & \xleftarrow{(\pi_*, \rho_*)} & K_1((\mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B}) \oplus_E \mathfrak{C}), \end{array}$$

where $\pi : (\mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B}) \oplus_E \mathfrak{C} \to \mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B}$ and $\rho : (\mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B}) \oplus_E \mathfrak{C} \to \mathfrak{C}$ by the same symbols as for $\mathfrak{A} \oplus_{\mathfrak{D}} \mathfrak{B}$, for convenience. □

Inductively, one can define a successive jointed sum of C^*-algebras $\mathfrak{A}_1, \cdots, \mathfrak{A}_n$ with quotients $\mathfrak{D}_1, \cdots, \mathfrak{D}_{n-1}$ as

$$\oplus_{\mathfrak{D}_i}^n \mathfrak{A}_i \equiv (\cdots((\mathfrak{A}_1 \oplus_{\mathfrak{D}_1} \mathfrak{A}_2) \oplus_{\mathfrak{D}_2} \mathfrak{A}_3) \cdots) \oplus_{\mathfrak{D}_{n-1}} \mathfrak{A}_n.$$

Note that the associativity for the successive jointed sums may not hold or not be defined in general. To have the associativity as in the 3-fold case, we further need to assume that the quotients are more common to have this as in the 3-fold case.

Proposition 4.3. *Let $\oplus_{\mathfrak{D}_i}^n \mathfrak{A}_i$ be a successive jointed sum C^*-algebra of contractible C^*-algebras $\mathfrak{A}_1, \cdots, \mathfrak{A}_n$ with quotients $\mathfrak{D}_1, \cdots, \mathfrak{D}_{n-1}$ that are contractible to $\mathbb{C}$. Then*

$$K_0(\oplus_{\mathfrak{D}_i}^n \mathfrak{A}_i) \cong 0 \quad \text{and} \quad K_1(\oplus_{\mathfrak{D}_i}^n \mathfrak{A}_i) \cong \mathbb{Z}^{n-1}.$$

Proof. We use induction by the same way as in the proof above. □

Corollary 4.4. *The jointed sum of two contractible C^*-algebras with a common quotient that is contractible to $\mathbb{C}$ is not contractible.*

As well, the successive jointed sum of n contractible C^*-algebras with successive common quotients that are contractible to $\mathbb{C}$ is not contractible.

Remark. Since a contractible C^*-algebra $\mathfrak{A}$ has K-theory groups zero, the Künneth formula in K-theory for C^*-algebras implies that any tensor product of $\mathfrak{A}$ with any other C^*-algebra $\mathfrak{B}$ has K-theory groups zero if $\mathfrak{A}$ or $\mathfrak{B}$ is in the bootstrap category.

What's more. As an interest, we obtain

Proposition 4.5. *Let $\mathfrak{A}$ be a contractible C^*-algebra. Then any C^*-tensor product $\mathfrak{A} \otimes \mathfrak{B}$ with any C^*-algebra $\mathfrak{B}$ is contractible.*

It follows that $K_j(\mathfrak{A} \otimes \mathfrak{B}) \cong 0$ for $j = 0, 1$.

Proof. There is a continuous homotopy (φ_t) between the identity map $\mathrm{id}_{\mathfrak{A}} : \mathfrak{A} \to \mathfrak{A}$ and the zero map $0 : \mathfrak{A} \to \mathfrak{A}$, with $\varphi_1 = \mathrm{id}_{\mathfrak{A}}$ and $\varphi_0 = 0$. For any simple tensor $a \otimes b \in \mathfrak{A} \otimes \mathfrak{B}$, we define maps $\psi_t : \mathfrak{A} \otimes \mathfrak{B} \to \mathfrak{A} \otimes \mathfrak{B}$ by $\psi_t(a \otimes b) = \varphi_t(a) \otimes b$, which extends to $*$-homomorphism from $\mathfrak{A} \otimes \mathfrak{B}$ to $\mathfrak{A} \otimes \mathfrak{B}$. Then (ψ_t) gives a continuous homotopy between the identity map $\mathrm{id}_{\mathfrak{A} \otimes \mathfrak{B}} : \mathfrak{A} \otimes \mathfrak{B}$ and the zero map $0 : \mathfrak{A} \otimes \mathfrak{B} \to \mathfrak{A} \otimes \mathfrak{B}$.

Indeed, any element $x \in \mathfrak{A} \otimes \mathfrak{B}$ is approximated by finite sums of simple tensors, so that $x = \lim_{n \to \infty} \sum_{k=1}^n a_k \otimes b_k \equiv \lim_{n \to \infty} s_n$. Then define

$$\psi_t(x) = \lim_{n \to \infty} \psi_t(s_n) = \lim_{n \to \infty} \psi_t(\sum_{k=1}^n a_k \otimes b_k) = \lim_{n \to \infty} \sum_{k=1}^n \varphi_t(a_k) \otimes b_k,$$

which is well defined. Then

$$\begin{aligned} &\|\psi_t(x) - \psi_s(x)\| \\ &\leq \|\psi_t(x) - \psi_t(s_n)\| + \|\psi_t(s_n) - \psi_s(s_n)\| + \|\psi_s(s_n) - \psi_s(x)\|, \end{aligned}$$

which is arbitrary small when n is large enough and $|t - s|$ is small enough. □

Remark. As for examples of noncommutative jointed sums, see the commutative cases in the previous sections. One (principal case) of noncommutaive cases can be also obtained as taking tensor products of C^*-algebras $\mathfrak{A}_i$ with commutative C^*-algebras $C_0(X_i)$ and taking their jointed sums, with quotients (of $\mathfrak{A}_i$ or $C_0(X_i)$) involved to be assumed. If the K-theory groups of $\mathfrak{A}_i$ are computable, then so are the K-theory groups of the jointed sums. As the other cases, tensor products may be replaced by other operations such as crossed products of C^*-algebras with suitable actions and free products of C^*-algebras.

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