MULTIPLIERS WITH CLOSED RANGE ON FRÉCHET ALGEBRAS

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Abstract. In this paper, we determine several equivalent conditions pertaining to closed range multipliers defined on a semisimple Fréchet locally m-convex algebra. Moreover, we give a complete description of the point spectrum and the residual spectrum of multipliers.

1. Introduction

The investigation of closed range multipliers, in the context of commutative semisimple Banach algebras was initiated by Glicksberg [8] in 1971, whereby he raised the following question: If T is a multiplier on a commutative semisimple Banach algebra A, whether a factorization $T = PB$, where P is an idempotent and B an invertible multiplier, is necessary and sufficient to ensure the closedness of TA? This problem was partially resolved by Host and Parreau [12] for a particular situation of the group algebra $L^1(G)$, where G is a locally compact abelian group. Various equivalent conditions have been determined in [17] for a multiplier T defined on a semisimple Banach algebra to have closed range.

It is quite natural to ask whether the above characterization of closed range multipliers holds for a semisimple Fréchet locally m-convex algebra A. In this paper, we consider this problem and establish several equivalent conditions pertaining to closed range multipliers on A. Precisely, we prove that if A has a bounded approximate identity, then TA is a closed ideal with a bounded approximate identity if and only if T admits a factorization $T = PB$ with P an idempotent and B an invertible multiplier. Moreover, if A is also a Fréchet locally C*-algebra then T has closed range if and only if $T^2A = TA$. Also, in this case, T is injective if and only if it is surjective.

Finally, we discuss the spectral properties of multipliers defined on a simisimple commutative Fréchet locally m-convex algebra A. The investigation of spectral properties of a multiplier T defined on $L^1(G)$ was initiated by Zafran [22]. Successively this problem was studied by several other authors in the framework of commutative semisimple Banach algebras. We study this problem in the more abstract situation of (non-normed) topological algebras. We show that if the maximal ideal space $\Delta(A)$ is discrete, then the point spectrum is completely characterized by $\sigma_p(T) = \mu^T(\Delta(A))$. Under the assumption that socle of A is dense in A, we establish that the residual spectrum of T is empty.

2. Closed range multipliers

Before investigating certain features of a multiplier with closed range, we need to establish our preliminaries. A Hausdorff topological algebra A whose topology is generated by a family $\{p_\alpha : \alpha \in \Lambda\}$ of seminorms is called a *locally convex* algebra. Moreover, if each seminorm p_α is also submultiplicative, i.e.,

$$p_\alpha(xy) \leq p_\alpha(x)\, p_\alpha(y)\, , \text{for all } x, y \in A,$$

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then A is called a *locally m-convex* algebra. Usually, a complete metrizable locally convex (resp. locally m-convex) algebra is called a *Fréchet locally convex* (resp. *Fréchet locally m-convex*) algebra.

Given a semisimple Fréchet locally convex algebra A, then following [13], a mapping $T : A \to A$ is said to be a *multiplier* if $x\,(Ty) = (Tx)y$ holds for all $x, y \in A$. We denote the set of all multipliers on A by $M(A)$. Since A is semisimple, any $T \in M(A)$ turns out to be linear and the identity $x\,(Ty) = T(xy)$ holds for any $x, y \in A$. Using the closed graph theorem, the definition of a multiplier, and the semisimplicity of A, one can show that all multipliers are necessarily continuous and hence bounded (see for instance, [13], Corollary 2.3). Moreover, $M(A)$ is a closed subalgebra of $B(A)$ with respect to the strong operator topology, where $B(A)$ denotes the algebra of all continuous (or bounded) linear operators on A. Also, $M(A)$ is commutative (see for instance, [13], Theorem 2.4) and has an identity element. An application of the identity $x\,(Ty) = T(xy)$ for all $x,\ y \in A$, yields that both TA and $\ker T$ are two sided ideals of A, where TA and $\ker T$ denote the range and kernel of T, respectively.

In this work, we want to study closed range multipliers on A. In [12], Host and Parreau have established that if $A = L^1(G)$, where G is a locally compact abelian group, and if T is a multiplier on $L^1(G)$, then TA is closed if and only if $T = PB$, where P is an idempotent and T an invertible multiplier. Thus they partially resolved the interesting problem due to Glicksberg [8] whether the factorization $T = PB$ is necessary and sufficient to ensure the closedness of TA for any multiplier T on a semisimple commutative Banach algebra A. Various equivalent conditions have been determined in [1] , [17] and [21] under which a multiplier T has closed range. Our aim is to consider this problem for a more general situation in (non-normed) topological algebras.

We recall that an operator $T \in B(A)$ has a *generalized inverse* (abbreviated as *g-inverse*), if there is an operator $S \in B(A)$ such that $T = TST$ and $S = STS$. The operator T is also called *relatively regular* [10]. We want to make a few observations about these operators.

Remark 1. *(i) There is no loss of generality in requiring only that $T = TST$. In fact, if $T = TST$, then $S' = STS$ will satisfy $T = TS'T$, as well as $S' = S'TS'$.*

(ii) If $T = TST$ and $S = STS$, then TS and ST are idempotents and hence projections for which $TS\,(A) = T\,(A)$ and $\ker T = \ker ST$. Indeed, $(TS)^2 = TSTS = TS$ and $(ST)^2 = STST = ST$. Moreover, from $T(A) = TST(A) \subseteq TS(A) \subseteq T(A)$ and $\ker T \subseteq \ker(ST) \subseteq \ker(TST) = \ker T$, we obtain $TS(A) = T(A)$ and $\ker(ST) = (I - ST)A = \ker T$, where I denotes the identity element in $B(A)$.

(iii) Generally speaking, a generalized inverse of T is rarely uniquely determined. For instance, if $T = TST$, then S can be anything on $\ker(T)$. But there is at most one generalized inverse which commutes with the given $T \in B(A)$. In fact, if S and S' are g-inverses of T, both commuting with T, then $TS' = TSTS' = ST$, and hence $S' = S'TS' = S'TS = STS = S$.

The following result has been proved in [21].

Theorem 2.1. *Let A be a semisimple Fréchet locally m-convex algebra and $T \in M(A)$. Then the following statements are equivalent.*

(1) *T has a g-inverse $S \in B(A)$ such that $ST = TS$.*
(2) *T has a g-inverse $S \in B(A)$ such that $TS \in M(A)$.*
(3) *T has a g-inverse $S \in B(A)$ such that TS commutes with T.*
(4) *T has a g-inverse $S \in M(A)$.*
(5) *$TA \oplus \ker T = A$.*
(6) *$T^2A = TA$ and $\ker T^2 = \ker T$.*
(7) *$T = PB = BP$, where $B \in M(A)$ is invertible and $P \in M(A)$ is idempotent.*

(8) *T is decomposably regular in $M(A)$, i.e., $T = TCT$, where C is an invertible multiplier.*

We see from the preceding theorem that if T$\in M(A)$ has a commuting g-inverse then this must be a multiplier. One fact about multipliers on semisimple algebras that we shall use below is that they satisfy the relation $\ker T^2 = \ker T$. In fact, if $T^2x = 0$ then $0{=}T^2x^2 = T(xTx) = (Tx)^2$, hence $Tx = 0$. An immediate consequence of this is that $TA \cap \ker T = \{0\}$.

Corollary 2.2. *Let A be a semisimple Fréchet locally m-convex algebra and $T \in M(A)$. If $T^2A = TA$, then TA is closed.*

Proof. For the proof see [21]. □

We remark that the converse of Corollary 2.2 may not be true even in the case of general Banach algebras. For instance, consider the disc algebra $A = A(D)$ of all complex valued continuous functions on the closed unit disc D which are analytic in the interior of D. Let $g \in A(D)$ be such that $g(z) = z$ for each $z \in D$, and let T_g be the corresponding multiplication operator. Clearly, $T_g \in M(A)$ and $T_gA = \{f \in A : f(0) = 0\}$, $T_g^2A = \{f \in A : f(0) = f'(0) = 0\}$. Obviously T_gA is closed, but $T_gA \neq T_g^2A$.

Let A be a Fréchet locally m-convex algebra whose topology is generated by a family $\{p_n : n \in \mathbb{N}\}$ of submultiplicative seminorms. A net $\{e_\alpha : \alpha \in I\}$ in A is called a *bounded approximate identity* (abbreviated as *bai*) if $p_n(e_\alpha) \leq 1$ for all $n \in \mathbb{N}$ and for all $\alpha \in I$, $\lim_\alpha e_\alpha x = \lim_\alpha xe_\alpha = x$ for all $x \in A$. Following Inoue [15], A is called a *Fréchet locally C^*-algebra* if it has an involution $*$ satisfying $p_n(x^*x) = (p_n(x))^2$ for all $n \in \mathbb{N}$ and $x \in A$. It is well-known that every Fréchet locally C*-algebra has a bai (see [15, Theorem 2.6] and [6, Theorem 4.5]).

Theorem 2.3. *Let A be a semisimple Fréchet locally m-convex algebra with a bounded approximate identity and $T \in M(A)$. Then TA is a closed ideal with a bounded approximate identity if and only if T admits a factorization $T = PB$, where P is an idempotent multiplier and B an invertible multiplier.*

Proof. Let $\{e_\alpha\}$ be a bounded approximate identity of A. Assume that $T \in M(A)$ has a factorization $T = PB$, where $P \in M(A)$ is idempotent and $B \in M(A)$ is invertible. Since $TA = PA$, it follows immediately that TA is a closed ideal. Also, the bounded net $\{Pe_\alpha\}$ is subset of TA. Hence $xPe_\alpha = P(xe_\alpha) \to Px = x$, for all $x \in TA$.

Conversely assume that TA is a closed ideal with a bounded approximate identity. Then using the generalized version of the Cohen's factorization theorem ([5], p. 610), for each $x \in TA$, there exist y, z in TA such that $x = yz$, i.e., $TA = (TA)^2$ which implies $T^2A \subseteq TA = (TA)^2$. On the other hand, for any $x, y \in A$, we have $(Tx)(Ty) = T(xTy) = T^2(xy) \in T^2A$, and so $(TA)^2 \subseteq T^2A$. Hence $TA = T^2A$. The desired factorization $T = PB$ follows from the preceding theorem. □

Corollary 2.4. *Let A be a semisimple Fréchet locally m-convex algebra with a bounded approximate identity and $T \in M(A)$. Then the conditions (1) to (8) of Theorem 2.1 are equivalent to the following condition: (9) TA is a closed ideal with a bounded approximate identity.*

Note that every Fréchet locally C*-algebra is semisimple (cf. [6, Corollary 5.6] and [7, Lemma 8.14(ii)]). Now we remark that Theorem 3.6 [21] follows immediately as a simple corollary of the preceding theorem. Precisely, we have:

Corollary 2.5. *Let A be a Fréchet locally C^*-algebra and $T \in M(A)$. Then TA is closed if and only if $T^2A = TA$.*

Corollary 2.6. *Let A be a semisimple Fréchet locally m-convex algebra and $T \in M(A)$. If $T^2A = TA$, then T is injective if and only if it is surjective.*

Proof. Let T be surjective. Since $TA \cap \ker T$=$\{0\}$, it follows that $\ker T = \{0\}$, that is, T is injective. Conversely, assume that $\ker T = \{0\}$. Since, by assumption, $T^2A = TA$, it follows from Theorem 2.1 that $TA \oplus \ker T = A$. Hence $TA = A$, that is, T is surjective. □

Now we see, by virtue of Corollary 2.4, that if T is a multiplier on a semisimple Fréchet locally m-convex algebra with a bounded approximate identity such that TA is a closed ideal with a bounded approximate identity, then T is injective if and only if it is surjective. In particular, we obtain a result of [20] which states that a closed range multiplier on a Fréchet locally C*-algebra is injective if and only if it is surjective.

3. Spectral Properties of Multipliers

In this section we investigate certain spectral properties of multipliers defined on a semisimple commutative Fréchet locally m-convex algebra A. Denote the set of all non-zero continuous multiplicative linear functionals on A by $\Delta(A)$. In what follows, we assume that $\Delta(A)$ is non-empty and point-separating, without mentioning it explicitly. For any $x \in A$, define the Gelfand transform $\widehat{x}$ of x by $\widehat{x}(f) = f(x)$ for each $f \in \Delta(A)$. The space $\Delta(A)$ is equipped with the Gelfand topology, i.e., the induced topology inherited from the weak* topology of A*. We shall use the following result of [13] frequently.

Theorem 3.1. *There is a continuous function $\mu^T : \Delta(A) \to \mathbb{C}$ corresponding to each $T \in M(A)$ defined by $\mu^T(f) = f \circ T(x)$, where x is chosen such that f(x)=1, satisfying the relation $(\widehat{Ty})(f) = \widehat{y}(f)\mu^T(f)$, for all $y \in A$ and all $f \in \Delta(A)$.*

Now we need to recall the definition of the socle of a semisimple commutative Fréchet locally m-convex algebra A, an ideal that plays an important role in our subsequent discussion. A *minimal idempotent* of A is a non-zero idempotent e such that eAe is a division algebra. Note that if e is a minimal idempotent element, then $eAe = \mathbb{C}e$ ([3], p. 292). The set of all minimal idempotents of A is denoted by E_A. It is well-known that an ideal J of A is a minimal ideal if and only if $J = eA$ for some $e \in E_A$ (see for instance, [4]). The *socle* of A, denoted by soc(A), is defined as the sum of all minimal ideals of A, or (0) if there are none. In what follows, we assume that the ideal soc(A) does exist, without mentioning it explicitly. The socle of A can be characterized in a simple way as:

$$\mathrm{soc}(A) = \left\{ \sum_{k=1}^{n} e_k A : e_k \in E_A,\ n \in \mathbb{N} \right\} = \mathrm{span}(E_A).$$

An important class of topological algebras consists of those which have a dense socle. For instance, consider the algebra $A = H(D)$ of all holomorphic functions defined on the open disc $D = \{z \in \mathbb{C} : |z| < 1\}$ with point-wise addition and scalar multiplication. With the Cauchy-Hadamard product and the compact-open topology, it is a semisimple commutative Fréchet locally m-convex algebra possessing an orthogonal basis $\{e_n : n \geq 0\}$, where $e_n(z) = z^n$ for $z \in D$. The element $e(z) = \sum_{n=0}^{\infty} z^n$ is the identity element of $H(D)$. Note that e_nA is a minimal ideal of A, for all $n \in \mathbb{N}$. Moreover, A is the direct sum of these minimal ideals, i.e., soc(A) is dense in A (see [14], Chapter III, p. 97).

Similarly, the algebra $A = s$ of all complex sequences with coordinate-wise operations is a semisimple commutative Fréchet locally m-convex algebra with identity and possessing an orthogonal basis $\{\mathrm{e}_n : n \geq 1\}$(see [14], Example 3.4, Chapter II). In this case, soc(A) is also dense in A. In fact, the socle is dense in every Hausdorff topological algebra possessing an orthogonal basis. Moreover, $\Delta(A)$ is homeomorphic with the discrete space of natural numbers $\mathbb{N}$ (see [14], Theorem 3.12, Chapter III). We now prove the following:

Theorem 3.2. *Let A be a semisimple commutative Fréchet locally m-convex algebra. If $\overline{\mathrm{soc}(A)} = A$, then $\Delta(A)$ is discrete.*

Proof. First we observe that $\widehat{A} = \{\widehat{a} : a \in A\}$ separates the points of $\Delta(A)$. In fact, if $f, g \in \Delta(A)$ such that $f \neq g$, then there exists $x_0 \in A$ with $f(x_0) \neq g(x_0)$. Therefore, it implies that $\widehat{x_0}(f) \neq \widehat{x_0}(g)$. Hence there is no $h \in \Delta(A)$ at which $\widehat{x}$ vanishes for all $x \in \mathrm{soc}(A)$. Thus if $\mathrm{f}_0 \in \Delta(A)$, then there exists an element $x \in \mathrm{soc}(A)$ for which $\widehat{x}(f_0) = 1$. Therefore, $\{h \in \Delta(A) : |\widehat{x}(h) - \widehat{x}(f_0)| < \frac{1}{2}\} = \{f_0\}$ is a weak*-neighborhood of f. This implies that $\Delta(A)$ is discrete. $\square$

We denote by $\mathcal{C}_c(\Delta(A))$ the algebra of all $\mathbb{C}$-valued continuous functions on $\Delta(A)$ endowed with the topology of compact convergence. Now by combining Theorem 3.2 with [9, Theorem 4.2], we get:

Corollary 3.3. *Let A be a unital semisimple commutative Fréchet locally m-convex algebra. If $\overline{\mathrm{soc}(A)} = A$, then $A = \mathcal{C}_c(\Delta(A))$, with respect to a topological algebraic isomorphism.*

A locally m-convex (resp. Fréchet locally m-convex) algebra A whose topology is generated by a family $\{p_\alpha : \alpha \in \Lambda\}$ of submultiplicative seminorms is called a *uniform locally m-convex* (resp. *uniform Fréchet locally m-convex*) algebra if $p_\alpha\left(x^2\right) = \left(p_\alpha(x)\right)^2$, for all $x \in A$, $\alpha \in \Lambda$. Every uniform locally m-convex algebra is commutative and semisimple (see [18, p. 275, Lemma 5.1]). Moreover, from [9, Corollary 5.4(ii)] and Theorem 3.2, we get:

Corollary 3.4. *A unital uniform Fréchet locally m-convex algebra with dense socle is a Banach algebra.*

We showed in Section 2 that the converse of Corollary 2.2 may not be true even in the case of Banach algebras, but it is true for Fréchet locally C*-algebras (see Corollary 2.5). A similar result proved in [2] states that if A is a semisimple commutative Fréchet locally m-convex algebra and $T \in M(A)$, then T^2A is closed if and only if $TA \oplus \ker T$ is closed. Note that a Fréchet locally m-convex algebra is simply called a Fréchet algebra in [2]. Now we remark that Theorem 5 [2] follows directly from Theorem 2.1. More precisely, we have:

Corollary 3.5. *Let A be a semisimple commutative Fréchet locally m-convex algebra with $T \in M(A)$ and $\overline{\mathrm{soc}(A)} = A$. Then T is a product of an idempotent multiplier and an invertible multiplier if and only $TA \oplus \ker T = A$.*

Observe that two conditions on A, it being a commutative algebra and having the dense socle, in Theorem 5 [2] can be relaxed by virtue of Theorem 2.1.

In the sequel, we denote by $\sigma_p(T)$ and $\sigma_r(T)$ the point spectrum and the residual spectrum of T, respectively. Recall that A is said to be *regular* if for each closed subset E of $\Delta(A)$ in the Gelfand topology and $f_0 \in \Delta(A) \setminus E$, there exists an element x in A such that $\widehat{x}(f_0) = 1$ and $\widehat{x}(f) = 0$ for all $f \in E$ (see for instance, [18], p. 332). We remark that if $\Delta(A)$ is discrete, then clearly A is regular. We recall that the *ascent* $p(T)$ of an operator T is defined as the smallest non-negative integer p, whenever it exists, such that $\ker T^p = \ker T^{p+1}$.

Theorem 3.6. *Let A be a semisimple commutative Fréchet locally m-convex algebra and $T \in M(A)$. Then*

(1) $\sigma_p(T) \subseteq \mu^T(\Delta(A)) \subseteq \sigma_p(T) \cup \sigma_r(T)$.

(2) *For any $\lambda \in \sigma(T)$ we have $p(\lambda I - T) \leq 1$.*

Proof. (1) Let $\lambda \in \sigma_p(T)$. Then there exists a none-zero element x of A such that $(\lambda I - T)(x) = 0$. Therefore, $((\widehat{\lambda I - T})(x)) = (\lambda - \mu^T)\widehat{x} = \widehat{0}$. Since A is semisimple and $\widehat{x} \neq \widehat{0}$ there exists $f_0 \in \Delta(A)$ such that $\widehat{x}(f_0) \neq 0$. Thus it follows, from above that $(\lambda - \mu^T) f_0 = 0$, and so $\mu^T(f_0) = \lambda$. That is, $\lambda \in \mu^T(\Delta(A))$.

To prove the second inclusion, let T^* denote the topological dual of T. Then for each $f \in \Delta(A)$, we have $(T^* f)x = f(Tx) = (\widehat{Tx})(f) = \mu^T(f)\widehat{x}(f) = \mu^T(f) f(x)$, (using Theorem 3.1), for all x$\in A$. Therefore, $T^* f = \mu^T(f) f$, and hence $\mu^T(f)$ is an eigenvalue of T^*. Since the inclusion $\sigma_p(T^*) \subseteq \sigma_p(T) \cup \sigma_r(T)$ holds by virtue of Theorem 2.16.5 [11], the desired inclusion follows immediately.

(2) Let $x \in \ker(\lambda I - T)^2$, where $x \neq 0$. Since $(\lambda I - T)^2 \in M(A)$ and $\mu^{(\lambda I - T)^2} = (\lambda - \mu^T)^2$, it follows that $0 = ((\widehat{\lambda I - T})^2(x))(f) = (\lambda - \mu^T)^2(f) \cdot \widehat{x}(f)$, for all $f \in \Delta(A)$ (using Theorem 3.1). Hence $(\lambda - \mu^T)(f) \cdot \widehat{x}(f) = 0$ for each $f \in \Delta(A)$. Therefore, $(\widehat{\lambda I - T})(x) = \widehat{0}$. Since A is semisimple, $(\lambda I - T)(x) = 0$, and so $x \in \ker(\lambda I - T)$. Thus $\ker(\lambda I - T)^2 \subseteq \ker(\lambda I - T)$. Since the reverse inclusion is trivial, we conclude that $p(\lambda I - T) \leq 1$. □

Remark 2. *To every $T \in M(A)$ the corresponding function μ^T may not be bounded, in general. However, if $M(A)$ is a Q-algebra, then the function μ^T is bounded since $\mu^T(\Delta(A)) \subseteq \sigma_p(T) \cup \sigma_r(T) \subseteq \sigma(T)$ and every element in a Q-algebra has compact spectrum* [19]. *Note that it would be interesting to investigating whether property Q on A could pass onto $M(A)$ and vice versa?*

Now we give a complete description of the point spectrum of $T \in M(A)$.

Theorem 3.7. *Let A be a semisimple commutative Fréchet locally m-convex algebra and $T \in M(A)$. If $\Delta(A)$ is discrete, then we have $\sigma_p(T) = \mu^T(\Delta(A))$.*

Proof. By virtue of Theorem 3.6, it remains only to show that $\mu^T(\Delta(A)) \subseteq \sigma_p(T)$. Let f_0 be fixed in $\Delta(A)$. Since, by assumption $\Delta(A)$ is discrete and hence A is regular, there exists an element x in A such that $\widehat{x}(f_\circ) = 1$ and $\widehat{x}$ vanishes identically on the set $\Delta(A) \backslash \{f_0\}$. Therefore, $([\widehat{\mu^T(f_0)I - T}]x)(f) = (\mu^T(f_0) - \mu^T(f)) \cdot \widehat{x}(f) = 0$ for each $f \in \Delta(A)$ and so $[\mu^T(f_0) I - T]x = 0$, because A is semisimple. Since $x \neq 0$, we obtain $\mu^T(f_0) \in \sigma_p(T)$. Hence $\sigma_p(T) = \mu^T(\Delta(A))$. □

Under the assumption that $\overline{\mathrm{soc}(A)} = A$, we now give a complete description of the residual spectrum of $T \in M(A)$.

Theorem 3.8. *Let A be a semisimple commutative Fréchet locally m-convex algebra with dense socle. Then $\sigma_r(T) = \emptyset$.*

Proof. Assume on the contrary that $\sigma_r(T) \neq \emptyset$. Let $\lambda \in \sigma_r(T)$. Then by Theorem 3.7, $\lambda \notin \sigma_p(T)$ implies that $\lambda \neq \mu^T(f)$ for each $f \in \Delta(A)$. For any $x \in E_A$ there exists $f_0 \in \Delta(A)$ such that $\widehat{x}(f_0) = 1$ and $\widehat{x}$ vanishes identically on $\Delta(A) \backslash \{f_0\}$. Set $y = (\lambda - \mu^T(f_0))^{-1} x$, then we have $[(\widehat{\lambda I - T})y](f) = \widehat{x}(f)$ for all f in $\Delta(A)$ and so $\underline{(\lambda I - T)y} = x$, that is, $\mathrm{E}_A \subseteq \underline{(\lambda I - T)(A)} \subseteq A$. Since, by hypothesis, we have $A = \overline{\mathrm{span}\{E_A\}}$ which implies $A = \overline{(\lambda I - T)(A)}$ and so $\lambda \notin \sigma_r(T)$, a contradiction. Hence $\sigma_r(T) = \emptyset$. □

Finally we give an application of our previous results: Let A denote a Hausdorff topological algebra with an orthogonal basis $\{x_i\}$. Then A is commutative ([14], Corollary 1.4, Chapter III), proper ([14], Proposition 1.6, Chapter III), semisimple ([14], Corollary 2.5, Chapter III), and has dense socle ([14], Theorem 4.3, Chapter III). Also, each coordinate functional λ_i determined by the basis $\{x_i\}$ via $x = \sum_{i=1}^{\infty} \lambda_i(x)x_i$, is continuous, i.e., $\{x_i\}$ is a Schauder basis ([14] Theorem 1.12, Chapter III). Further, each λ_i is a multiplicative linear functional ([14], p. 79). Moreover, $\Delta(A)$ is homeomorphic with the discrete space of natural numbers $\mathbb{N}$ ([14] Theorem 3.12, Chapter III). To each $T \in M(A)$, there corresponds a sequence $\{\mu_i^T\}$ of complex numbers defined by $\mu_i^T = \mu^T(\lambda_i)$ for all $i \geq 1$, and moreover it is completely described by: $Tx = \sum_{i=1}^{\infty} \lambda_i(x)\mu_i^T x_i$, for all $x \in A$ ([14], p. 225).

Corollary 3.9. *Let A be a locally m-convex algebra with an orthogonal basis $\{x_i\}$ and $T \in M(A)$. Then we have $\sigma_p(T) = \{\mu_i^T : i \geq 1\}$ and $\sigma_r(T) = \emptyset$.*

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