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SOME COALITIONAL GAMES WITH THE SHAPLEY VALUE

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Abstract. Inthispaper,acharacteristic function depending on the state of a pair's relationship is introduced to a coalitional game with the Shapley value. By applying the characteristic function, we draw some theorems where a specific player makes his reward the maximum or the minimum. Furthermore, some properties in two concrete models are shown and various strategies of each player are discussed in two simulations. Especially, it is investigated how a player with low original reward should cooperate with other players in order to make his portion the maximum.

1 Introduction

If three people obtain reward when they cooperate, there exists a problem how to divide reward to them. Ordinary person simply thinks it should be divided reward by three evenly. But with different potential or skills, that division way is not proper from the perspective of each person's satisfaction. In real life, if you think about your wage in the company, this wages are divided by your experience, your role, and the significance of your position. We think that the system is rational in our real life.

In this research, based on the Shapley value L.S. Shapley introduced, we will discuss how to divide reward in a coalitional game depending on the state of player's relationship. It was well known that L.S. Shapley won the 2012 Nobel Memorial Prize in Economic Sciences. In the coalitional game, it is clear that when relationship among all players should be good, their sum of reward becomes the maximum. But we are not sure that a specific player can get the most reward from that relationship. For a specific player, there exists the strategy what kind of relationship the player makes to other players. Here, we give each relationship between two players, and we define the value of characteristic function depending on the state of that relationship. If the relationship is good, the value of characteristic function goes high. If the relationship is bad, that value goes down. We define a characteristic function being like this situation and discuss the strategy of each player.

2 Definition of a characteristic function

Let S be the set of relationship between two players, S= $\{s_1, s_2, s_3, \ldots, s_m\}$.
For any $i < j$, let $s_i \gg s_j$. $\gg$ means that the relationship of s_i is better than that of s_j .

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All elements of S are the following relations, $s_1 \gg s_2 \gg s_3 \gg \ldots\ldots \gg s_i \gg \ldots\ldots \gg s_m$. Of course, s_1 means the best relationship and s_m means the worst relationship. Let P be the set of players, $P = \{p_1, p_2, p_3, \ldots, p_n\}$.

When the relationship of two players p_i and p_j is s_h, we make the characteristic function giving reward based on the relationship.

[Definition 1]

That characteristic function is defined as

$v(p_i \cup p_j, s_h)$, where for k<l, $v(p_i \cup p_j, s_k) \geqq v(p_i \cup p_j, s_l)$.

When we have the characteristic function with three people, we define

$v(p_i \cup p_j \cup p_k) = \frac{1}{2}\{v(p_i \cup p_j, s') + v(p_i \cup p_k, s'') + v(p_k \cup p_j, s''')\}$,

where s' is the relationship between p_i and p_j, s'' is the relationship between p_i and p_k, and s''' is the relationship between p_j and p_k.

For four players, we define as follows,

$v(p_i \cup p_j \cup p_k \cup p_l) = \frac{1}{3}\{v(p_i \cup p_j \cup p_k) + v(p_j \cup p_k \cup p_l) + v(p_i \cup p_k \cup p_l) + v(p_i \cup p_j \cup p_l)\}$.

We can define the same things to others following this.

By applying the characteristic function depending on the relationship, we discuss the strategy that each player makes the Shapley value the maximum.

[Definition 2]

For the coalitional game (P, v), the Shapley value of player p_i is given

$$f(p_i) = \sum_{P'} \frac{j!\,(n-j-1)!}{n!}\{v(P' \cup P_i) - v(P')\}$$

where n is the number of players in the set P, P' represents any set except player p_i, $P' \cup p_i$ is the set P' adding player p_i, j represents the number of players in the set P', and $\sum_{P'}$ can give us the sum of all of the combination of P'.

[Example]

In the coalitional game of three players, let be $P = \{p_1, p_2, p_3\}$ and $S = \{s_1, s_2, s_3, \ldots, s_m\}$.

It is assumed the reward to be able to get alone as follows,

$v(p_1) = q_1, \quad v(p_2) = q_2, \quad v(p_3) = q_3$.

We can get the Shapley value of each player as follows,

$f(p_1) = \frac{2!}{3!}\{v(p_1) - v(\phi)\} + \frac{1}{3!}\{v(p_1 \cup p_2, s') - v(p_2)\} + \frac{1}{3!}\{v(p_1 \cup p_3, s'') - v(p_3)\} + \frac{2!}{3!}\{v(p_1 \cup p_2 \cup p_3) - v(p_2 \cup p_3, s''')\}$

$= \frac{1}{6}\{2q_1 - (q_2, +q_3)\} + \frac{1}{6}\{2(v(p_1 \cup p_2, s') + v(p_1 \cup p_3, s'')) - v(p_2 \cup p_3, s''')\}$

$f(p_2) = \frac{1}{6}\{2q_2 - (q_1 + q_3)\} + \frac{1}{6}\{2(v(p_1 \cup p_2, s') + v(p_2 \cup p_3, s''')) - v(p_1 \cup p_3, s'')\}$

$f(p_3) = \frac{1}{6}\{2q_3 - (q_1 + q_2)\} + \frac{1}{6}\{2(v(p_2 \cup p_3, s''') + v(p_1 \cup p_3, s'')) - v(p_1 \cup p_2, s')\}$

where s' , s'' and s''' are elements of $S=\{s_1, s_2, s_3,\ldots, s_m\}$.

Since the property of $v(p_i \cup p_j , s_k) \geqq v(p_i \cup p_j , s_l)$ for $k<l$, the strategy to make $f(p_1)$ be the maximum is $(s',s'',s''') = (s_1, s_1, s_m)$.

We get as follows similarly, the strategy to make $f(p_2)$ be the maximum is $(s',s'',s''') = (s_1, s_m, s_1)$,the strategy to make $f(p_3)$ be the maximum is $(s',s'',s''') = (s_m, s_1, s_1)$.

Even if it extends a player to n persons from three persons, it is clear that the same structure is held.

[Theorem 1]

When the relationship of $p_i \cup p_j$ for every j ($j\neq i$) is s_1 and the relationship of $p_j \cup p_k$ is s_m ($j\neq i$ and $k\neq i$), $f(p_i)$ becomes the maximum.

[Theorem 2]

When the relationship of $p_i \cup p_j$ for every j ($j\neq i$) is s_m and the relationship of $p_j \cup p_k$ is s_1 ($j\neq i$ and $k\neq i$), $f(p_i)$ becomes the minimum.

[Proof]

Two theorems can be quickly derives from Definition 1 and Definition 2.

From Theorem 1, when one chooses good relationship of a pair with oneself and does worse relationship of other pair except oneself, one can make one's reward the maximum. Conversely, to make reward of specific player the minimum is by having a bad relationship of a pair with the player, and also relationship of others except the player needs to be good.

3 Simulation Model I

Let $P=\{A,B,C\}$ be a set of 3 players, $S=\{g,n,w\}$ be the state set of relationship between two players. Let g be "good" of relationship, n be "neutral", and w be "worse".

Each of the 3 players can choose a element of the state set and the selection is carried out to their strategies.

The characteristic function v is defined as follows.

$$v(A)=a,\quad v(B)=b,\quad v(C)=c,$$

$$v(A\cup B,s)=\begin{cases} 2(a+b), & s=g \\ \frac{3}{2}(a+b), & s=n \\ a+b, & s=w \end{cases}$$

$$v(A\cup C,t)=\begin{cases} 2(a+c), & t=g \\ \frac{3}{2}(a+c), & t=n \\ a+c, & t=w \end{cases}$$

$$v(B\cup C,u)=\begin{cases} 2(b+c), & u=g \\ \frac{3}{2}(b+c), & u=n \end{cases}$$

b+c, u=w

$$v(A\cup B\cup C)=\frac{1}{2}\{v(A\cup B,s)+v(A\cup C,t)+v(B\cup C,u)\}$$

The Shapley value of each player can be calculated by using the dividend.

$$f(A)=\frac{1}{6}(2a-b-c)+\frac{1}{6}\{2v(A\cup B,s)+2v(A\cup C,t)-v(B\cup C,u)\}$$
$$f(B)=\frac{1}{6}(2b-a-c)+\frac{1}{6}\{2v(A\cup B,s)+2v(B\cup C,u)-v(A\cup C,t)\}$$
$$f(C)=\frac{1}{6}(2c-a-b)+\frac{1}{6}\{2v(A\cup C,t)+2v(B\cup C,u)-v(A\cup B,s)\}$$

[Property Ⅰ]

When relationships between three players are all "good", the sum of all players reward becomes the maximum. But the strategy which makes each individual's reward the maximum can be expressed by Theorem Ⅰ.

Each strategy of player A, player B, and player C is (s,t,u)=(g,g,w), (s,t,u)=(g,w,g), and (s,t,u)=(w,g,g), respectively.

[Property Ⅱ]

When relationships between three players are all "worse", the sum of all players reward becomes the minimum. But the strategy which makes each individual's reward the minimum can be expressed by Theorem Ⅱ.

Each strategy of player A, player B and player C is (s,t,u)=(w,w,g),(s,t,u)=(w,g,w), or (s,t,u)=(g,w,w), respectively.

[Property Ⅲ]

When (s,t,u)=(n,*,g) or (s,t,u)=(w,*,w), f(A) does not depend on b.

When (s,t,u)=(*,n,g) or (s,t,u)=(*,w,w), f(A) does not depend on c.

The symbol* denotes an arbitrary state of relationship.

Especially, when (s,t,u)=(n,n,g), $f(A)=\frac{3}{4}a$ does not depend on both b and c.

[Property Ⅳ]

When (s,t,u)=(n,g,*) or (s,t,u)=(w,w,*), f(B) does not depend on a.

When (s,t,u)=(*,g,n) or (s,t,u)=(*,w,w), f(B) does not depend on c.

Especially, when (s,t,u)=(n,g,n), $f(B)=\frac{3}{4}b$ does not depend on both a and c.

[Property Ⅴ]

When (s,t,u)=(g,*,n) or (s,t,u)=(w,*,w), f(C) does not depend on b.

When (s,t,u)=(g,n,*) or (s,t,u)=(w,w,*), f(C) does not depend on a.

Especially, when (s,t,u)=(g,n,n), $f(C)=\frac{3}{4}c$ does not depend on both a and b.

[Property Ⅵ]
When (s,t,u)=(w,w,w), f(A)=a, f(B)=b, and f(C)=c.

[Property Ⅶ]
When (s,t,u)=(w,*,g) or (s,t,u)=(*,w,g), f(A) is decreasing in b and is decreasing in c, respectively. Especially, when (s,t,u)=(w,w,g), f(A) is decreasing in both b and c.

[Property Ⅷ]
When (s,t,u)=(g,w,*) or (s,t,u)=(*,g,w), f(B) is decreasing in a and is decreasing in c, respectively. Especially, when (s,t,u)=(w,g,w), f(B) is decreasing in both a and c.

[Property Ⅸ]
When (s,t,u)=(g,w,*) or (s,t,u)=(g,*,w), f(C) is decreasing in b and is decreasing in c, respectively. Especially, when (s,t,u)=(g,w,w), f(B) is decreasing in both a and b.

[Numerical analysis of Model Ⅰ]
When v(A)=a=4, v(B)=b=3, and v(C)=c=2, the Shapley value of each player can be calculated in each strategy.
From Property Ⅰ, when you choose good relationship of a pair with yourself and does worse relationship of a pair except yourself, you can make your reward the maximum.
The number of strategies which 3 players take the state is 27. Figure 1-1 represents that it arranges in many order with reward f(A) of player A.
Figure 1-2 and Figure 1-3 are also the same.

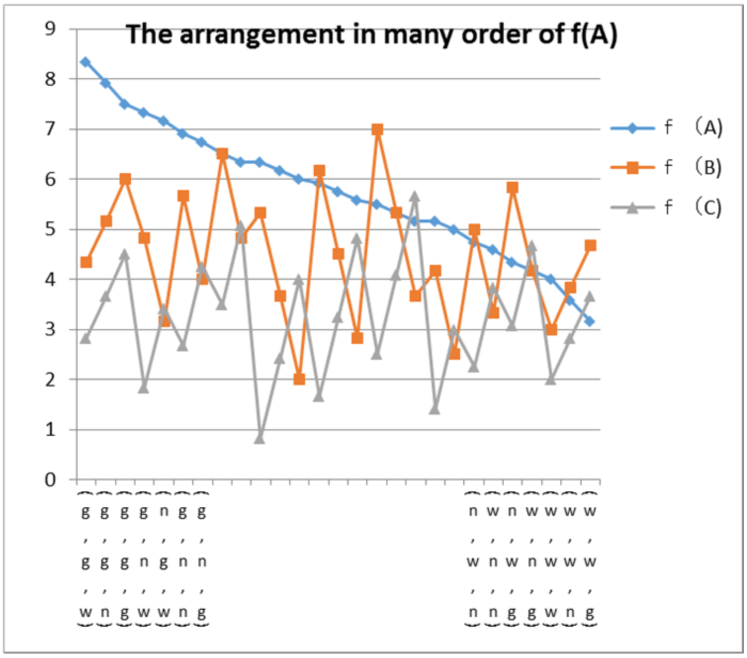

Figure 1-1

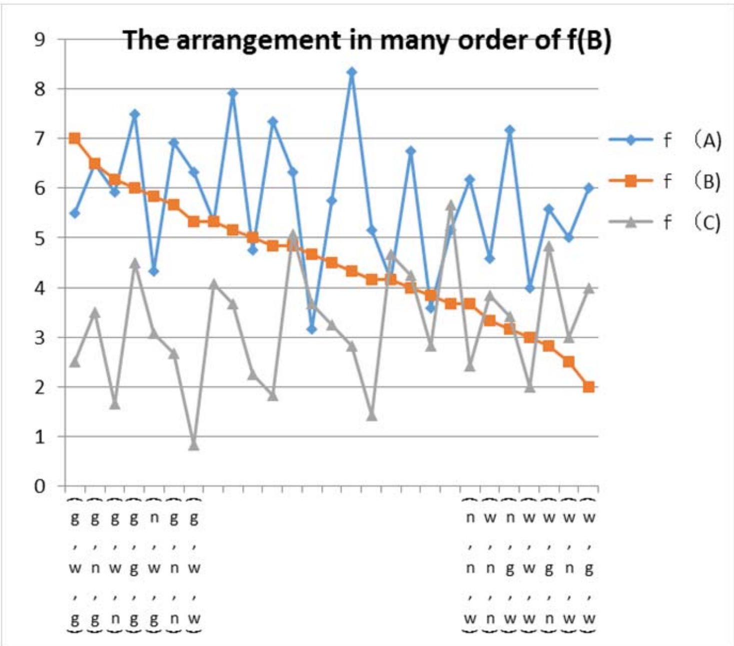

Figure 1-2

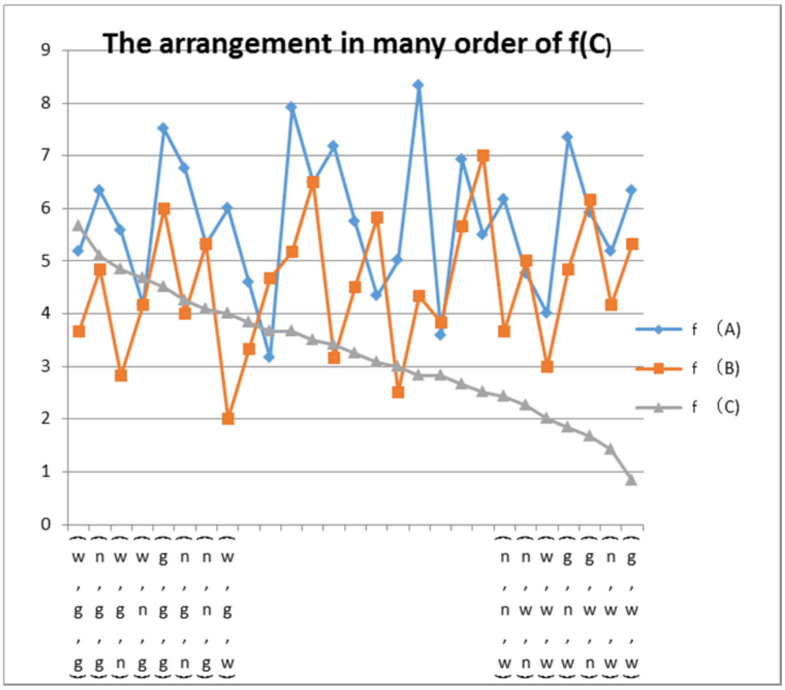

Figure 1-3

When relationships among three players are all "good", the strategy (s,t,u)=(g,g,g) is not necessarily best for each player. In the arrangement of f(A),f(B), and f(C), the ranking of the strategy (s,t,u)=(g,g,g) is 3rd,4th and 5th,respectively. When original reward v(C) of player C is the lowest value of three players, a player like player C is called "low potential player". For low potential player like player C, (s,t,u)=(g,g,g) is not so an important strategy. If the value of v(C) becomes small, the importance of the strategy (s,t,u)=(g,g,g) will fall for player C.

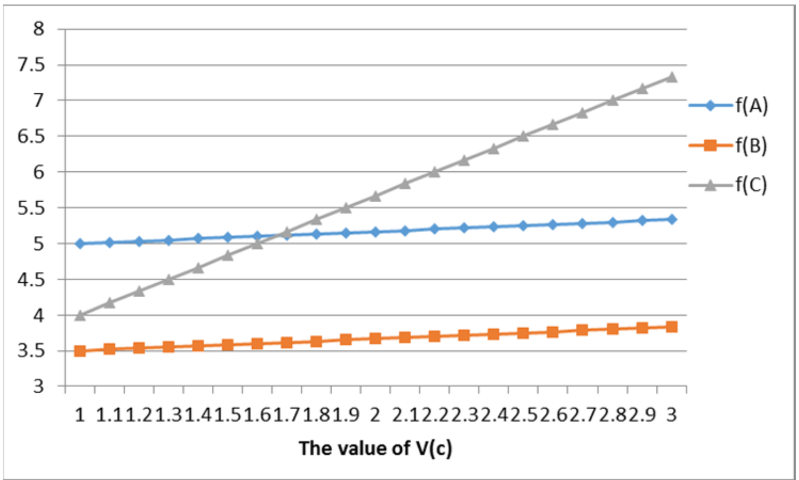

Figure 1-4

If player C takes the strategy (s,t,u)=(w,g,g) which makes one's reward the maximum, it is investigated how f(A), f(B), and f(C) will change by the variable of v(C)=c.

Figure 1-4 represents the changes of f(A), f(B), and f(C) where v(A)=4, v(B)=3 and v(C) is changed to 3 from 1. If v(C) exceeds 1.7, f(C) will become the maximum among three players.

4 Simulation Model Ⅱ

In Model Ⅰ, the characteristic function v depends on the sum of 2 player's reward and is linear function of v(A), v(B) and v(C). In Model Ⅱ, the characteristic function v changes the product of 2 player's reward. The reward of each player will become large if a good relationship is chosen.

The characteristic function v is defined as follows.

$$v(A)=a,\quad v(B)=b,\quad v(C)=c,$$

$$v(A\cup B,s)=\begin{cases} 2ab, & s=g \\ \frac{3}{2}ab, & s=n \\ a+b, & s=w \end{cases}$$

$$v(A\cup C,t)=\begin{cases} 2ac, & t=g \\ \frac{3}{2}ac, & t=n \\ a+c, & t=w \end{cases}$$

$$v(B\cup C,u)=\begin{cases} 2bc, & u=g \\ \frac{3}{2}bc, & u=n \\ b+c, & u=w \end{cases}$$

$$v(A\cup B\cup C)=\frac{1}{2}\{v(A\cup B,s)+v(A\cup C,t)+v(B\cup C,u)\}$$

Since this characteristic function v satisfies the conditions of Theorem Ⅰ and Ⅱ, Model Ⅱ is keeping the same properties as Property Ⅰ, Property Ⅱ and Property Ⅵ in Model Ⅰ.

[Numerical analysis of Model Ⅱ]

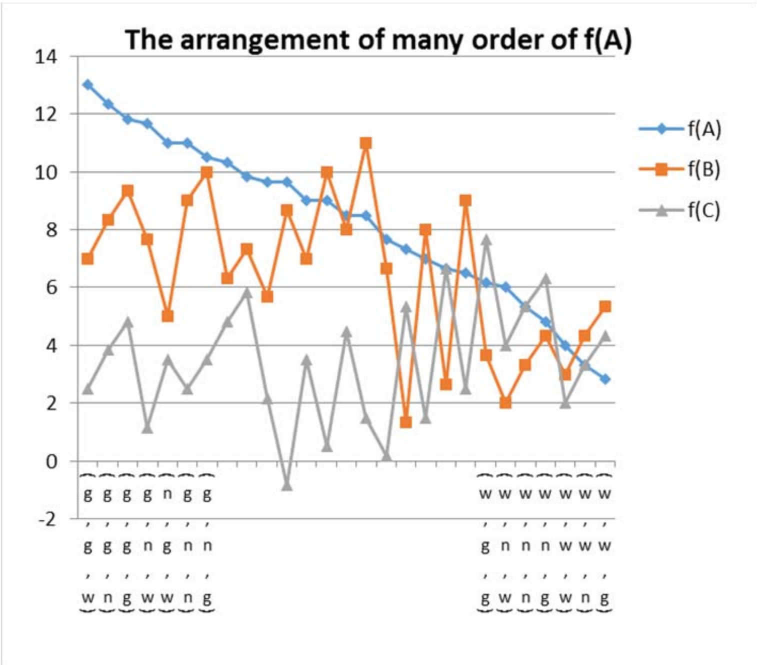

Figure 2-1

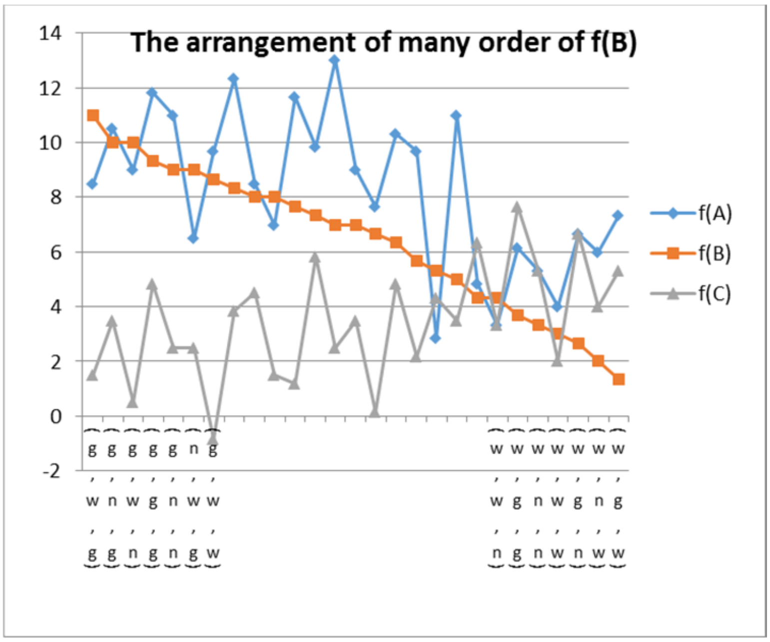

Figure 2-2

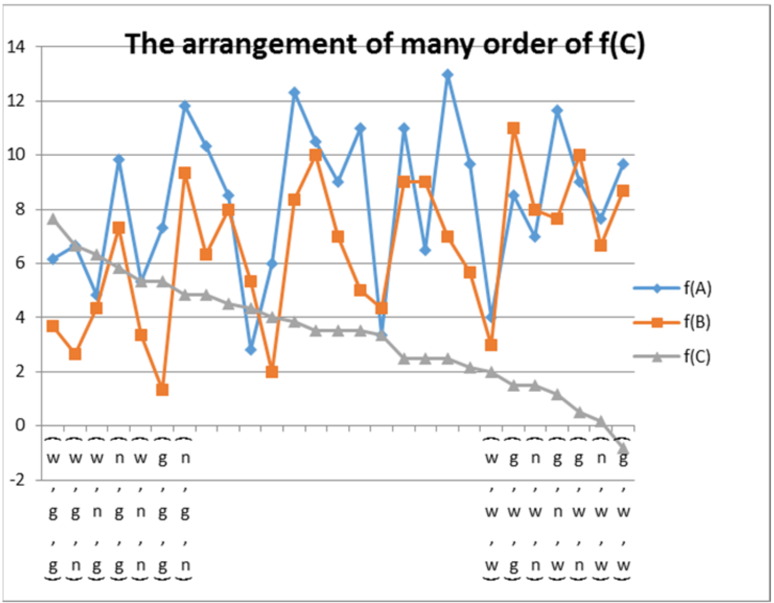

Figure 2-3

In the arrangement of many order of f(C), the ranking of the strategy (s,t,u)=(g,g,g) is 6th. Furthermore, the importance of this strategy will fall for player C.

[Property X]

When $a \geqq b \geqq c > \frac{1}{2}$ and (s,t,u)=(w,w,g), f(A) is decreasing in both b and c.
When $a \geqq b \geqq c > \frac{1}{2}$ and (s,t,u)=(w,g,w), f(B) is decreasing in both a and c.
When $a \geqq b \geqq c > \frac{1}{2}$ and (s,t,u)=(g,w,w), f(C) is decreasing in both a and b.

[Property XI]

Except (s,t,u)=(w,w,g) ,(s,t,u)=(w,g,w), and (s,t,u)=(g,w,w), let $a \geqq b \geqq c > \frac{1}{2}$,then f(A), f(B), and f(C) are increasing in all a, b and c.

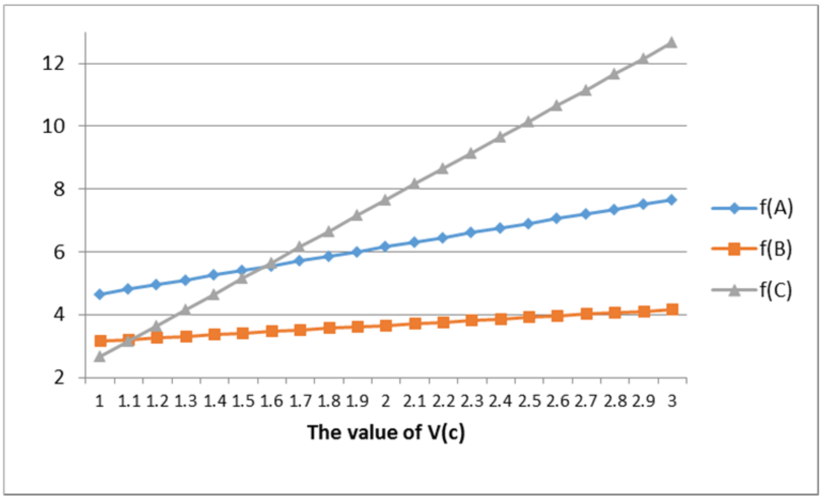

Figure 2-4

It is investigated how f(A), f(B) and f(C) will change by the variable of v(C)=c like Model

Ⅰ. In spite of the best strategy (s,t,u)=(w,g,g) for player C, it becomes the lowest at v(C)=1. When v(C) exceeds 1.6, f(C) will become the top of three players. The structure of characteristic function v is disadvantageous for the low potential player.

5 Comparison of Model Ⅰ and Model Ⅱ

In Model Ⅰ and Model Ⅱ,since 3 players can choose three kinds of relationships each one ,the number of their strategies is 27. We investigate the reward distribution of 3 players in all strategies. In Model Ⅰ and Model Ⅱ, the order of strategies make a small difference for each player. In particular, the change of the order is large for the low potential player. In all strategies, the average of each player's distribution percentage is as follows.

In Model Ⅰ,when v(A)=4, v(B)=3, and v(C)=2, we can see the share of f(A) in v(A∪B∪C). We take the average of that and let it be ASR(Average Share Rate).

ASR of f(A): ASR of f(B): ASR of f(C) =42.6% : 33.3% : 24.1% , and

the average of all of f(A): the average of all of f(B): the average of all of f(C) = 5.75 : 4.5 : 3.25 .

Compared with an original reward, the average of each player becomes large comparatively. Especially, when an original reward, v(C)=2 becomes the average 3.25, the satisfaction of player C may be high.

When relationships among three players are all "worse", f(A),f(B), and f(C) depend on each original reward only from Property Ⅵ. The structure of cooperative relation will not exist at all.

In Model Ⅱ, when v(A)=4, v(B)=3 and v(C)=2,

ASR of f(A): ASR of f(B): ASR of f(C) = 44.9%:34.9%:20.2%,

the average of all of f(A): the average of all of f(B): the average of all of f(C) =8.2777 : 6.4444 : 3.4444.

Compared with an original reward, the average of each player becomes large comparatively like Model Ⅰ. On the contrary to Model Ⅰ, the satisfactions of player A and player B will be high. The structure of the characteristic function is disadvantageous for the low potential player like player C.

6 Conclusion

It is clear that a high potential player is advantageous in the coalitional game with the Shapley value. Since a characteristic function depending on the state of a pair's relationship is introduced to a coalitional game, there exists the strategy where a specific player makes his reward the maximum or the minimum.

Since two models hold $v(p_i \cup p_j, s_k) \geqq v(p_i) + v(p_j)$ for any s_k, the Shapley value of each player becomes more than an original reward in a coalitional game. A low potential player is disadvantageous in two models. But when the low potential player C takes two strategies (s,t,u)=(w,g,g) or (s,t,u)=(w,n,g) in Model Ⅰ,four strategies (s,t,u)=(w,g,g), (s,t,u)=(w,g,n), (s,t,u)=(w,n,g), or (s,t,u)=(w,n,n) in Model Ⅱ,respectively, reward of the player will become the top of three players. The choice is increasing in spite of the disadvantageous structure of Model Ⅱ for the low potential player C.

In the future, we can extend to 4 players and 5 players from three players and may draw many properties from these models. In this paper, the characteristic function of more than three players was made from two players' relationship. We can make directly the relationship of more than three players and will discuss the structure of a complicated relationship.

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