EXISTENCE AND MEAN APPROXIMATION OF FIXED POINTS OF GENERALIZED HYBRID NON-SELF MAPPINGS IN HILBERT SPACES

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Abstract. In this paper, we prove a fixed point theorem for widely more generalized hybrid non-self mappings in Hilbert spaces. Furthermore, we prove mean convergence theorems of Baillon's type for widely more generalized hybrid non-self mappings in a Hilbert space.

1 Introduction Let H be a real Hilbert space and let C be a non-empty subset of H. In 2010, Kocourek, Takahashi and Yao [13] defined a class of nonlinear mappings in a Hilbert space. A mapping T from C into H is said to be generalized hybrid if there exist real numbers α and β such that

$$\alpha\|Tx-Ty\|^2+(1-\alpha)\|x-Ty\|^2 \le \beta\|Tx-y\|^2+(1-\beta)\|x-y\|^2$$

for any $x, y \in C$. We call such a mapping an (α, β)-generalized hybrid mapping. We observe that the class of the mappings covers the classes of well-known mappings. For example, an (α, β)-generalized hybrid mapping is nonexpansive [18] for $\alpha = 1$ and $\beta = 0$, i.e., $\|Tx-Ty\| \le \|x-y\|$ for all $x, y \in C$. It is nonspreading [15] for $\alpha = 2$ and $\beta = 1$, i.e., $2\|Tx-Ty\|^2 \le \|Tx-y\|^2+\|Ty-x\|^2$ for all $x, y \in C$. It is also hybrid [19] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e., $3\|Tx-Ty\|^2 \le \|x-y\|^2+\|Tx-y\|^2+\|Ty-x\|^2$ for all $x, y \in C$. They proved fixed point theorems for such mappings; see also Kohsaka and Takahashi [14] and Iemoto and Takahashi [9]. Moreover, they proved the following nonlinear ergodic theorem.

Theorem 1.1 ([13]). *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H, let T be a generalized hybrid mapping from C into itself which has a fixed point, and let P be the metric projection from H onto the set of fixed points of T. Then for any $x \in C$,*

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

is weakly convergent to a fixed point p of T, where $p = \lim_{n\to\infty} PT^n x$.

Furthermore, they defined a more broad class of nonlinear mappings than the class of generalized hybrid mappings. A mapping T from C into H is said to be super hybrid if there exist real numbers α, β and γ such that

$$\begin{aligned}&\alpha\|Tx-Ty\|^2+(1-\alpha+\gamma)\|x-Ty\|^2\\ &\quad\le(\beta+(\beta-\alpha)\gamma)\|Tx-y\|^2+(1-\beta-(\beta-\alpha-1)\gamma)\|x-y\|^2\\ &\qquad+(\alpha-\beta)\gamma\|x-Tx\|^2+\gamma\|y-Ty\|^2\end{aligned}$$

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for any $x, y \in C$. A generalized hybrid mapping with a fixed point is quasinonexpansive. However, a super hybrid mapping is not quasi-nonexpansive generally even if it has a fixed point. Very recently, the authors [11] also defined a class of nonlinear mappings in a Hilbert space which covers the class of contractive mappings and the class of generalized hybrid mappings defined by Kocourek, Takahashi and Yao [13]. A mapping T from C into H is said to be widely generalized hybrid if there exist real numbers $\alpha, \beta, \gamma, \delta, \varepsilon$ and ζ such that

$$\begin{aligned}\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ + \max\{\varepsilon\|x - Tx\|^2, \zeta\|y - Ty\|^2\} \leq 0\end{aligned}$$

for any $x, y \in C$. Furthermore, the authors [12] defined a class of nonlinear mappings in a Hilbert space which covers the class of super hybrid mappings and the class of widely generalized hybrid mappings. A mapping T from C into H is said to be widely more generalized hybrid if there exist real numbers $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ and η such that

$$\begin{aligned}\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ + \varepsilon\|x - Tx\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x - Tx) - (y - Ty)\|^2 \leq 0\end{aligned}$$

for any $x, y \in C$. Then we prove fixed point theorems for such new mappings in a Hilbert space. Furthermore, we prove nonlinear ergodic theorems of Baillon's type in a Hilbert space. It seems that the results are new and useful. For example, using our fixed point theorems, we can directly prove Browder and Petryshyn's fixed point theorem [5] for strictly pseudocontractive mappings and Kocourek, Takahashi and Yao's fixed point theorem [13] for super hybrid mappings. On the other hand, Hojo, Takahashi and Yao [8] defined a more broad class of nonlinear mappings than the class of generalized hybrid mappings. A mapping T from C into H is said to be extended hybrid if there exist real numbers α, β and γ such that

$$\begin{aligned}\alpha(1+\gamma)\|Tx - Ty\|^2 + (1 - \alpha(1+\gamma))\|x - Ty\|^2 \\ \leq (\beta + \alpha\gamma)\|Tx - y\|^2 + (1 - (\beta + \alpha\gamma))\|x - y\|^2 \\ -(\alpha - \beta)\gamma\|x - Tx\|^2 - \gamma\|y - Ty\|^2\end{aligned}$$

for any $x, y \in C$. Furthermore, they proved a fixed point theorem for generalized hybrid non-self mappings by using the extended hybrid mapping.

In this paper, using an idea of [8], we prove a fixed point theorem for widely more generalized hybrid non-self mappings in Hilbert spaces. Furthermore, we prove mean convergence theorems of Baillon's type for widely more generalized hybrid non-self mappings in a Hilbert space.

2 Preliminaries Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. Let A be a non-empty subset of H. We denote by $\overline{co}A$ the closure of the convex hull of A. In a Hilbert space, it is known that

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x - y\|^2 \tag{2.1}$$

for any $x, y \in H$ and for any $\alpha \in \mathbb{R}$; see [18]. Furthermore, in a Hilbert space, we obtain that

$$2\langle x - y, z - w\rangle = \|x - w\|^2 + \|y - z\|^2 - \|x - z\|^2 - \|y - w\|^2 \tag{2.2}$$

for any $x, y, z, w \in H$. Let C be a non-empty subset of H and let T be a mapping from C into H. We denote by $F(T)$ the set of fixed points of T. A mapping T from C into H with $F(T) \neq \emptyset$ is said to be quasi-nonexpansive if $\|x - Ty\| \leq \|x - y\|$ for any $x \in F(T)$ and for any $y \in C$. It is well-known that the set $F(T)$ of fixed points of a quasi-nonexpansive mapping T is closed and convex; see Ito and Takahashi [10]. It is not difficult to prove such a result in a Hilbert space; see, for instace, [21]. Let D be a non-empty closed convex subset of H and $x \in H$. Then, we know that there exists a unique nearest point $z \in D$ such that $\|x - z\| = \inf_{y \in D} \|x - y\|$. We denote such a correspondence by $z = P_D x$. The mapping P_D is said to be the metric projection from H onto D. It is known that P_D is nonexpansive and

$$\langle x - P_D x, P_D x - u \rangle \geq 0$$

for any $x \in H$ and for any $u \in D$; see [18] for more details. For proving a mean convergence theorem in this paper, we also need the following lemma proved by Takahashi and Toyoda [20].

Lemma 2.1. *Let D be a non-empty closed convex subset of H. Let P be the metric projection from H onto D. Let $\{u_n\}$ be a sequence in H. If $\|u_{n+1} - u\| \leq \|u_n - u\|$ for any $u \in D$ and for any $n \in \mathbb{N}$, then $\{Pu_n\}$ converges strongly to some $u_0 \in D$.*

Let l^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(l^\infty)^*$ (the dual space of l^∞). Then, we denote by $\mu(f)$ the value of μ at $f = (x_1, x_2, x_3, \ldots) \in l^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ on l^∞ is said to be a mean if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \ldots)$. A mean μ is said to be a Banach limit on l^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on l^∞. If μ is a Banach limit on l^∞, then for $f = (x_1, x_2, x_3, \ldots) \in l^\infty$,

$$\liminf_{n\to\infty} x_n \leq \mu_n(x_n) \leq \limsup_{n\to\infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, \ldots) \in l^\infty$ and $x_n \to a \in \mathbb{R}$, then we obtain $\mu(f) = \mu_n(x_n) = a$. See [17] for the proof of existence of a Banach limit and its other elementary properties. Using means and the Riesz theorem, we have the following result; see [16] and [17].

Lemma 2.2. *Let H be a Hilbert space, let $\{x_n\}$ be a bounded sequence in H and let μ be a mean on l^∞. Then there exists a unique point $z_0 \in \overline{co}\{x_n \mid n \in \mathbb{N}\}$ such that*

$$\mu_n \langle x_n, y \rangle = \langle z_0, y \rangle$$

for any $y \in H$.

Kawasaki and Takahashi [12] proved from Lemma 2.2 the following fixed point theorem.

Theorem 2.1. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into itself which satisfies the following condition* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma + \varepsilon + \eta > 0$ *and* $\zeta + \eta \geq 0$*;*

(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta + \zeta + \eta > 0$ *and* $\varepsilon + \eta \geq 0$.

Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z \mid n = 0, 1, \ldots\}$ is bounded. In particular, a fixed point of T is unique in the case of $\alpha + \beta + \gamma + \delta > 0$ on the conditions (1) *and* (2)*.*

As a direct consequence of Theorem 2.1, we obtain the following.

Theorem 2.2. *Let H be a real Hilbert space, let C be a bounded closed convex subset of H and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping from C into itself which satisfies the following condition* (1) *or* (2)*:*

(1) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\gamma+\varepsilon+\eta > 0$ *and* $\zeta+\eta \geq 0$*;*

(2) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\beta+\zeta+\eta > 0$ *and* $\varepsilon+\eta \geq 0$.

Then T has a fixed point. In particular, a fixed point of T is unique in the case of $\alpha+\beta+\gamma+\delta > 0$ on the conditions (1) *and* (2).

3 Fixed point theorem Let H be a real Hilbert space and let C be a non-empty subset of H. A mapping T from C into H was said to be widely more generalized hybrid if there exist $\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta \in \mathbb{R}$ such that

$$\begin{aligned}(3.1)\qquad &\alpha\|Tx-Ty\|^2+\beta\|x-Ty\|^2+\gamma\|Tx-y\|^2+\delta\|x-y\|^2\\ &+\varepsilon\|x-Tx\|^2+\zeta\|y-Ty\|^2+\eta\|(x-Tx)-(y-Ty)\|^2 \leq 0\end{aligned}$$

for any $x,y \in C$; see Introduction. Such a mapping T is said to be $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid; see [12]. An $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping is generalized hybrid in the sense of Kocourek, Takahashi and Yao [13] if $\alpha+\beta=-\gamma-\delta=1$ and $\varepsilon=\zeta=\eta=0$. Moreover it is an extension of widely generalized hybrid mappings in the sence of Kawasaki and Takahashi [11]. Using Theorem 2.2, we prove a fixed point theorem for widely more generalized hybrid non-self mappings in a Hilbert space.

Theorem 3.1. *Let H be a real Hilbert space, let C be a non-empty bounded closed convex subset of H and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping from C into H which satisfies the following condition* (1) *or* (2)*:*

(1) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\gamma+\varepsilon+\eta > 0$*, and there exists $\lambda \in \mathbb{R}$ such that $\lambda \neq 1$ and* $(\alpha+\beta)\lambda+\zeta+\eta \geq 0$*;*

(2) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\beta+\zeta+\eta > 0$*, and there exists $\lambda \in \mathbb{R}$ such that $\lambda \neq 1$ and* $(\alpha+\gamma)\lambda+\varepsilon+\eta \geq 0$.

Suppose that for any $x \in C$, there exist $m \in \mathbb{R}$ and $y \in C$ such that $0 \leq (1-\lambda)m \leq 1$ and $Tx = x+m(y-x)$. Then T has a fixed point. In particular, a fixed point of T is unique in the case of $\alpha+\beta+\gamma+\delta > 0$ on the conditions (1) *and* (2).

Proof. Let $S=(1-\lambda)T+\lambda I$. Since

$$\begin{aligned}Sx &= (1-\lambda)Tx+\lambda x\\ &= (1-\lambda)(x+m(y-x))+\lambda x\\ &= (1-(1-\lambda)m)x+(1-\lambda)my \in C\end{aligned}$$

for any $x \in C$, S is a mapping from C into itself. Since $\lambda \neq 1$, we obtain that $F(S)=F(T)$. Moreover from (2.1) we obtain that

$$\alpha\left\|\left(\frac{1}{1-\lambda}Sx-\frac{\lambda}{1-\lambda}x\right)-\left(\frac{1}{1-\lambda}Sy-\frac{\lambda}{1-\lambda}y\right)\right\|^2$$

$$
\begin{aligned}
&+\beta\left\|x-\left(\frac{1}{1-\lambda}Sy-\frac{\lambda}{1-\lambda}y\right)\right\|^2+\gamma\left\|\left(\frac{1}{1-\lambda}Sx-\frac{\lambda}{1-\lambda}x\right)-y\right\|^2\\
&+\delta\|x-y\|^2\\
&+\varepsilon\left\|x-\left(\frac{1}{1-\lambda}Sx-\frac{\lambda}{1-\lambda}x\right)\right\|^2+\zeta\left\|y-\left(\frac{1}{1-\lambda}Sy-\frac{\lambda}{1-\lambda}y\right)\right\|^2\\
&+\eta\left\|\left(x-\left(\frac{1}{1-\lambda}Sx-\frac{\lambda}{1-\lambda}x\right)\right)-\left(y-\left(\frac{1}{1-\lambda}Sy-\frac{\lambda}{1-\lambda}y\right)\right)\right\|^2\\
=\ &\alpha\left\|\frac{1}{1-\lambda}(Sx-Sy)-\frac{\lambda}{1-\lambda}(x-y)\right\|^2\\
&+\beta\left\|\frac{1}{1-\lambda}(x-Sy)-\frac{\lambda}{1-\lambda}(x-y)\right\|^2\\
&+\gamma\left\|\frac{1}{1-\lambda}(Sx-y)-\frac{\lambda}{1-\lambda}(x-y)\right\|^2+\delta\|x-y\|^2\\
&+\varepsilon\left\|\frac{1}{1-\lambda}(x-Sx)\right\|^2+\zeta\left\|\frac{1}{1-\lambda}(y-Sy)\right\|^2\\
&+\eta\left\|\frac{1}{1-\lambda}(x-Sx)-\frac{1}{1-\lambda}(y-Sy)\right\|^2\\
=\ &\frac{\alpha}{1-\lambda}\|Sx-Sy\|^2+\frac{\beta}{1-\lambda}\|x-Sy\|^2\\
&+\frac{\gamma}{1-\lambda}\|Sx-y\|^2+\left(-\frac{\lambda}{1-\lambda}(\alpha+\beta+\gamma)+\delta\right)\|x-y\|^2\\
&+\frac{\varepsilon+\gamma\lambda}{(1-\lambda)^2}\|x-Sx\|^2+\frac{\zeta+\beta\lambda}{(1-\lambda)^2}\|y-Sy\|^2\\
&+\frac{\eta+\alpha\lambda}{(1-\lambda)^2}\|(x-Sx)-(y-Sy)\|^2\leq 0.
\end{aligned}
$$

Then S is an $\left(\frac{\alpha}{1-\lambda},\frac{\beta}{1-\lambda},\frac{\gamma}{1-\lambda},-\frac{\lambda}{1-\lambda}(\alpha+\beta+\gamma)+\delta,\frac{\varepsilon+\gamma\lambda}{(1-\lambda)^2},\frac{\zeta+\beta\lambda}{(1-\lambda)^2},\frac{\eta+\alpha\lambda}{(1-\lambda)^2}\right)$-widely more generalized hybrid mapping. Furthermore, we obtain that

$$
\begin{aligned}
\frac{\alpha}{1-\lambda}+\frac{\beta}{1-\lambda}+\frac{\gamma}{1-\lambda}-\frac{\lambda}{1-\lambda}(\alpha+\beta+\gamma)+\delta &= \alpha+\beta+\gamma+\delta\geq 0,\\
\frac{\alpha}{1-\lambda}+\frac{\gamma}{1-\lambda}+\frac{\varepsilon+\gamma\lambda}{(1-\lambda)^2}+\frac{\eta+\alpha\lambda}{(1-\lambda)^2} &= \frac{\alpha+\gamma+\varepsilon+\eta}{(1-\lambda)^2}>0,\\
\frac{\zeta+\beta\lambda}{(1-\lambda)^2}+\frac{\eta+\alpha\lambda}{(1-\lambda)^2} &= \frac{(\alpha+\beta)\lambda+\zeta+\eta}{(1-\lambda)^2}\geq 0.
\end{aligned}
$$

Therefore by Theorem 2.2 we obtain $F(S)\neq\emptyset$.

Next suppose that $\alpha+\beta+\gamma+\delta>0$. Let p_1 and p_2 be fixed points of T. Then

$$
\begin{aligned}
&\alpha\|Tp_1-Tp_2\|^2+\beta\|p_1-Tp_2\|^2+\gamma\|Tp_1-p_2\|^2+\delta\|p_1-p_2\|^2\\
&\qquad+\varepsilon\|p_1-Tp_1\|^2+\zeta\|p_2-Tp_2\|^2+\eta\|(p_1-Tp_1)-(p_2-Tp_2)\|^2\\
&\quad=(\alpha+\beta+\gamma+\delta)\|p_1-p_2\|^2\leq 0
\end{aligned}
$$

and hence $p_1=p_2$. Therefore a fixed point of T is unique.

In the case of the condition (2), we can obtain the result by replacing the variables x and y. □

Example 3.1. Let $H = \mathbb{R}$, let $C = \left[0, \frac{\pi}{2}\right]$, let $Tx = (1+2x)\cos x - 2x^2$ and let $\alpha = 1$, $\beta = \gamma = 11$, $\delta = -22$, $\varepsilon = \zeta = -12$ and $\eta = 1$. Then T is an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into H, $\alpha+\beta+\gamma+\delta = 1 \geq 0$ and $\alpha+\gamma+\varepsilon+\eta = 1 > 0$. Let $\lambda = \frac{2+3\pi}{3(1+\pi)}$ and let $m = 1+\pi$. Then $0 \leq (1-\lambda)m = \frac{1}{3} < 1$ and $(\alpha+\beta)\lambda+\zeta+\eta = \frac{\pi-3}{1+\pi} \geq 0$. Let $y = x + \frac{(1+2x)(\cos x - x)}{1+\pi}$ for any $x \in C$. Then $Tx = x + m(y-x)$ and $y \in C$. Therefore by Theorem 3.1 T has a unique fixed point.

4 Nonlinear ergodic theorems

In this section, using the technique developed by Takahashi [16], we prove mean convergence theorems of Baillon's type in a Hilbert space. Before proving the results, we need the following lemmas.

Lemma 4.1. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into H which has a fixed point and satisfies the condition:*

$$\alpha+\gamma+\varepsilon+\eta > 0, \text{ or } \alpha+\beta+\zeta+\eta > 0.$$

Then $F(T)$ is closed.

Proof. Suppose that $\{x_n \mid n = 1, 2, \ldots\} \subset F(T)$ is convergent to $x \in H$. We show $x \in F(T)$. Putting $y = x_n$ in (3.1), we obtain that

$$\alpha\|Tx - Tx_n\|^2 + \beta\|x - Tx_n\|^2 + \gamma\|Tx - x_n\|^2 + \delta\|x - x_n\|^2 \\ +\varepsilon\|x - Tx\|^2 + \zeta\|x_n - Tx_n\|^2 + \eta\|(x - Tx) - (x_n - Tx_n)\}^2 \leq 0$$

and hence

$$(\alpha+\gamma)\|Tx - x_n\|^2 + (\beta+\delta)\|x - x_n\|^2 + (\varepsilon+\eta)\|x - Tx\|^2 \leq 0. \tag{4.1}$$

Letting $n \to \infty$, we obtain that

$$(\alpha+\gamma+\varepsilon+\eta)\|Tx - x\|^2 \leq 0. \tag{4.2}$$

Since $\alpha+\gamma+\varepsilon+\eta > 0$, from (4.2) we obtain that $x \in F(T)$. Therefore $F(T)$ is closed. Similarly, we can obtain the desired result for the case of $\alpha+\beta+\zeta+\eta > 0$. □

Lemma 4.2. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into H which has a fixed point and satisfies the condition* (1) *or* (2)*:*

(1) $\alpha+\beta+\gamma+\delta \geq 0$ *and* $\alpha+\gamma+\varepsilon+\eta > 0$*;*

(2) $\alpha+\beta+\gamma+\delta \geq 0$ *and* $\alpha+\beta+\zeta+\eta > 0$*.*

Then $F(T)$ is convex.

Proof. For $x_1, x_2 \in F(T)$ and $\lambda \in \mathbb{R}$ with $0 \leq \lambda \leq 1$, put $x = (1-\lambda)x_1 + \lambda x_2$. We show that $x \in F(T)$. Putting $y = x_1$ in (3.1), we obtain that

$$\alpha\|Tx - Tx_1\|^2 + \beta\|x - Tx_1\|^2 + \gamma\|Tx - x_1\|^2 + \delta\|x - x_1\|^2 \\ +\varepsilon\|x - Tx\|^2 + \zeta\|x_1 - Tx_1\|^2 + \eta\|(x - Tx) - (x_1 - Tx_1)\|^2 \leq 0$$

and hence

$$(\alpha+\gamma)\|Tx - x_1\|^2 + (\beta+\delta)\lambda^2\|x_1 - x_2\|^2 + (\varepsilon+\eta)\|x - Tx\|^2 \leq 0. \tag{4.3}$$

Similarly, putting $y = x_2$ in (3.1), we obtain that

$$(\alpha+\gamma)\|Tx - x_2\|^2 + (\beta+\delta)(1-\lambda)^2\|x_1 - x_2\|^2 + (\varepsilon+\eta)\|x - Tx\|^2 \le 0. \tag{4.4}$$

Therefore from (4.3) we obtain that

$$\begin{aligned}&(\alpha+\gamma)\|Tx - x_1\|^2 + (\beta+\delta)\lambda^2\|x_1 - x_2\|^2 \\ &\quad +(\varepsilon+\eta)(\|Tx - x_1\|^2 + \lambda^2\|x_1 - x_2\|^2 + 2\lambda\langle Tx - x_1, x_1 - x_2\rangle) \le 0.\end{aligned}$$

Thus we obtain that

$$\begin{aligned}&(\alpha+\gamma+\varepsilon+\eta)\|Tx - x_1\|^2 + (\beta+\delta+\varepsilon+\eta)\lambda^2\|x_1 - x_2\|^2 \\ &\quad +2(\varepsilon+\eta)\lambda\langle Tx - x_1, x_1 - x_2\rangle) \le 0.\end{aligned} \tag{4.5}$$

Similarly, from (4.4) we obtain that

$$\begin{aligned}&(\alpha+\gamma+\varepsilon+\eta)\|Tx - x_2\|^2 + (\beta+\delta+\varepsilon+\eta)(1-\lambda)^2\|x_1 - x_2\|^2 \\ &\quad -2(\varepsilon+\eta)(1-\lambda)\langle Tx - x_2, x_1 - x_2\rangle) \le 0.\end{aligned} \tag{4.6}$$

Using (2.1), (4.5), (4.6), $\alpha+\gamma+\varepsilon+\eta > 0$ and $\alpha+\beta+\gamma+\delta \ge 0$, we obtain that

$$\begin{aligned}
&\|Tx - x\|^2 \\
&= \|Tx - ((1-\lambda)x_1 + \lambda x_2)\|^2 \\
&= (1-\lambda)\|Tx - x_1\|^2 + \lambda\|Tx - x_2\|^2 - \lambda(1-\lambda)\|x_1 - x_2\|^2 \\
&\le (1-\lambda)\left(-\frac{(\beta+\delta+\varepsilon+\eta)\lambda^2}{\alpha+\gamma+\varepsilon+\eta}\|x_1 - x_2\|^2 \right. \\
&\qquad \left. -\frac{2(\varepsilon+\eta)\lambda}{\alpha+\gamma+\varepsilon+\eta}\langle Tx - x_1, x_1 - x_2\rangle\right) \\
&\quad +\lambda\left(-\frac{(\beta+\delta+\varepsilon+\eta)(1-\lambda)^2}{\alpha+\gamma+\varepsilon+\eta}\|x_1 - x_2\|^2 \right. \\
&\qquad \left. +\frac{2(\varepsilon+\eta)(1-\lambda)}{\alpha+\gamma+\varepsilon+\eta}\langle Tx - x_2, x_1 - x_2\rangle\right) \\
&\quad -\lambda(1-\lambda)\|x_1 - x_2\|^2 \\
&= -\frac{(\beta+\delta+\varepsilon+\eta)\lambda^2(1-\lambda)}{\alpha+\gamma+\varepsilon+\eta}\|x_1 - x_2\|^2 \\
&\quad -\frac{(\beta+\delta+\varepsilon+\eta)\lambda(1-\lambda)^2}{\alpha+\gamma+\varepsilon+\eta}\|x_1 - x_2\|^2 \\
&\quad +\frac{2(\varepsilon+\eta)\lambda(1-\lambda)}{\alpha+\gamma+\varepsilon+\eta}\|x_1 - x_2\|^2 \\
&\quad -\lambda(1-\lambda)\|x_1 - x_2\|^2 \\
&= -\frac{(\alpha+\beta+\gamma+\delta)\lambda(1-\lambda)}{\alpha+\gamma+\varepsilon+\eta}\|x_1 - x_2\|^2 \le 0
\end{aligned}$$

and hence $x \in F(T)$. Thus $F(T)$ is convex. Similarly, we can obtain the desired result in the case of $\alpha+\beta+\gamma+\delta \ge 0$ and $\alpha+\beta+\zeta+\eta > 0$. □

Lemma 4.3. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping from C into H which has a fixed point and satisfies the condition* (1) *or* (2)*:*

(1) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\gamma > 0$ *and* $\varepsilon+\eta \geq 0$*;*

(2) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\beta > 0$ *and* $\zeta+\eta \geq 0$.

Then T *is quasi-nonexpansive.*

Proof. From (3.1) we have that for any $x \in C$ and for any $y \in F(T)$,

$$\begin{aligned}\alpha\|Tx-Ty\|^2 + \beta\|x-Ty\|^2 + \gamma\|Tx-y\|^2 + \delta\|x-y\|^2 \\ +\varepsilon\|x-Tx\|^2 + \zeta\|y-Ty\|^2 + \eta\|(x-Tx)-(y-Ty)\|^2 \\ = (\alpha+\gamma)\|Tx-y\|^2 + (\beta+\delta)\|x-y\|^2 + (\varepsilon+\eta)\|x-Tx\|^2 \leq 0.\end{aligned}$$

From $\alpha+\gamma > 0$ we obtain that

$$\|Tx-y\|^2 \leq -\frac{\beta+\delta}{\alpha+\gamma}\|x-y\|^2 - \frac{\varepsilon+\eta}{\alpha+\gamma}\|x-Tx\|^2.$$

Since $-\frac{\beta+\delta}{\alpha+\gamma} \leq 1$ from $\alpha+\beta+\gamma+\delta \geq 0$ and $-\frac{\varepsilon+\eta}{\alpha+\gamma} \leq 0$ from $\varepsilon+\eta \geq 0$, we obtain that

$$\|Tx-y\|^2 \leq \|x-y\|^2$$

and hence

$$\|Tx-y\| \leq \|x-y\|.$$

Thus T is quasi-nonexpansive. Similarly, we can obtain the desired result for the case of $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\beta > 0$ and $\zeta+\eta \geq 0$. □

Moreover we obtain the following.

Lemma 4.4. *Let* H *be a real Hilbert space, let* C *be a non-empty closed convex subset of* H *and let* T *be an* $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$*-widely more generalized hybrid mapping from* C *into* H *which has a fixed point and satisfies the condition* (1) *or* (2)*:*

(1) $\alpha+\beta+\gamma+\delta \geq 0$, *and there exists* $\lambda \in \mathbb{R}$ *such that* $0 \leq (\alpha+\gamma)\lambda+\varepsilon+\eta < \alpha+\gamma+\varepsilon+\eta$*;*

(2) $\alpha+\beta+\gamma+\delta \geq 0$, *and there exists* $\lambda \in \mathbb{R}$ *such that* $0 \leq (\alpha+\beta)\lambda+\zeta+\eta < \alpha+\beta+\zeta+\eta$.

Then $(1-\lambda)T+\lambda I$ *is quasi-nonexpansive.*

Proof. Let $S = (1-\lambda)T+\lambda I$. As in the proof of Theorem 3.1, we have that S is an $\left(\frac{\alpha}{1-\lambda}, \frac{\beta}{1-\lambda}, \frac{\gamma}{1-\lambda}, -\frac{\lambda}{1-\lambda}(\alpha+\beta+\gamma)+\delta, \frac{\varepsilon+\gamma\lambda}{(1-\lambda)^2}, \frac{\zeta+\beta\lambda}{(1-\lambda)^2}, \frac{\eta+\alpha\lambda}{(1-\lambda)^2}\right)$-widely more generalized hybrid mapping from C into H and $F(S) = F(T)$. Furthermore, we obtain that

$$\begin{aligned}\frac{\alpha}{1-\lambda}+\frac{\beta}{1-\lambda}+\frac{\gamma}{1-\lambda}-\frac{\lambda}{1-\lambda}(\alpha+\beta+\gamma)+\delta &= \alpha+\beta+\gamma+\delta \geq 0, \\ \frac{\alpha}{1-\lambda}+\frac{\gamma}{1-\lambda} &= \frac{\alpha+\gamma}{1-\lambda} > 0, \\ \frac{\varepsilon+\gamma\lambda}{(1-\lambda)^2}+\frac{\eta+\alpha\lambda}{(1-\lambda)^2} &= \frac{(\alpha+\gamma)\lambda+\varepsilon+\eta}{(1-\lambda)^2} \geq 0.\end{aligned}$$

By Lemma 4.3 S is quasi-nonexpansive. Similarly, we can obtain the desired result for the case of the condition (2). □

Now we first obtain the following mean convergence theorem for widely more generalized hybrid mappings in a Hilbert space.

Theorem 4.1. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into H which has a fixed point and satisfies the condition* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma > 0$ *and* $\varepsilon + \eta \geq 0$*;*

(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$ *and* $\zeta + \eta \geq 0$.

Then for any $x \in C(T; 0) = \{z \mid T^n z \in C, \ \forall n \in \mathbb{N} \cup \{0\}\}$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

is weakly convergent to a fixed point p of T, where P is the metric projection from H onto $F(T)$ and $p = \lim_{n\to\infty} PT^n x$.

Proof. Let $x \in C(T; 0)$. We first consider the case of the condition (2). Since $F(T)$ is non-empty and by Lemma 4.3 T is quasi-nonexpansive, we obtain that

$$\|T^{n+1}x - y\| \leq \|T^n x - y\|$$

for any $n \in \mathbb{N} \cup \{0\}$ and for any $y \in F(T)$ and hence $\{T^n x\}$ is bounded for any $x \in C(T; 0)$. Since

$$\|S_n x - y\| \leq \frac{1}{n} \sum_{k=0}^{n-1} \|T^k x - y\| \leq \|x - y\|$$

for any $n \in \mathbb{N} \cup \{0\}$ and for any $y \in F(T)$, $\{S_n x \mid n = 0, 1, \ldots\}$ is also bounded. Therefore there exist a strictly increasing sequence $\{n_i\}$ and $p \in H$ such that $\{S_{n_i} x \mid i = 0, 1, \ldots\}$ is weakly convergent to p. Since C is closed and convex, C is weakly closed. Thus $p \in C$. We show that $p \in F(T)$. Since T is an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into H, we obtain that

$$\begin{aligned}
&\alpha\|Tz - T^{k+1}x\|^2 + \beta\|z - T^{k+1}x\|^2 + \gamma\|Tz - T^k x\|^2 + \delta\|z - T^k x\|^2 \\
&\quad +\varepsilon\|z - Tz\|^2 + \zeta\|T^k x - T^{k+1}x\|^2 + \eta\|(z - Tz) - (T^k x - T^{k+1}x)\|^2 \leq 0
\end{aligned}$$

for any $k \in \mathbb{N} \cup \{0\}$ and for any $z \in C$. By (2.2) we obtain that

$$\begin{aligned}
&\|(z - Tz) - (T^k x - T^{k+1}x)\|^2 \\
&\quad = \|z - Tz\|^2 + \|T^k x - T^{k+1}x\|^2 - 2\langle z - Tz, T^k x - T^{k+1}x\rangle \\
&\quad = \|z - Tz\|^2 + \|T^k x - T^{k+1}x\|^2 + \|z - T^k x\|^2 + \|Tz - T^{k+1}x\|^2 \\
&\qquad -\|z - T^{k+1}x\|^2 - \|Tz - T^k x\|^2.
\end{aligned}$$

Thus we obtain that

$$\begin{aligned}
&(\alpha + \eta)\|Tz - T^{k+1}x\|^2 + (\beta - \eta)\|z - T^{k+1}x\|^2 + (\gamma - \eta)\|Tz - T^k x\|^2 \\
&\quad +(\delta + \eta)\|z - T^k x\|^2 + (\varepsilon + \eta)\|z - Tz\|^2 + (\zeta + \eta)\|T^k x - T^{k+1}x\|^2 \leq 0.
\end{aligned}$$

From

$$\begin{aligned}&(\gamma-\eta)\|Tz-T^kx\|^2\\&\quad=(\alpha+\gamma)(\|z-Tz\|^2+\|z-T^kx\|^2-2\langle z-Tz,z-T^kx\rangle)\\&\qquad-(\alpha+\eta)\|Tz-T^kx\|^2,\end{aligned}$$

we obtain that

$$\begin{aligned}&(\alpha+\eta)\|Tz-T^{k+1}x\|^2+(\beta-\eta)\|z-T^{k+1}x\|^2\\&\quad+(\alpha+\gamma)(\|z-Tz\|^2+\|z-T^kx\|^2-2\langle z-Tz,z-T^kx\rangle)\\&\quad-(\alpha+\eta)\|Tz-T^kx\|^2+(\delta+\eta)\|z-T^kx\|^2\\&\quad+(\varepsilon+\eta)\|z-Tz\|^2+(\zeta+\eta)\|T^kx-T^{k+1}x\|^2\le 0.\end{aligned}$$

and hence

$$\begin{aligned}&(\alpha+\eta)(\|Tz-T^{k+1}x\|^2-\|Tz-T^kx\|^2)+(\beta-\eta)\|z-T^{k+1}x\|^2\\&\quad+(\alpha+\gamma+\delta+\eta)\|z-T^kx\|^2-2(\alpha+\gamma)\langle z-Tz,z-T^kx\rangle\\&\quad+(\alpha+\gamma+\varepsilon+\eta)\|z-Tz\|^2+(\zeta+\eta)\|T^kx-T^{k+1}x\|^2\le 0.\end{aligned}$$

By $\alpha+\beta+\gamma+\delta\ge 0$, we obtain that

$$-(\beta-\eta)=-(\beta+\delta)+\delta+\eta\le\alpha+\gamma+\delta+\eta.$$

From this inequality and $\zeta+\eta\ge 0$, we obtain that

$$\begin{aligned}&(\alpha+\eta)(\|Tz-T^{k+1}x\|^2-\|Tz-T^kx\|^2)\\&\quad+(\beta-\eta)(\|z-T^{k+1}x\|^2-\|z-T^kx\|^2)\\&\quad-2(\alpha+\gamma)\langle z-Tz,z-T^kx\rangle+(\alpha+\gamma+\varepsilon+\eta)\|z-Tz\|^2\le 0\end{aligned}$$

for any $k\in\mathbb{N}\cup\{0\}$ and for any $z\in C$. Summing up these inequalities with respect to $k=0,1,\dots,n-1$ and dividing by n, we obtain that

$$\begin{aligned}&\frac{\alpha+\eta}{n}(\|Tz-T^nx\|^2-\|Tz-x\|^2)+\frac{\beta-\eta}{n}(\|z-T^nx\|^2-\|z-x\|^2)\\&\quad-2(\alpha+\gamma)\langle z-Tz,z-S_nx\rangle+(\alpha+\gamma+\varepsilon+\eta)\|z-Tz\|^2\le 0.\end{aligned}$$

Replacing n by n_i, we obtain that

$$\begin{aligned}&\frac{\alpha+\eta}{n_i}(\|Tz-T^{n_i}x\|^2-\|Tz-x\|^2)+\frac{\beta-\eta}{n_i}(\|z-T^{n_i}x\|^2-\|z-x\|^2)\\&\quad-2(\alpha+\gamma)\langle z-Tz,z-S_{n_i}x\rangle+(\alpha+\gamma+\varepsilon+\eta)\|z-Tz\|^2\le 0.\end{aligned}$$

Letting $i\to\infty$, we obtain that

$$-2(\alpha+\gamma)\langle z-Tz,z-p\rangle+(\alpha+\gamma+\varepsilon+\eta)\|z-Tz\|^2\le 0.$$

Putting $z=p$, we obtain that

$$(\alpha+\gamma+\varepsilon+\eta)\|p-Tp\|^2\le 0.$$

Since $\alpha+\gamma+\varepsilon+\eta>0$, we obtain that $Tp=p$. Since by Lemmas 4.1 and 4.2 $F(T)$ is closed and convex, the metric projection P from H onto $F(T)$ is well-defined. By Lemma 2.1 there exists $q \in F(T)$ such that $\{PT^n x \mid n = 0, 1, \dots\}$ is convergent to q. To complete the proof, we show that $q = p$. Note that the metric projection P satisfies

$$\langle z - Pz, Pz - u \rangle \geq 0$$

for any $z \in H$ and for any $u \in F(T)$; see [17]. Therefore

$$\langle T^k x - PT^k x, PT^k x - y \rangle \geq 0$$

for any $k \in \mathbb{N} \cup \{0\}$ and for any $y \in F(T)$. Since P is the metric projection and T is quasi-nonexpansive, we obtain that

$$\begin{aligned} \|T^n x - PT^n x\| &\leq \|T^n x - PT^{n-1} x\| \\ &\leq \|T^{n-1} x - PT^{n-1} x\|, \end{aligned}$$

that is, $\{\|T^n x - PT^n x\| \mid n = 0, 1, \dots\}$ is non-increasing. Therefore we obtain

$$\begin{aligned} \langle T^k x - PT^k x, y - q \rangle &\leq \langle T^k x - PT^k x, PT^k x - q \rangle \\ &\leq \|T^k x - PT^k x\| \cdot \|PT^k x - q\| \\ &\leq \|x - Px\| \cdot \|PT^k x - q\|. \end{aligned}$$

Summing up these inequalities with respect to $k = 0, 1, \dots, n-1$ and dividing by n, we obtain

$$\left\langle S_n x - \frac{1}{n} \sum_{k=0}^{n-1} PT^k x, y - q \right\rangle \leq \frac{\|x - Px\|}{n} \sum_{k=0}^{n-1} \|PT^k x - q\|.$$

Since $\{S_{n_i} x \mid i = 0, 1, \dots\}$ is weakly convergent to p and $\{PT^n x \mid n = 0, 1, \dots\}$ is convergent to q, we obtain that

$$\langle p - q, y - q \rangle \leq 0.$$

Putting $y = p$, we obtain

$$\|p - q\|^2 \leq 0$$

and hence $q = p$. This completes the proof.

Similarly, we can obtain the desired result for the case of the condition (1). □

Moreover we obtain the following.

Theorem 4.2. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into H which has a fixed point and satisfies the condition* (1) *or* (2)*:*

(1) $\alpha+\beta+\gamma+\delta \geq 0$, *and there exists* $\lambda \in \mathbb{R}$ *such that* $0 \leq (\alpha+\gamma)\lambda+\varepsilon+\eta < \alpha+\gamma+\varepsilon+\eta$*;*

(2) $\alpha+\beta+\gamma+\delta \geq 0$, *and there exists* $\lambda \in \mathbb{R}$ *such that* $0 \leq (\alpha+\beta)\lambda+\zeta+\eta < \alpha+\beta+\zeta+\eta$*.*

Then for any $x \in C(T; \lambda) = \{z \mid ((1-\lambda)T + \lambda I)^n z \in C, \ \forall n \in \mathbb{N} \cup \{0\}\}$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} ((1-\lambda)T + \lambda I)^k x$$

is weakly convergent to a fixed point p of T, where P is the metric projection from H onto $F(T)$ and $p = \lim_{n\to\infty} P((1-\lambda)T + \lambda I)^n x$.

Proof. Let $S = (1-\lambda)T + \lambda I$. As in the proof of Theorem 3.1, we have that S is an $\left(\frac{\alpha}{1-\lambda}, \frac{\beta}{1-\lambda}, \frac{\gamma}{1-\lambda}, -\frac{\lambda}{1-\lambda}(\alpha+\beta+\gamma)+\delta, \frac{\varepsilon+\gamma\lambda}{(1-\lambda)^2}, \frac{\zeta+\beta\lambda}{(1-\lambda)^2}, \frac{\eta+\alpha\lambda}{(1-\lambda)^2}\right)$-widely more generalized hybrid mapping from C into H and

$$\begin{aligned}\frac{\alpha}{1-\lambda}+\frac{\beta}{1-\lambda}+\frac{\gamma}{1-\lambda}-\frac{\lambda}{1-\lambda}(\alpha+\beta+\gamma)+\delta &= \alpha+\beta+\gamma+\delta \geq 0,\\ \frac{\alpha}{1-\lambda}+\frac{\gamma}{1-\lambda} &= \frac{\alpha+\gamma}{1-\lambda} > 0,\\ \frac{\varepsilon+\gamma\lambda}{(1-\lambda)^2}+\frac{\eta+\alpha\lambda}{(1-\lambda)^2} &= \frac{(\alpha+\gamma)\lambda+\varepsilon+\eta}{(1-\lambda)^2} \geq 0.\end{aligned}$$

By Theorem 4.1 $\{S_n x\}$ is weakly convergent to $p \in F(S) = F(T)$. Since by Lemmas 4.1 and 4.2 $F(S)$ is closed and convex, the metric projection P from H onto $F(S)$ is well-defined. Since by Lemma 4.4 S is quasi-nonexpansive, we obtain that

$$\|S^{n+1}x - y\| \leq \|S^n x - y\|$$

for any $n \in \mathbb{N} \cup \{0\}$ and for any $y \in F(S)$. Therefore we can obtain the desired result as in the proof of Theorem 4.1.

Similarly, we can obtain the desired result for the case of the condition (2). □

Theorem 4.3. *Let H be a real Hilbert space, let C be a non-empty bounded closed convex subset of H and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping from C into H which satisfies the following condition* (1) *or* (2)*:*

(1) $\alpha+\beta+\gamma+\delta \geq 0$, *and there exists* $\lambda \in \mathbb{R}$ *such that* $0 \leq (\alpha+\gamma)\lambda+\varepsilon+\eta < \alpha+\gamma+\varepsilon+\eta$*;*

(2) $\alpha+\beta+\gamma+\delta \geq 0$, *and there exists* $\lambda \in \mathbb{R}$ *such that* $0 \leq (\alpha+\beta)\lambda+\zeta+\eta < \alpha+\beta+\zeta+\eta$.

Suppose that for any $x \in C$, there exist $m \in \mathbb{R}$ and $y \in C$ such that $0 \leq (1-\lambda)m \leq 1$ and $Tx = x + m(y-x)$. Then for any $x \in C$,

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1}((1-\lambda)T+\lambda I)^k x$$

is weakly convergent to a fixed point p of T, where P is the metric projection from H onto $F(T)$ and $p = \lim_{n\to\infty} P((1-\lambda)T+\lambda I)^n x$.

Proof. Let $S = (1-\lambda)T + \lambda I$. Since $S = (1-\lambda)T+\lambda I$ is a mapping from C into itself, we have $C(T;\lambda) = \{z \mid ((1-\lambda)T+\lambda I)^n z \in C,\ \forall n \in \mathbb{N}\cup\{0\}\} = C$. Therefore by Theorem 4.2 we obtain the desired result. □

Example 4.1. Let $H = \mathbb{R}$, let $C = \left[0, \frac{\pi}{2}\right]$, let $Tx = (1+2x)\cos x - 2x^2$ and let $\alpha = 1$, $\beta = \gamma = 11$, $\delta = -22$, $\varepsilon = \zeta = -12$ and $\eta = 1$. Then T is an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping from C into H, $\alpha+\beta+\gamma+\delta = 1 \geq 0$ and $\alpha+\gamma+\varepsilon+\eta = 1 > 0$. Let $\lambda = \frac{2+3\pi}{3(1+\pi)}$ and $m = 1+\pi$. Then $0 \leq (1-\lambda)m = \frac{1}{3} < 1$ and $0 \leq (\alpha+\gamma)\lambda+\varepsilon+\eta = \frac{\pi-3}{1+\pi} < 1 = \alpha+\gamma+\varepsilon+\eta$. Let $y = x + \frac{(1+2x)(\cos x - x)}{1+\pi}$ for any $x \in C$. Then $Tx = x + m(y-x)$ and $y \in C$. Therefore by Theorem 4.3 for any $x \in C$,

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1}((1-\lambda)T+\lambda I)^k x$$

is weakly convergent to a fixed point p of T, where P is the metric projection from H onto $F(T)$ and $p = \lim_{n\to\infty} P((1-\lambda)T+\lambda I)^n x$.

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