ELEMENTARY PROOFS OF OPERATOR MONOTONICITY OF SOME FUNCTIONS II

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Abstract. In the previous paper we gave elementary proofs of operator monotonicity of the representing function of the weighted arithmetic mean and some other related functions. In this note, we show some extensions and applications of those results.

1 Introduction. A (bounded linear) operator A acting on a Hilbert space H is said to be positive, denoted by $A \geq 0$, if $(Av, v) \geq 0$ for all $v \in H$. The definition of positivity induces the order $A \geq B$ for self-adjoint operators A and B on H. A real-valued function f on $(0, \infty)$ is *operator monotone,* if $0 \leq f(A) \leq f(B)$ for operators A and B on H such that $0 \leq A \leq B$. Thus, throughout this paper, we assume that operator monotone functions are positive and their domains are $(0, \infty)$. As a typical example, $x \mapsto x^p$ $(0 \leq p \leq 1)$ is an operator monotone function, which is well-known as *Löwner-Heinz theorem* (*LH*).

For convenience sake, we state the main facts shown in our previous paper with elementary proofs:

Proposition 1.1 (cf. [11, Theorem 1.2], [1], [2], [3], [4], [5], [8], [9], [13]). *The function*

$$a_p(x) = \left(\frac{1+x^p}{2}\right)^{\frac{1}{p}}, \ p \neq 0 \quad \left(a_0(x) = x^{\frac{1}{2}}\right)$$

is operator monotone if (and only if) $-1 \leq p \leq 1$.

Proposition 1.2 (cf. [11, Theorem 1.1], [13], [1]). *The function*

$$s_p(x) = \left(\frac{p(x-1)}{x^p-1}\right)^{\frac{1}{1-p}}, \ p \neq 0, 1 \quad \left(s_0(x)\left(= \lim_{p\to 0} s_p(x)\right) = \frac{x-1}{\log x}, \ s_1(x) = \frac{1}{e}x^{\frac{x}{x-1}}\right)$$

is operator monotone if $-2 \leq p \leq 2$.

Proposition 1.3 ([11, Theorem 1.3], [5], [9], [6], [2], [3]). *The function*

$$k_p(x) = \frac{p-1}{p} \cdot \frac{x^p-1}{x^{p-1}-1}, \ p \neq 0, 1 \quad \left(k_0(x) = \frac{x\log x}{x-1}, \ k_1(x) = \frac{x-1}{\log x}\right)$$

is operator monotone if $-1 \leq p \leq 2$.

In this paper, we give some extensions of those propositions and their applications. As an application of the extension of Proposition 1.2, we give a slight extensions of Uchiyama's example in [15] related to Petz-Hasegawa theorem [14].

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2 Preliminaries. By Kubo-Ando theory [12], an operator mean σ is defined as a binary relation of positive operators, satisfying the following properties in common:

(monotonicity)	$A \le C, B \le D \Longrightarrow A\sigma B \le C\sigma D$,
(transformer inequality)	$C(A\sigma B)C \le (CAC)\sigma(CBC)$,
(normality)	$A\sigma A = A$,
(strong operator semi-continuity)	$A_n \downarrow A, B_n \downarrow B \Longrightarrow A_n\sigma B_n \downarrow A\sigma B$.

Sometimes for the definition of an operator mean we must assume operators to be invertible. Without any assumption for invertibility every mean is well-defined as the (strong operator) limits of $(A+\varepsilon I)\sigma(B+\varepsilon I)$ as $\varepsilon \downarrow 0$ instead of $A\sigma B$. (I is the identity operator.)

Every operator mean σ corresponds a unique operator monotone function, that is, its representing function f_σ which is defined by $f_\sigma(x) = 1\sigma x$. Conversely, if f is an operator monotone function with $f(1) = 1$, then the definition of the operator mean corresponding to f is given by

$$A\sigma B = A^{\frac{1}{2}} f\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right) A^{\frac{1}{2}}$$

for positive invertible operators A and B.

For our discussion, we use the following basic facts:

(I) For an operator mean σ and for two operator monotone functions g and h, if we define $g\sigma h$ by

$$(g\sigma h)(x) = g(x) f_\sigma\left(\frac{h(x)}{g(x)}\right),$$

then $g\sigma h$ is operator monotone.

(II) For a strictly positive function f on $(0,\infty)$, define $f^\circ(x) := xf(1/x)$ (transpose), $f^*(x) := 1/f(1/x)$ (adjoint) and $f^\perp(x) := x/f(x)$ (dual), then the four functions $f, f^\circ, f^*, f^\perp$ are equivalent to one another with respect to operator monotonicity ([12], [10]).

(III) For a continuous path f_t $(0 \le t \le 1)$ of operator monotone functions, its integral mean $\tilde{f}$ defined by

$$\tilde{f}(x) = \int_0^1 f_t(x)dt$$

is an operator monotone function ([2], [3]).

3 Main results. Applying (I) to the operator mean σ_{a_p} corresponding to the operator monotone function $a_p(x)$ (*notice* $a_p(1) = 1$), as an extension of Proposition 1.1, we showed in [11]:

Lemma 3.1 (cf. [11, Lemma 3.1], [13])**.** *Let f, g be operator monotone functions, then $f\sigma_{a_p}g = \left(\frac{f^p+g^p}{2}\right)^{\frac{1}{p}}$ (or equivalently, $(f^p+g^p)^{\frac{1}{p}}$) is operator monotone for $-1 \le p \le 1$, $p \neq 0$. Further, if $f_1, ..., f_n$ are operator monotone functions, then $(\sum_{i=1}^n f_i^p)^{\frac{1}{p}}$ is operator monotone. In particular, $(\sum_{i=1}^n(\alpha_i+\beta_i x)^p)^{\frac{1}{p}}$ $(\alpha_i, \beta_i \ge 0)$ is operator monotone.*

Similarly as σ_{a_p}, let σ_{s_p} and σ_{k_p} be the operator means corresponding to the operator monotone functions s_p and k_p, respectively. Then we obtain the following result:

Theorem 3.2 (cf. [11, Theorem 1.1], [13], [1])**.** *For operator monotone functions f, g $(f \neq g)$, the function*

$$f\sigma_{s_p}g = \left(\frac{p(f-g)}{f^p-g^p}\right)^{\frac{1}{1-p}}, \quad p \neq 0, 1 \quad \left(f\sigma_{s_0}g = \frac{f-g}{\log f - \log g}, f\sigma_{s_1}g = \frac{f}{e}\cdot\left(\frac{g}{f}\right)^{\frac{f}{g-f}}\right)$$

is operator monotone if $-2 \le p \le 2$.

Proof. By Proposition 1.2 (for $p \ne 0, 1,$) we have

$$f\sigma_{s_p}g = f \cdot \left(1\sigma_{s_p}\frac{g}{f}\right) = f \cdot \left(\frac{p(\frac{g}{f}-1)}{(\frac{g}{f})^p - 1}\right)^{\frac{1}{1-p}} = \left(\frac{p(f-g)}{f^p - g^p}\right)^{\frac{1}{1-p}}.$$

□

Similarly, we can show:

Theorem 3.3 (cf. [11, Theorem 1.3], [5], [9], [6], [2], [3])**.** *For operator monotone functions* f, g $(f \ne g)$, *the function*

$$f\sigma_{k_p}g = \frac{p-1}{p}\cdot\frac{f^p - g^p}{f^{p-1} - g^{p-1}}, \quad p \ne 0, 1, \left(f\sigma_{k_0}g = \frac{f(\log f - \log g)}{f-g},\ f\sigma_{k_1}g = \frac{f-g}{\log f - \log g}\right)$$

is operator monotone if $-1 \le p \le 2$.

In [11], the following fact was shown, as an extension of Proposition 1.3:

Lemma 3.4 (cf. [11, Theorem 3.2])**.** *For* $-1 \le p \le 1,\ 0 \le s \le 1$, *the function*

$$u_{p,s}(x) = \frac{p}{p+s}\cdot\frac{x^{p+s}-1}{x^p - 1},\ p \ne 0, -s \quad \left(u_{0,s}(x) = \frac{x^s - 1}{\log x^s},\ u_{-s,s}(x) = \frac{\log x^{-s}}{x^{-s}-1}\right)$$

is operator monotone.

For the operator mean corresponding to the function $u_{p,s}$, we can obtain the following theorem which is an extension of Theorem 3.3 (and also Lemma 3.4):

Theorem 3.5. *For operator monotone functions* f, g $(f \ne g)$, *and for* $-1 \le p \le 1,\ 0 \le s \le 1$, *the function*

$$(*) \quad f\sigma_{u_{p,s}}g = \frac{p}{p+s}\cdot\frac{f^{p+s} - g^{p+s}}{f^p - g^p},\ p \ne 0, -s$$

$$\left(f\sigma_{u_{0,s}}g = \frac{f^s - g^s}{\log f^s - \log g^s},\ f\sigma_{u_{-s,s}}g = \frac{f^{-s} - g^{-s}}{\log f^{-s} - \log g^{-s}}\right) \text{ is operator monotone.}$$

Example (cf. [15, Example 2.4])**.** For $-1 \le p \le 1,\ 0 \le q - p \le 1,\ p \ne 0,\ q \ne 0$ (and for $a \ge 0$),

$$\frac{p}{q}\cdot\frac{x^q - a^q}{x^p - a^p} \text{ is operator monotone.}$$

We can obtain this fact, by putting $f = x,\ g = a$, and $q = p + s$ in $(*)$.

As an application of Proposition 1.2, we showed an alternative proof of the following result due to Petz and Hasegawa [14], [6]:

Proposition 3.6 (cf. [11, Theorem 3.4])**.** *For* $-1 \le p \le 2$

$$h_p(x) = \frac{p(1-p)(x-1)^2}{(x^p - 1)(x^{1-p} - 1)},\ p \ne 0, 1 \quad \left(h_0(x) = h_1(x) = \frac{x-1}{\log x}\right)$$

is operator monotone.

As an extension of this fact and an application of Theorem 3.2, though the range of p is reduced, we have:

Theorem 3.7. *If f, g, k, l $(f \neq g, k \neq l)$ are operator monotone functions, then for $0 < p < 1$,*

$$\frac{(f-g)(k-l)}{(f^p - g^p)(k^{1-p} - l^{1-p})} \quad \textit{is operator monotone.}$$

Proof. Since $f\sigma_{s_p}g$ and $k\sigma_{s_{1-p}}l$ are operator monotone, we see $\frac{1}{p(1-p)} \cdot (f\sigma_{s_p}g)\sharp_p(k\sigma_{s_{1-p}}l) = \frac{(f-g)(k-l)}{(f^p-g^p)(k^{1-p}-l^{1-p})}$ is operator monotone. □

Example (cf. [15, Theorem 2.7])**.** Putting $f = k = x$ and $g = a,\ l = b$ $(a, b \geq 0)$, we see that $\frac{(x-a)(x-b)}{(x^p-a^p)(x^{1-p}-b^{1-p})}$ is operator monotone.

Further, we have:

Theorem 3.8. *For $-1 \leq p \leq 2,\ a, b \geq 0$*

$$(**) \qquad h_p(a, b; x) = \frac{p(1-p)(x-a)(x-b)}{(x^p - a^p)(x^{1-p} - b^{1-p})} \quad \textit{is operator monotone.}$$

Proof. We may prove the theorem for $p \neq 0, \pm 1, 2$ and $a, b > 0$. For the case $0 < p < 1$, then $(**)$ is clear. There remain the two cases:

(i) If $1 < p < 2$, then we put $p = q + 1$, so that $0 < q < 1$. We have:

$$\begin{aligned} h_p(a, b; x) = h_{q+1}(a, b; x) &= (-q)(q+1) \cdot \frac{(x-a)(x-b)}{(x^{q+1} - a^{q+1})(x^{-q} - b^{-q})} \\ &= \frac{q(q+1)b^q x^q (x-a)(x-b)}{(x^{q+1} - a^{q+1})(x^q - b^q)}. \end{aligned}$$

Now since $0 < q < 1$, we see that $\left(\frac{q(x-b)}{x^q-b^q}\right)^{\frac{1}{1-q}}$ is operator monotone by Proposition 1.2. Further, since $1 < q+1 < 2$, we see that

$$(\eta(a, b; x) :=) \ \left(\frac{(q+1)(x-a)}{x^{q+1} - a^{q+1}}\right)^{\frac{1}{1-(q+1)}} = \left(\frac{(q+1)(x-a)}{x^{q+1} - a^{q+1}}\right)^{-\frac{1}{q}}$$

is operator monotone by Proposition 1.2, so that its dual $(\eta^{\perp}(a, b; x) =) \ x \cdot \left(\frac{(q+1)(x-a)}{x^{q+1}-a^{q+1}}\right)^{\frac{1}{q}}$ is operator monotone. Hence

$$\left\{ \left(\frac{q(x-b)}{x^q - b^q}\right)^{\frac{1}{1-q}} \sharp_q \ x \cdot \left(\frac{(q+1)(x-a)}{x^{q+1} - a^{q+1}}\right)^{\frac{1}{q}} \right\} \times b^q = h_p(a, b; x)$$

is operator monotone.

(ii) If $-1 < p < 0$, then putting $p = -q$, we can similarly prove $(**)$. □

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