A CHARACTERIZATION OF ω_1-STRONGLY COUNTABLE-DIMENSIONAL SPACES IN TERMS OF K-APPROXIMATIONS

Masahiro Matsumoto

Received October 9, 2012; revised January 7, 2013

Abstract. We give a characterization of ω_1-strongly countable-dimensional metrizable spaces in terms of K-approximations. A characterization of locally finite-dimensional metrizable spaces is also obtained.

1 Introduction The purpose of this paper is to characterize a class of ω_1-strongly countable-dimensional metrizable spaces in terms of K-approximations. A concept of a K-approximation is due to Dydak-Mishra-Shukla.

Definition 1.1. (Dydak-Mishra-Shukla [1; Definition of K-approximations 1.1]) Let X be a normal space, let K be a metric simplicial complex (i.e., a simplicial complex equipped with the metric topology) and let $f : X \to K$ be a continuous mapping. A continuous mapping $g : X \to K$ is a *K-approximation* of f provided for each simplex Δ of K and each $x \in X$, $f(x) \in \Delta$ implies $g(x) \in \Delta$. g is an *n-dimensional* (respectively, *finite-dimensional*) *K-approximation* of f if it is a *K-approximation* and $g(X) \subset K^{(n)}$ (respectively, $g(X) \subset K^{(m)}$ for some m).

Dydak-Mishra-Shukla gave a characterization of n-dimensional spaces in terms of K-approximations. If every finite open cover of a normal space X has a finite open refinement of order $\leq n+1$, then X has *covering dimension* $\leq n$, $\dim X \leq n$.

Theorem 1.2. (Dydak-Mishra-Shukla [1; Theorem 2.2]) *Let n be an integer. For a paracompact space X the following conditions are equivalent:*

(a) $\dim X \leq n$.

(b) *For every metric simplicial complex K and every continuous mapping $f : X \to K$ there is an n-dimensional K-approximation g of f.*

(c) *For every metric simplicial complex K and every continuous mapping $f : X \to K$ there is an n-dimensional K-approximation g of f such that $g|f^{-1}(K^{(n)}) = f|f^{-1}(K^{(n)})$.*

Also, Dydak-Mishra-Shukla characterized finitistic-dimensional spaces. A normal space X is *finitistic* if every open cover of X has an open refinement of finite order.

Theorem 1.3. (Dydak-Mishra-Shukla [1; Theorem 2.1]) *For a paracompact space X the following conditions are equivalent:*

(a) *X is finitistic.*

(b) *For every metric simplicial complex K and every continuous mapping $f : X \to K$ there is a finite-dimensional K-approximation g of f.*

(c) *For every integer $m \geq -1$, every metric simplicial complex K and every continuous mapping $f : X \to K$ there is a finite-dimensional K-approximation g of f such that $g|f^{-1}(K^{(m)}) = f|f^{-1}(K^{(m)})$.*

[1]AMS Subject Classification: Primary 54F45; Secondary 54E35.

[2]Key words and phrases: K-approximation, ω_1-strongly countable-dimension, locally finite-dimension, strong small transfinite dimension

In [5], Y. Hattori extended Theorem 1.2 to strong large transfinite dimensional spaces. A normal space X is said to have *strong large transfinite dimension* if X has both large transfinite dimension and strong small transfinite dimension (see Definition 2.3). For a space X we denote $\mathcal{D}(X) = \{D \mid D$ is a closed discrete subset of $X\}$.

Theorem 1.4. (Y. Hattori [5; Theorem]) *For a metrizable space X the following conditions are equivalent:*

(a) *X has a strong large transfinite dimension.*

(b) *There is a function $\varphi : \mathcal{D}(X) \to \omega$ such that for every metric simplicial complex K and every continuous mapping $f : X \to K$ there is a K-approximation g of f such that $g(D) \subset K^{(\varphi(D))}$ for each $D \in \mathcal{D}(X)$.*

(c) *For every integer $m \geq -1$, there is a function $\psi : \mathcal{D}(X) \to \omega$ such that for every metric simplicial complex K and every continuous mapping $f : X \to K$ there is a finite-dimensional K-approximation g of f such that $g(D) \subset K^{(\psi(D))}$ for each $D \in \mathcal{D}(X)$ and $g|f^{-1}(K^{(m)}) = f|f^{-1}(K^{(m)})$.*

A normal space X is *strongly countable-dimensional* if X can be represented as a countable union of closed finite-dimensional subspaces.

Theorem 1.5. (Y. Hattori [5; Corollary]) *For a paracompact space X the following conditions are equivalent:*

(a) *X is a strongly countable-dimensional space.*

(b) *There is a function $\varphi : X \to \omega$ such that for every metric simplicial complex K and every continuous mapping $f : X \to K$ there is a K-approximation g of f such that $g(x) \in K^{(\varphi(x))}$ for each $x \in X$.*

(c) *For every integer $m \geq -1$, there is a function $\psi : X \to \omega$ such that for every metric simplicial complex K and every continuous mapping $f : X \to K$ there is a K-approximation g of f such that $g(x) \in K^{(\psi(x))}$ for each $x \in X$ and $g|f^{-1}(K^{(m)}) = f|f^{-1}(K^{(m)})$.*

2 Characterizations

In this section, we give a characterization of ω_1-strongly countable-dimensional metrizable spaces in terms of K-approximations. A characterization of locally finite-dimensional metrizable spaces is also obtained.

A notion of a locally finite-dimensional space is well known (cf. [2]).

Definition 2.1. A metrizable space X is *locally finite-dimensional* if for every point $x \in X$ there exists an open subspace U of X such that $x \in U$ and $\dim U < \infty$.

The first infinite ordinal number is denoted by ω and ω_1 is the first uncountable ordinal number. Z. Shmuely introduced and studied ω_1-strongly countable-dimensional spaces ([8]).

Definition 2.2. A metrizable space X is called an ω_1-*strongly countable-dimensional space* if $X = \bigcup\{P_\xi \mid 0 \leq \xi < \xi_0\}$, $\xi_0 < \omega_1$, where P_ξ is an open subset of $X - \bigcup\{P_\eta \mid 0 \leq \eta < \xi\}$ and $\dim P_\xi < \infty$.

For a metrizable space X and a non-negative integer n, we put

$$P_n(X) = \bigcup\{U \mid U \text{ is an open subspace of } X \text{ and } \dim U \leq n\}.$$

We notice that for each ordinal number α, we can put $\alpha = \lambda(\alpha) + n(\alpha)$, where $\lambda(\alpha)$ is a limit ordinal number or 0 and $n(\alpha)$ is a non-negative integer. Strong small transfinite dimension is studied by Y. Hattori (cf. [3]).

Definition 2.3. Let X be a metrizable space and α either an ordinal number ≥ 0 or the integer -1. Then *strong small transfinite dimension sind* of X is defined as follows:

(1) $\operatorname{sind} X = -1$ if and only if $X = \emptyset$.

(2) $\operatorname{sind} X \leq \alpha$ if X is expressed in the form $X = \bigcup\{P_\xi \mid \xi < \alpha\}$, where $P_\xi = P_{n(\xi)}(X - \bigcup\{P_\eta \mid \eta < \lambda(\xi)\})$.

Furthermore, if $\operatorname{sind} X$ is defined, we say that X has *strong small transfinite dimension*.

Clearly, a metrizable space X is locally finite-dimensional if and only if $\operatorname{sind} X \leq \omega$ (cf. [2; Proposition 5.5.3]). And X is ω_1-strongly countable-dimensional if and only if there is a $\xi_0 < \omega_1$ such that $\operatorname{sind} X \leq \xi_0$.

Let X be a metrizable space, let α be an ordinal number and let $\mathcal{F} = \{X_\beta \mid 0 \leq \beta \leq \alpha\}$ be a family of subsets of X. We say that $\mathcal{F}$ is a *closed α-sequence in X* if

(f-1) X_β is closed in X for $\beta \leq \alpha$,

(f-2) $X_0 = X$,

(f-3) $X_\beta \supset X_{\beta'}$ for $\beta \leq \beta' \leq \alpha$,

(f-4) $X_\beta = \bigcap\{X_{\beta'} \mid \beta' < \beta\}$ if β is a limit.

The *power set* of X shall be denoted by $\mathcal{P}(X)$.

Let $N : X \to \mathcal{P}(X)$ be a function and let $\mathcal{F} = \{X_\beta \mid 0 \leq \beta \leq \alpha\}$ be a closed α-sequence in X. We say that N is an *$\mathcal{F}$-neighborhood function* provided that $N(x)$ is an open neighborhood of x in $X_{\beta(x)}$ for each $x \in X$, where $\beta(x) = \max\{\beta \mid x \in X_\beta,\ 0 \leq \beta \leq \alpha\}$.

Remark 2.4. ([6; Remark 2.5]) Let $\{X_\beta \mid 0 \leq \beta \leq \alpha\}$ be a closed α-sequence in X. Then we shall show that for every point x of X, there is a maximum element $\beta(x)$ of $\{\beta \mid x \in X_\beta\}$. Indeed, if $x \in X_{\lambda(\alpha)}$, then $\beta(x) = \max\{\beta \mid x \in X_\beta,\ \lambda(\alpha) \leq \beta \leq \alpha\}$. Now, we suppose that $x \notin X_{\lambda(\alpha)}$, there is a minimum element $\beta_0 > 0$ of $\{\beta \mid x \notin X_\beta\}$. Assume that β_0 is limit. By the condition (f-4), $x \in \bigcap\{X_\beta \mid \beta < \beta_0\} = X_{\beta_0}$. This contradicts the definition of β_0. Therefore β_0 is not limit and hence $\beta(x) = \beta_0 - 1$.

Theorem 2.8 is a main theorem. Thus we characterize the class of ω_1-strongly countable-dimensional metrizable spaces in terms of K-approximations. To prove this theorem, we need Theorem 2.5.

Theorem 2.5. *Let α be an ordinal number with $\alpha < \omega_1$ and let n be a non-negative integer. The following conditions are equivalent for a metrizable space X:*

(a) $\operatorname{sind} X \leq \omega\alpha + n$.

(b) *There are a closed α-sequence $\mathcal{F} = \{X_\beta \mid 0 \leq \beta \leq \alpha\}$ in X, an $\mathcal{F}$-neighborhood function $N : X \to \mathcal{P}(X)$ and a function $\varphi : X \to \omega$ satisfing the following conditions: $X_\alpha = \emptyset$ if $n = 0$, $\varphi(X_\alpha) = n - 1$, and for every metric simplicial complex K and every continuous mapping $f : X \to K$ there is a K-approximation g of f such that $g(N(x)) \subset K^{(\varphi(x))}$ for each $x \in X$.*

(c) *There are a closed α-sequence $\mathcal{F} = \{X_\beta \mid 0 \leq \beta \leq \alpha\}$ in X and an $\mathcal{F}$-neighborhood function $N : X \to \mathcal{P}(X)$, and for every integer $m \geq -1$ there is a function $\psi : X \to \omega$ satisfing the following conditions: $X_\alpha = \emptyset$ if $n = 0$, $\varphi(X_\alpha) = n - 1$, and for every metric simplicial complex K and every continuous mapping $f : X \to K$ there is a K-approximation g of f such that $g(N(x)) \subset K^{(\psi(x))}$ for each $x \in X$ and $g|f^{-1}(K^{(m)}) = f|f^{-1}(K^{(m)})$.*

To prove this theorem, we need the following lemmas. Essentially, the following lemma is the same as [4; Lemma 1.5]. By a minor modification in the proof of [4; Lemma 1.5], we obtain the following lemma.

Lemma 2.6. ([4; Lemma 1.5], [7; Lemma 1]) *Let n be a non-negative integer and let $\{F_m \mid m = 0,\ 1, ...\}$ be a closed cover of a metrizable space X such that $\dim F_m \le (n-1)+m$, $F_m \subset F_{m+1}$ for $m = 0,\ 1, ...$ Then for every open cover $\mathcal{U}$ of X, there are a sequence $\mathcal{V}_1$, $\mathcal{V}_2$, ... of discrete families of open subsets of X and an open cover $\mathcal{W}$ of X which satisfy the following conditions:*

(1) $\bigcup\{\mathcal{V}_k \mid k \in \mathbb{N}\}$ *is a cover of* X.

(2) $\bigcup\{\mathcal{V}_k \mid k \in \mathbb{N}\}$ *refines* $\mathcal{U}$.

(3) *If $W \in \mathcal{W}$ satisfies $W \cap F_m \ne \emptyset$, then W meets at most one member of $\mathcal{V}_k$ for $k \le (n+0)+(n+1)+...+(n+m)$ and meets no member of $\mathcal{V}_k$ for $k > (n+0)+(n+1)+...+(n+m)$.*

Lemma 2.7. ([1; Corollary 1.7]) *Let $f : X \to K$ be a map from a normal space X to a metric simplicial complex K so that $f(A) \subset K^{(n)}$ for some subset A of X. There is a K-approximation g of f so that $g|U$ is an n-dimensional K-approximation of $f|U$ for some open neighborhood U of A in X and $g|A = f|A$.*

Proof of Theorem 2.5. (a) $\Rightarrow$ (b) : Let $\operatorname{sind} X \le \omega\alpha + n$. We use the constraction in [6; Theorem 2.4]. We put

$$Y_\gamma = X - \bigcup\{P_\xi \mid \xi < \gamma\} \quad \text{for} \quad \gamma \le \omega\alpha + n$$

and

$$X_\beta = Y_{\omega\beta} \quad \text{for} \quad \beta \le \alpha.$$

Clearly, $\mathcal{F} = \{X_\beta \mid 0 \le \beta \le \alpha\}$ is a closed α-sequence in X satisfing $X_\alpha = \emptyset$ if $n = 0$.

Notice that $P_{\omega\beta+m}$ is an open subset of X_β such that $P_{\omega\beta+m} \subset P_{\omega\beta+(m+1)}$ for $m = 0, 1, ...$ Also $P_{\omega\alpha+(n-1)}$ is a closed subset of X. Hence for each $\beta \le \alpha$ there is a family $\{W_{\omega\beta+m} \mid m = 0,\ 1, ...\}$ of open subsets of X_β such that

(1) $\overline{W_{\omega\beta+m}} \subset P_{\omega\beta+m}$,

(2) $\overline{W_{\omega\beta+m}} \subset W_{\omega\beta+(m+1)}$,

(3) $\bigcup_{m=0}^{\infty} W_{\omega\beta+m} = \bigcup_{m=0}^{\infty} P_{\omega\beta+m}$.

Since $\{\beta \mid 0 \le \beta < \alpha\}$ is countable, there is a mapping h from ω onto $\{\beta \mid 0 \le \beta < \alpha\}$. For each $m = 0, 1, ...$, we put

$$V_0 = P_{\omega\alpha+(n-1)},$$

$$V_1 = P_{\omega\alpha+(n-1)} \cup W_{\omega h(1)+(n-1)+1},$$

$$V_2 = P_{\omega\alpha+(n-1)} \cup W_{\omega h(1)+(n-1)+2} \cup W_{\omega h(2)+(n-1)+2},$$

...

$$V_m = P_{\omega\alpha+(n-1)} \cup W_{\omega h(1)+(n-1)+m} \cup W_{\omega h(2)+(n-1)+m} \cup ... \cup W_{\omega h(m)+(n-1)+m},$$

...

Then $V_0, V_1, ...$ are subsets of X satisfing the following conditions:

(4) $\overline{V_m} \subset V_{m+1}$.

(5) $\dim \overline{V_m} \le (n-1) + m$.

(6) $X = \bigcup_{m=0}^{\infty} V_m$.

Let $x \in X$. Put $n_0 = \min\{m \mid x \in V_m\}$.

Clearly, if $n_0 = 0$ then $x \in V_0 = P_{\omega\alpha+(n-1)} \subset X_\alpha$. Now we shall show that if $n_0 > 0$, then $x \in W_{\omega\beta(x)+(n-1)+n_0} \subset X_{\beta(x)}$. By the definition of n_0, $x \in V_{n_0} = P_{\omega\alpha+(n-1)} \cup W_{\omega h(1)+(n-1)+n_0} \cup W_{\omega h(2)+(n-1)+n_0} \cup ... \cup W_{\omega h(n_0)+(n-1)+n_0}$. Since $x \notin P_{\omega\alpha+(n-1)}$ by $n_0 > 0$, there is a natural number i such that $x \in W_{\omega h(i)+(n-1)+n_0}$. Hence $x \in W_{\omega h(i)+(n-1)+n_0} \subset P_{\omega h(i)+(n-1)+n_0} \subset X_{h(i)} - X_{h(i)+1}$. Also since $\beta(x) = \max\{\beta \mid x \in X_\beta\} < \alpha$, $x \in X_{\beta(x)} - X_{\beta(x)+1}$. Hence $h(i) = \beta(x)$ and hence $x \in W_{\omega\beta(x)+(n-1)+n_0} \subset X_{\beta(x)}$.

We put

$$N(x) = \begin{cases} P_{\omega\alpha+(n-1)}, & \text{if } n_0 = 0, \\ W_{\omega\beta(x)+(n-1)+n_0}, & \text{if } n_0 > 0. \end{cases}$$

Since $N(x)$ is an open neighborhood of x in $X_{\beta(x)}$, $N : X \to \mathcal{P}(X)$ is an $\mathcal{F}$-neighborhood function.

Put $\varphi(x) = (n+0) + (n+1) + ... + (n+n_0) - 1$. Then $\varphi(X_\alpha) = n-1$.

The latter half of the proof is similar to the proof of [5; Theorem]. Let K be a metric simplicial complex and let $f : X \to K$ be a continuous mapping. By Lemma 2.6, there are a sequence $\mathcal{U}_1, \mathcal{U}_2, ...$ of discrete families of open subsets of X and an open cover $\mathcal{W}$ of X which satisfy the following conditions:

(7) $\bigcup_{k=1}^{\infty} \mathcal{U}_k$ is a cover of X.

(8) $\bigcup_{k=1}^{\infty} \mathcal{U}_k$ refines $\{f^{-1}(St(v,K)) \mid v \in K^{(0)}\}$.

(9) If $W \in \mathcal{W}$ satisfies $W \cap \overline{V_m} \neq \emptyset$, then W meets at most one member of $\mathcal{U}_k$ for $k \leq (n+0)+(n+1)+...+(n+m)$ and meets no member of $\mathcal{U}_k$ for $k > (n+0)+(n+1)+...+(n+m)$.

Then $\mathcal{U} = \bigcup_{k=1}^{\infty} \mathcal{U}_k$ is a locally finite open cover of X by (7) and (9). For each $U \in \mathcal{U}$ there is $v(U) \in K^{(0)}$ such that $U \subset f^{-1}(St(v(U),K))$ by (8). For each $v \in K^{(0)}$ we put $Q_v = \bigcup\{U \in \mathcal{U} \mid v(U) = v\}$, and $\mathcal{Q} = \{Q_v \mid v \in K^{(0)}\}$. Then $\mathcal{Q}$ is a locally finite open cover of X such that $Q_v \subset f^{-1}(St(v,K))$ for each $v \in K^{(0)}$. Let $\{\kappa_v \mid v \in K^{(0)}\}$ be a partition of unity subordinated to $\mathcal{Q}$. We define $g : X \to K$ as $g(x) = \sum_{v \in K^{(0)}} \kappa_v(x)v$. Then g is a K-approximation of f.

Now, let $x \in X$. Notice that $N(x) \subset V_{n_0} \subset \overline{V_{n_0}}$. By (9), $\mathrm{ord}_y\, \mathcal{Q} \leq \mathrm{ord}_y\, \mathcal{U} \leq \varphi(x)+1$ for each $y \in N(x)$. Hence $g(y) \in K^{(\varphi(x))}$ for each $y \in N(x)$ and hence $g(N(x)) \subset K^{(\varphi(x))}$.

(b) $\Rightarrow$ (a) : We use the proof of [6; Theorem 2.4]. We shall show that for every $\beta \leq \alpha$

$$(10)\quad X - \bigcup\{P_\xi \mid \xi < \omega\beta\} \subset X_\beta.$$

The validity of (10) is clear for $\beta = 0$. To prove (10) by transfinite induction we assume (10) for $\gamma < \beta$. Let $x \notin X_\beta$. Notice that $\beta(x) < \beta$.

If $x \in \bigcup\{P_\xi \mid \xi < \omega\beta(x)\}$, then $x \in \bigcup\{P_\xi \mid \xi < \omega\beta\}$ by $\beta(x) < \beta$.

We shall also show that if $x \in X - \bigcup\{P_\xi \mid \xi < \omega\beta(x)\}$, then $x \in \bigcup\{P_\xi \mid \xi < \omega\beta\}$. Since $N(x)$ is an open neighborhood of x in $X_{\beta(x)}$, by the induction hypothesis, $N(x) \cap (X - \bigcup\{P_\xi \mid \xi < \omega\beta(x)\})$ is an open neighborhood of x in $X - \bigcup\{P_\xi \mid \xi < \omega\beta(x)\}$. There is an open neighborhood $V(x)$ of x in $X - \bigcup\{P_\xi \mid \xi < \omega\beta(x)\}$ such that

$$\overline{V(x)} \subset N(x) \cap (X - \bigcup\{P_\xi \mid \xi < \omega\beta(x)\}).$$

Let $\mathcal{U}$ be a finite open cover of $\overline{V(x)}$. Given $U \in \mathcal{U}$, choose an open subset $\tilde{U}$ of X such that $\tilde{U} \cap \overline{V(x)} = U$. Put $\tilde{\mathcal{U}} = \{\tilde{U} \mid U \in \mathcal{U}\} \cup \{X - \overline{V(x)}\}$. We index a covering $\tilde{\mathcal{U}}$ as $\tilde{\mathcal{U}} = \{U_v \mid v \in S\}$. We use the proof of [1; Theorem 2.1]. Choose a partition of unity $\{\alpha_v \mid v \in S\}$ of X with $\alpha_v^{-1}(0,1] \subset U_v$ for all $v \in S$ and notice that $f(y) = \sum_{v \in S} \alpha_v(y)v$ defines a map $f : X \to K$, where K is the full complex with S as its set of vertices. Then, by (b), there is a K-approximation g of f such that $g(N(y)) \subset K^{(\varphi(y))}$ for each $y \in X$. Notice that $g^{-1}(St(v,K)) \subset U_v$ for all $v \in S$ and $\tilde{\mathcal{V}} = \{g^{-1}(St(v,K)) \mid v \in S\}$ is an open cover of

X. In particular, $g(\overline{V(x)}) \subset g(N(x)) \subset K^{(\varphi(x))}$. Then $\mathcal{V} = \{\tilde{V} \cap \overline{V(x)} \mid \tilde{U} \in \tilde{\mathcal{V}}\}$ is a finite open cover of $\overline{V(x)}$ such that $\mathcal{V}$ is a refinement of $\mathcal{U}$ and $\sup\{\mathrm{ord}_y \mathcal{V} \mid y \in \overline{V(x)}\} \leq \varphi(x)+1$. Hence

$$(11) \quad \dim V(x) \leq \dim \overline{V(x)} \leq \varphi(x).$$

We use the proof of [6; Theorem 2.4].

$$x \in V(x) \subset P_{\varphi(x)}(X - \bigcup\{P_\xi \mid \xi < \omega\beta(x)\}) = P_{\omega\beta(x)+\varphi(x)}$$

$$\subset \bigcup\{P_\xi \mid \xi < \omega(\beta(x)+1)\} \subset \bigcup\{P_\xi \mid \xi < \omega\beta\}.$$

Thus, (10) holds.

In particular,

$$(12) \quad X - \bigcup\{P_\xi \mid \xi < \omega\alpha\} \subset X_\alpha.$$

We use the proof of [6; Theorem 2.4]. We shall show that

$$(13) \quad X - \bigcup\{P_\xi \mid \xi < \omega\alpha\} \subset \bigcup\{P_\xi \mid \omega\alpha \leq \xi < \omega\alpha + n\}.$$

If $n = 0$ then $X_\alpha = \emptyset$, and hence $X - \bigcup\{P_\xi \mid \xi < \omega\alpha\} = \emptyset$ by (12).

Assume that $n > 0$. Let $x \in X - \bigcup\{P_\xi \mid \xi < \omega\alpha\}$. Since $x \in X_\alpha$ by (12), $\beta(x) = \alpha$. Hence $N(x)$ is an open neighborhood of x in X_α. By (12), $N(x) \cap (X - \bigcup\{P_\xi \mid \xi < \omega\alpha\})$ is an open neighborhood of x in $X - \bigcup\{P_\xi \mid \xi < \omega\alpha\}$. There is an open neighborhood $V(x)$ of x in $X - \bigcup\{P_\xi \mid \xi < \omega\alpha\}$ such that

$$\overline{V(x)} \subset N(x) \cap (X - \bigcup\{P_\xi \mid \xi < \omega\alpha\}).$$

By the proof of (11), $\dim V(x) \leq \dim \overline{V(x)} \leq \varphi(x)$. Furthermore $\varphi(x) = n-1$ by $x \in X_\alpha$. Hence,

$$x \in V(x) \subset P_{\varphi(x)}(X - \bigcup\{P_\xi \mid \xi < \omega\alpha\}) = P_{\omega\alpha+\varphi(x)}$$

$$\subset \bigcup\{P_\xi \mid \omega\alpha \leq \xi \leq \omega\alpha + \varphi(x)\} \subset \bigcup\{P_\xi \mid \omega\alpha \leq \xi < \omega\alpha + n\}.$$

Thus, (13) holds.

Therefore $X = \bigcup\{P_\xi \mid 0 \leq \xi < \omega\alpha + n\}$ and hence $\mathrm{sind}\, X \leq \omega\alpha + n$.

(b) $\Rightarrow$ (c) : The proof is similar to the proof of [5; Theorem]. For completeness, we give the proof. Let $m \geq -1$. In addition, let $\varphi : X \to \omega$ be as in (b). We put $\psi(x) = \max\{m, \varphi(x)\}$ for each $x \in X$. Let K be a metric simplicial complex and let $f : X \to K$ be a continuous mapping. By Lemma 2.7, there are an open subset U of X and a K-approximation g_1 of f such that $f^{-1}(K^{(m)}) \subset U$, $g_1|f^{-1}(K^{(m)}) = f|f^{-1}(K^{(m)})$ and $g_1|U$ is an m-dimensional K-approximation of $f|U$. Then, by (b), there is a K-approximation g_2 of g_1 such that $g_2(N(x)) \subset K^{(\varphi(x))}$ for each $x \in X$. Let $\kappa : X \to [0,1]$ be a continuous mapping such that $\kappa(f^{-1}(K^{(m)})) = 1$ and $\kappa(X - U) = 0$. We define $g(x) = \kappa(x)g_1(x) + (1-\kappa(x))g_2(x)$ for each $x \in X$. Then g is a K-approximation of f and $g(N(x)) \subset K^{(\psi(x))}$ for each $x \in X$.

(c) $\Rightarrow$ (b) is obvious. □

We obtain the Main Theorem 2.8 and Theorem 2.9.

Theorem 2.8. *The following conditions are equivalent for a metrizable space X:*

(*a*) *X is an ω_1-strongly countable-dimensional space.*

(*b*) *There are an ordinal number $\alpha < \omega_1$, a closed α-sequence $\mathcal{F} = \{X_\beta \mid 0 \leq \beta \leq \alpha\}$ in X, an $\mathcal{F}$-neighborhood function $N : X \to \mathcal{P}(X)$ and a function $\varphi : X \to \omega$ satisfing the following condition: For every metric simplicial complex K and every continuous mapping $f : X \to K$ there is a K-approximation g of f such that $g(N(x)) \subset K^{(\varphi(x))}$ for each $x \in X$.*

(*c*) *There are an ordinal number $\alpha < \omega_1$, a closed α-sequence $\mathcal{F} = \{X_\beta \mid 0 \leq \beta \leq \alpha\}$ in X and an $\mathcal{F}$-neighborhood function $N : X \to \mathcal{P}(X)$, and for every integer $m \geq -1$ there is a function $\psi : X \to \omega$ satisfing the following condition: For every metric simplicial complex K and every continuous mapping $f : X \to K$ there is a K-approximation g of f such that $g(N(x)) \subset K^{(\psi(x))}$ for each $x \in X$ and $g|f^{-1}(K^{(m)}) = f|f^{-1}(K^{(m)})$.*

Proof. (b) $\Rightarrow$ (a) : Refer to the proof of [6; Theorem 2.9]. By the proof of (13) of Theorem 2.5, $X - \bigcup\{P_\xi \mid \xi < \omega\alpha\} \subset \bigcup\{P_\xi \mid \omega\alpha \leq \xi < \omega\alpha + \omega\}$. Hence $\operatorname{sind} X \leq \omega\alpha + \omega$, and hence X is an ω_1-strongly countable-dimensional space.

The implications (a) $\Rightarrow$ (b), (b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (b) are obvious by proofs of Theorem 2.5 and [5; Theorem]. □

Notice that if $N : X \to \mathcal{P}(X)$ is an $\{X\}$-neighborhood function then $N(x)$ is an open neighborhood of x in X for each $x \in X$.

Theorem 2.9. *The following conditions are equivalent for a metrizable space X:*

(*a*) *X is a locally finite-dimensional space.*

(*b*) *There are an $\{X\}$-neighborhood function $N : X \to \mathcal{P}(X)$ and a function $\varphi : X \to \omega$ satisfing the following condition: For every metric simplicial complex K and every continuous mapping $f : X \to K$ there is a K-approximation g of f such that $g(N(x)) \subset K^{(\varphi(x))}$ for each $x \in X$.*

(*c*) *There is an $\{X\}$-neighborhood function $N : X \to \mathcal{P}(X)$, and for every integer $m \geq -1$ there is a function $\psi : X \to \omega$ satisfing the following condition: For every metric simplicial complex K and every continuous mapping $f : X \to K$ there is a K-approximation g of f such that $g(N(x)) \subset K^{(\psi(x))}$ for each $x \in X$ and $g|f^{-1}(K^{(m)}) = f|f^{-1}(K^{(m)})$.*

References

[1] J. Dydak, S. N. Mishra and R. A. Shukla, *On finitistic spaces*, Topology Appl. 97 (1999), no. 3, 217-229.

[2] R. Engelking, *Theory of Dimensions, Finite and Infinite*, Heldermann Verlag (1995).

[3] Y. Hattori, *On spaces related to strongly countable-dimensional spaces*, Math. Japonica 28 (1983), no. 5, 583-593.

[4] Y. Hattori, *On special metrics characterizing topological properties*, Fund. Math. 126 (1986), no. 2, 133-145.

[5] Y. Hattori, *K-approximations and strongly countable-dimensional spaces*, Proc. Japan Acad. Ser. A Math. Sci. 75 (1999), no. 7, 115-117.

[6] M. Matsumoto, *A characterization of ω_1-strongly countable-dimensional matrizable spaces*, Sci. Math. Jpn. 66 (2007), no. 3, 335-343.

[7] J. Nagata, *Topics in dimension theory*, General Topology and its Relations to Modern Analysis and Algebra (Proc. Fifth Prague Topology Symposium 1981), Heldermann Verlag, Berlin (1982), 497-507.

[8] Z. Shmuely, *On strongly countable-dimensional sets*, Duke Math. J. 38 (1971), 169-173.

Communicated by *Yasunao Hattori*

Masahiro Matsumoto
Mathematics and Statistics, Division of Premedical Sciences, Premedical Sciences, Dokkyo Medical University,
Kitakobayashi 880, Mibu, Tochigi, 321-0293, Japan
e-mail: m-masa@dokkyomed.ac.jp