SUBSPACE-OPERATIONS ON TOPOLOGICAL SPACES *

Haruo Maki, G.Sai Sundara Krishnan and Hayao Ogata

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Abstract. Some concepts of *operations on a subspace-topology* are presented on a subspace of a given topological space (Definitions 2.1, 2.3) and some operation-open sets on the subspace are introduced (Definition 3.2). And we investigate some relationships among families of such operation-open sets (Theorem 3.9, Corollary 3.11). As an application, we have an operation-closure formula for a subspace (cf. Theorem 4.5).

1 Introduction

In 1979, Kasahara [2] introduced the concepts of *operation* in topological spaces and *operation-closed graph* of a function; and he unified several characterizations of compact spaces, nearly-compact spaces and H-closed spaces. In 1983, Janković [1] introduced and studied the concept of *operation-closures* of a subset, *operation-closed sets* (in the sense of Janković) in a topological space and several related topics. After the works above, in 1991, Ogata [4] introduced the concept of *operation-open sets* and investigated the related topological properties of the associated family of all the operation-open sets with a given topology and a given operation. Moreover he introduced the concept of *operation-T_i spaces*, where $i \in \{0, 1/2, 1, 2\}$.

Throughout the present paper, for a nonempty set X, (X, τ) always denote a topological space on which no separation axioms are assumed unless explicitly stated. In the present paper, we use the notation in Ogata's papers [4], [5]: a function $\gamma : \tau \to P(X)$ is called an *operation on* τ, if $U \subset U^\gamma$ holds for every set $U \in \tau$, where $U^\gamma := \gamma(U)$ (the value of U by γ) and $P(X)$ denotes the power set of X. Let τ^γ be the family of all γ-open sets in (X, τ).

The purpose of the present paper is to present and study the concept of *subspace-operations* (i.e., operations on subspaces) and subspace-operation-open sets for a given operation; and also we study some topological properties of such subspace-operation-open sets (cf. Section 2, Section 3). In Section 2, two operations $\gamma_H^O : \tau|H \to P(H)$ (Definition 2.1 below) and $\gamma_{(H)} : \tau_{(H)} \to P(H)$ (Definition 2.3 below) are introduced and studied. In Section 3, the concept of operation-open sets relative to H is introduced (Definition 3.2) and basic properties are investigated (cf. Theorem 3.5). And we investigate some relationships among the family τ_H^γ of all operation γ-open sets relative to H, the family of all γ_H^O-open sets on $\tau|H$, the family $\tau^\gamma|H$ and the subspace topology $\tau|H$ (Theorem 3.8(ii), Theorem 3.9, Theorem 3.10, Corollary 3.11). In Section 4, we give an operation-closure formula for such subspace-operation γ_H^O on $\tau|H$ and a given operaion γ on τ (Theorem 4.5).

2 Some operations on subspaces

In the present section, we introduce the following two *subspace-operations* (Definition 2.1, Definition 2.3). Namely, we define some concepts of operations, say γ_H^O and $\gamma_{(H)}$, on a subspace $(H, \tau|H)$ of a topological space (X, τ) for a given operation $\gamma : \tau \to P(X)$ and a subset H of X (note: we assume $H \in \tau$ if $\gamma \neq$ "*id*" in the concept of Definition 2.1 below and $\tau|H := \{U \cap H |\ U \in \tau\}$). Let "*id*" : $\tau \to P(X)$ be the identity operation defined by "*id*"$(U) = U$ for every $U \in \tau$.

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Definition 2.1 Let $\gamma : \tau \to P(X)$ be an operation on τ. Suppose that H is an open subset of (X, τ) if $\gamma \neq$ "*id*". An operation $\gamma_H^O : \tau|H \to P(H)$ is well defined as follows:
• if $\gamma \neq$ "*id*" $: \tau \to P(X)$, then $\gamma_H^O(U \cap H) := (U \cap H)^\gamma \cap H$ for every $U \cap H \in \tau|H$, where $(U \cap H)^\gamma := \gamma(U \cap H)$ (the value of γ at $U \cap H \in \tau$); and
• if $\gamma =$ "*id*" $: \tau \to P(X)$, then $\gamma_H^O(U \cap H) := U \cap H$ for every $U \cap H \in \tau|H$.
The former is well defined, because of $U \cap H \in \tau$ by assumption. This operation γ_H^O is said to be *the restriction of* γ *on* $\tau|H$.

Remark 2.2 When we consider the operation $\gamma_H^O : \tau|H \to P(H)$, we assume $H \in \tau$ if $\gamma \neq$ "*id*". And, if $\gamma =$ "*id*" $: \tau \to P(X)$, then we do not assume that $H \in \tau$. Namely, even if $H \notin \tau$, by definition, for any subset H of (X, τ), "*id*"$_H^O : \tau|H \to P(H)$ is the identity operation on $\tau|H$. Indeed, "*id*"$_H^O(U) = U = U^{\text{"}id\text{"}}$ for any $U \in \tau|H$.

We note that, in the following Definition 2.3, the openness of H is not assumed.

Definition 2.3 Let $(H, \tau|H)$ be a subspace of a topological space (X, τ).
(i) Let $\tau_{(H)}$ denotes the following family of subsets of H:
• $\tau_{(H)} := \{U |\ U \subset H, U \in \tau\}$.
(ii) (cf. Remark 2.4 (ii) below) For an operation $\gamma : \tau \to P(X)$, the following operation $\gamma_{(H)}$ on the family $\tau_{(H)}$ is well defined:
• $\gamma_{(H)} : \tau_{(H)} \to P(H)$ is defined by $\gamma_{(H)}(U) := U^\gamma \cap H \in P(H)$ for every $U \in \tau_{(H)}$, where $U^\gamma := \gamma(U)$ (the value of γ at $U \in \tau_{(H)} \subset \tau$).

Remark 2.4 (i) The following properties are well known.
(i-1) $\tau_{(H)} \subset \tau|H \subset P(H)$.
(i-2) If H is open in (X, τ), then $\tau_{(H)}$ is a topology of H and $\tau_{(H)} = \tau|H$.
(ii) Let $\gamma : \tau \to P(X)$ be an operation on τ. The following properties are shown.
(ii-1) (cf. Definition 2.3(ii)) $\gamma_{(H)} : \tau_{(H)} \to P(H)$ is an operation on $\tau_{(H)}$, because $\gamma_{(H)}(U) = U^\gamma \cap H \supset U \cap H = U$ hold for a subset $U \in \tau_{(H)}$.
(ii-2) If H is open in (X, τ), then $\gamma_{(H)} = \gamma_H^O : \tau|H \to P(H)$.
(iii) A correspondence from $\tau|H$ into $P(H)$, say $f : \tau|H \to P(H)$, defined by $f(W \cap H) := W^\gamma \cap H$ is not well defined, where $W \in \tau$ and $\gamma : \tau \to P(X)$ is a given operation on τ. Indeed, for some topological space (X, τ) and a subset H of X, we can take two open sets W and S of (X, τ) such that $W \cap H = S \cap H$ and $W^\gamma \cap H \neq S^\gamma \cap H$; thus f is not well defined.
For example, let $X := \{a, b, c\}, \tau := \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$ and $H := \{a, c\}$. And let $\gamma : \tau \to P(X)$ be a given operation defined by $\gamma(U) := U$ if $b \in U; \gamma(U) := Cl(U)$ if $b \notin U$. Then, we take $W := \{a, b\} \in \tau, S := \{a\} \in \tau$; then $W \cap H = \{a\} = S \cap H$ and $W^\gamma \cap H = \{a, b\}^\gamma \cap H = \{a, b\} \cap H = \{a\}$ and $S^\gamma \cap H = \{a\}^\gamma \cap H = Cl(\{a\}) \cap H = \{a, c\} \cap H = \{a, c\}$. Thus, $W^\gamma \cap H \neq S^\gamma \cap H$ and so $f(W \cap H) \neq f(S \cap H)$, even if $W \cap H = S \cap H$ holds.
(iii)' We note that our operation $\gamma_H^O : \tau|H \to P(H)$ of Definition 2.1 is well defined. Indeed, in the case where $\gamma \neq$ "*id*", we assume $H \in \tau$ and so we have $W \cap H \in \tau$ for any $W \in \tau$ and $(W \cap H)^\gamma$ is well defined; and hence $\gamma_H^O(W \cap H) := (W \cap H)^\gamma \cap H$ is well defined. We note the difference of definitions of γ_H^O (cf. Definition 2.1) and f of (iii) above. In the case where $\gamma =$ "*id*", even if $H \notin \tau$, we have $\gamma_H^O(W \cap H) = W \cap H$ and so γ_H^O is well defined.

3 Operation-open sets relative to subspaces In the present section, we introduce the concept of γ*-open sets relative to a subset* H, where $\gamma : \tau \to P(X)$ is a given operation on τ, and we investigate general properties of them. For an operation $\gamma : \tau \to P(X)$ on τ, we recall the definition of γ*-open sets* in (X, τ) as follow (cf. [4, Definition 2.2]). It is well known that every γ-open set of (X, τ) is open in (X, τ), where γ is an operation on τ.

Definition 3.1 [4] (i) A nonempty subset A of (X,τ) is said to be *γ-open* in (X,τ), if for each point $x \in A$ there exists a subset $U \in \tau$ such that $x \in U$ and $U^\gamma \subset A$. We suppose that the emptyset $\emptyset$ is γ-open in (X,τ). A subset F of (X,τ) is called *γ-closed* in (X,τ) if $X \setminus F$ is γ-open in (X,τ) in the sense of the above.

(ii) Let τ^γ denotes the collection of all γ-open sets of (X,τ). Namely,
• $A \in \tau^\gamma$ if and only if $A = \emptyset$ or for any point $x \in A$ there exists a subset $U \in \tau$ such that $x \in U$ and $U^\gamma \subset A$.
(Note: the notation τ^γ above is denoted by τ_γ in [4] (e.g.,[3])).

Definition 3.2 Let $(H, \tau|H)$ be a subspace of (X,τ) and $\gamma : \tau \to P(X)$ be an operation on τ.

(i) A nonempty subset A of a subspace $(H,\tau|H)$ is said to be *γ-open relative to H*, if for each point $x \in A$ there exists a subset $U \in \tau$ such that $x \in U$ and $U^\gamma \cap H \subset A$. Suppose that the empty set $\emptyset$ is γ-open relative to H. A subset F of $(H,\tau|H)$ is said to be *γ-closed relative to H*, if $H \setminus F$ is γ-open relative to H.

(ii) Let τ_H^γ denotes the collection of all γ-open sets relative to H. Namely,
• $A \in \tau_H^\gamma$ if and only if $A = \emptyset$ or for each point $x \in A$ there exists an open set U of (X,τ) such that $x \in U$ and $U^\gamma \cap H \subset A$.

Remark 3.3 In Definition 3.1 and Definition 3.2, we assume that $\gamma : \tau \to P(X)$ is the identity operation, say "*id*", i.e., "*id*"$(U) = U$ for every $U \in \tau$. Then, we have the following property:
(i) $\tau^{\text{“}id\text{”}} = \tau$; (ii) $\tau_H^{\text{“}id\text{”}} = \tau|H$.

We recall the following well known definition.

Definition 3.4 (i) ([2], e.g., [4, Definition 2.5]) An operation $\gamma : \tau \to P(X)$ is *regular* on τ [2] (e.g., [4]) if for every open neighbourhoods U and V of each point $x \in X$ there exists an open set W such that $x \in W$ and $W^\gamma \subset U^\gamma \cap V^\gamma$.

(ii) An operation $\gamma : \tau \to P(X)$ is said to be *monotone* if $A^\gamma \subset B^\gamma$ whenever $A \subset B$, $A \in \tau$ and $B \in \tau$.

It is well known that: every monotone operation is regular; and if $\gamma : \tau \to P(X)$ is regular on τ, then the collection τ^γ of all γ-open sets forms a topology of X (cf. [4, Proposition 2.9]).

Theorem 3.5 *Let $\gamma : \tau \to P(X)$ be an operation on τ and H a subset of X.*

(i) *The union of any family of γ-open sets relative to H is a γ-open set relative to H.*

(ii) *If $\gamma : \tau \to P(X)$ is regular on τ, then the intersection of two γ-open sets relative to H is also γ-open relative to H.*

(iii) *If $\gamma : \tau \to P(X)$ is regular on τ, then the family τ_H^γ forms a topology of H.*

Proof. **(i)** Let $\{A_i |\ i \in \Omega\}$ be a family of γ-open sets relative to H, where Ω is an index set. Put $A := \bigcup\{A_i |\ i \in \Omega\}$. Let $x \in A$. There exists a γ-open set A_i relative to H such that $x \in A_i$, where $i \in \Omega$. Then, there exists a subset $U(i) \in \tau$ such that $x \in U(i)$ and $U(i)^\gamma \cap H \subset A_i \subset A$. We prove that A is γ-open relative to H.

(ii) Let B and E be γ-open sets relative to H. Let $x \in B \cap E$. There exist two open neighbourhoods U and V of the point x such that $U^\gamma \cap H \subset B$ and $V^\gamma \cap H \subset E$. Since γ is regular on τ, there exists an open neighbourhood W of x such that $W^\gamma \subset U^\gamma \cap V^\gamma$; then $W^\gamma \cap H \subset (U^\gamma \cap H) \cap (V^\gamma \cap H) \subset B \cap E$. Therefore, $B \cap E$ is γ-open relative to H.

(iii) For the set H, we have $H \in \tau_H^\gamma$. Indeed, for a point $x \in H, X$ is the open neighbourhood of x such that $X^\gamma \cap H \subset H$. Since $\emptyset \in \tau_H^\gamma$ by definition, using (i) and (ii) above, it is concluded that τ_H^γ is a topology of H. □

We need the following notation.

Definition 3.6 (i) For a subset H of (X, τ) and an operation $\gamma : \tau \to P(X)$,

• $\tau^\gamma|H := \{V \cap H \in P(H)|\ V$ is γ-open in (X, τ), i.e., $V \in \tau^\gamma\}$ (cf. Definiton 3.1).

(ii) Suppose that H is open in (X, τ) if $\gamma \neq$ "*id*". For an operation $\gamma_H^O : \tau|H \to P(H)$ (cf. Definition 2.1),

• $(\tau|H)^{\gamma_H^O} := \{A \in P(H)|A$ is γ_H^O-open in $(H, \tau|H)\}$ (cf. Definition 2.1, Definition 3.1 for a topological subspace $(H, \tau|H)$).

Remark 3.7 In Definition 3.6, especially we assume that $\gamma : \tau \to P(X)$ is the identity operation. Note: we do not assume that $H \in \tau$(cf. Definition 2.1, Definition 3.6 (ii)). Then, we have the following properties:
(i) $\tau^{\text{“}id\text{”}}|H = \tau|H$;
(ii) $(\tau|H)^{\text{“}id\text{”}_H^O} = \{A \in P(H)|A$ is "*id*"$_H^O$-open in $(H, \tau|H)\} = \{A \in P(H)|A$ is "*id*"-open in $(H, \tau|H)\} = \tau|H$, because "*id*"$_H^O$ is the identity operation by Definition 2.1.

The following Theorem 3.8, Theorem 3.9, Theorem 3.10 and Corollary 3.11 show some properties on the relations among the families $\tau^\gamma|H$, τ_H^γ, $(\tau|H)^{\gamma_H^O}$ and $\tau|H$ under some assumptions.

Theorem 3.8 (i) *Let $\gamma : \tau \to P(X)$ be an operation on τ and let H subsets of X.*

(i-1) *If a subset B of (X, τ) is γ-open in (X, τ), then $B \cap H$ is γ-open relative to H; namely,*

• $\tau^\gamma|H \subset \tau_H^\gamma$ *holds* (cf. Definition 3.2(ii)).

(i-2) *Every γ-open set relative to H is open in $(H, \tau|H)$; namely,*

• $\tau_H^\gamma \subset \tau|H$ *holds.*

(ii) • $\tau^\gamma|H \subset \tau_H^\gamma \subset \tau|H$ *hold for any subset H of (X, τ) and any operation $\gamma : \tau \to P(X)$.*

Proof. **(i)** **(i-1)** Let $x \in B \cap H$. It follows from assumption that there exists an open subset U of (X, τ) such that $x \in U$ and $U^\gamma \subset B$ and hence $U^\gamma \cap H \subset B \cap H$. Thus, $B \cap H$ is γ-open relative to H (cf. Definition 3.2(i)). Let $V \in \tau^\gamma|H$. There exists a subset $B \in \tau^\gamma$ such that $V = B \cap H$; and so $V \in \tau_H^\gamma$ (cf. the former result above and Definition 3.2(ii)). Thus we have the implication $\tau^\gamma|H \subset \tau_H^\gamma$.

(i-2) Let V be a nonempty γ-open set relative to H. For each point $x \in V$, there exists a subset $U(x) \in \tau$ such that $x \in U(x)$ and $U(x)^\gamma \cap H \subset V$. Using the above subsets $U(x)$ for each point $x \in V$, we define a family $\mathcal{U}_V := \{U(x) \cap H|\ x \in V, x \in U(x), U(x) \in \tau, U(x)^\gamma \cap H \subset V\}$. First we claim that $(*1)$ $V = \bigcup\{U|\ U \in \mathcal{U}_V\}$ holds. By the definition of $\mathcal{U}_V$, it is obtained that $(*2)$ $V \subset \bigcup\{U|\ U \in \mathcal{U}_V\}$. Conversely, we have $(*3)$ $\bigcup\{U|\ U \in \mathcal{U}_V\} \subset V$. Indeed, let $y \in \bigcup\{U|\ U \in \mathcal{U}_V\}$; then there exists a subset $W \in \mathcal{U}_V$ such that $y \in W$. This means that there exists a subset $U(x) \in \tau$ such that $W = U(x) \cap H, x \in V, x \in U(x), U(x)^\gamma \cap H \subset V$; and so $y \in H, y \in U(x) \subset U(x)^\gamma$ and $y \in U(x)^\gamma \cap H$, because γ is an operation on τ. Since $U(x)^\gamma \cap H \subset V$, we have $y \in V$; and hence we have $(*3)$ above. By $(*2)$ and $(*3)$ above, the property $(*1)$ is obtained. Finally, by definitions, it is shown that $V = \bigcup\{U|\ U \in \mathcal{U}_V\} = U_1 \cap H$ and $U_1 \in \tau$, where $U_1 := \bigcup\{U(x)|\ x \in V, x \in U(x), U(x) \in \tau, U(x)^\gamma \cap H \subset V\}$. Thus, V is open in $(H, \tau|H)$. Using Definition 3.2(ii), we have the implication $\tau_H^\gamma \subset \tau|H$.

(ii) By (i-1) (resp. (i-2)), it is obtained that $\tau^\gamma|H \subset \tau_H^\gamma$ (resp. $\tau_H^\gamma \subset \tau|H$) hold. □

Theorem 3.9 (i) *Suppose that* $\gamma \neq$ "*id*" *and* H *is open in* (X, τ). *Then, every* γ_H^O*-open set in* $(H, \tau|H)$ *is* γ*-open relative to* H*; namely,*

- $(\tau|H)^{\gamma_H^O} \subset \tau_H^\gamma$ *holds* (cf. Definition 2.1).

(ii) *Suppose that* $\gamma : \tau \to P(X)$ *is a monotone operation such that* $\gamma \neq$ "*id*" *and* H *is open in* (X, τ). *Under the assumptions above, we have the following properties.*

(ii-1) *Every* γ*-open set relative to* H *is* γ_H^O*-open in* $(H, \tau|H)$.

(ii-2) • $\tau_H^\gamma \subset (\tau|H)^{\gamma_H^O}$ *holds; and hence* • $\tau_H^\gamma = (\tau|H)^{\gamma_H^O}$ *holds under the assumption of* (ii).

Proof. **(i)** Let A be a γ_H^O-open set in $(H, \tau|H)$ (i.e., $A \in (\tau|H)^{\gamma_H^O}$), where $\gamma_H^O : \tau|H \to P(H)$ is an operation on $\tau|H$ and $\gamma \neq$ "*id*". Let $x \in A$. There exists a subset $W \in \tau|H$ such that $x \in W$ and $\gamma_H^O(W) = W^\gamma \cap H \subset A$ (Note: $\gamma \neq$ "*id*" and using the assumption that H is open, we have $W \in \tau$). Thus, for the point $x \in A$, $W \in \tau$ such that $W^\gamma \cap H \subset A$. This shows that A is γ-open relative to H, i.e., $A \in \tau_H^\gamma$.

(ii) (ii-1) Let A be a γ-open set relative to H (i.e., $A \in \tau_H^\gamma$). Let $x \in A$. There exists a subset $U(x) \in \tau$ such that $x \in U(x)$ and $U(x)^\gamma \cap H \subset A$. Since $\gamma \neq$ "*id*", $H \in \tau$ and γ is monotone, we have $\gamma_H^O(U(x) \cap H) = (U(x) \cap H)^\gamma \cap H \subset U(x)^\gamma \cap H \subset A$. This shows that for the point $x \in A$, we have $\gamma_H^O(U(x) \cap H) \subset A$ and $U(x) \cap H \in \tau|H$; and so A is γ_H^O-open in $(H, \tau|H)$ (i.e., $A \in (\tau|H)^{\gamma_H^O}$).

(ii-2) By (ii-1) above, Definition 3.2(ii) and Definition 3.1(ii) for $(H, \tau|H)$, it is obtained that $\tau_H^\gamma \subset (\tau|H)^{\gamma_H^O}$. Using (i), we have the required equality under the assumption of (ii). □

Theorem 3.10 *Let* $A \subset H \subset X$ *and* $\gamma : \tau \to P(X)$ *be an operation on* τ *such that* $\gamma \neq$ "*id*".

(i) *If* A *is* γ*-open in* (X, τ) *and* H *is open in* (X, τ), *then* A *is* γ_H^O*-open in* $(H, \tau|H)$ (i.e., $A \in (\tau|H)^{\gamma_H^O}$).

(i)' *If* H *is* γ*-open in* (X, τ) *and* $\gamma : \tau \to P(X)$ *is a regular operation, then* • $\tau^\gamma|H \subset (\tau|H)^{\gamma_H^O}$ *holds.*

(ii) *If* $\gamma : \tau \to P(X)$ *is a regular operation on* τ, A *is* γ_H^O*-open in* $(H, \tau|H)$ *and* H *is* γ*-open in* (X, τ), *then* A *is* γ*-open in* (X, τ) (i.e., $A \in \tau^\gamma$).

(ii)' *If* H *is* γ*-open in* (X, τ) *and* $\gamma : \tau \to P(X)$ *is a regular operation on* τ, *then* • $(\tau|H)^{\gamma_H^O} \subset \tau^\gamma|H$ *holds.*

(iii) *If* $\gamma : \tau \to P(X)$ *is regular on* τ *and* H *is* γ*-open in* (X, τ), *then*

- $(\tau|H)^{\gamma_H^O} = \tau^\gamma|H$ *holds.*

Proof. **(i)** Let $x \in A$. There exists a subset U of X such that $x \in U, U \in \tau$ and $U^\gamma \subset A$. We have $x \in U \cap H = U$ and $U \in \tau|H$; and so $\gamma_H^O(U) = U^\gamma \cap H \subset A \cap H = A$. Thus, we show $A \in (\tau|H)^{\gamma_H^O}$.

(i)' Let $A \in \tau^\gamma|H$; then there exists a subset B of X such that $B \in \tau^\gamma$ and $A = B \cap H$. Since γ is regular, τ^γ forms a topology of X ([4, Proposition 2.9]). Thus, we have $B \cap H \in \tau^\gamma$ and so $A \in \tau^\gamma$, because $B \in \tau^\gamma$ and $H \in \tau^\gamma$. By (i) above, it is obtained that $A \in (\tau|H)^{\gamma_H^O}$ (because of the fact that $\tau^\gamma \subset \tau$ in general and so $H \in \tau$). Thus, we prove $\tau^\gamma|H \subset (\tau|H)^{\gamma_H^O}$.

(ii) Let $x \in A$. There exists a subset $U \in \tau$ such that $x \in U, \gamma_H^O(U \cap H) = (U \cap H)^\gamma \cap H \subset A$, because $A \in (\tau|H)^{\gamma_H^O}$, $\gamma \neq$ "*id*", $U \cap H \in \tau|H$ and $U \cap H \in \tau$. Since $H \in \tau^\gamma$ and $x \in A \subset H$, for the point $x \in H$, there exists a subset $V \in \tau$ such that $x \in V$ and $V^\gamma \subset H$. By the regularity of γ, for two open subsets $U \cap H$ and V containing x, there exists a subset $W \in \tau$ such that $x \in W$ and $W^\gamma \subset (U \cap H)^\gamma \cap V^\gamma$ (cf. the place between Remark 3.3 and Theorem 3.5); and so $W^\gamma \subset (U \cap H)^\gamma \cap H \subset A$. Therefore, for each point $x \in A$, we have a subset W such that $W \in \tau, x \in W$ and $W^\gamma \subset A$; and so A is γ-open in (X, τ) (i.e., $A \in \tau^\gamma$).

(ii)' Let $A \in (\tau|H)^{\gamma_H^O}$; then it follows from (ii) above that $A \in \tau^\gamma$ and $A = A \cap H \in \tau^\gamma|H$. Thus, we have the implication $(\tau|H)^{\gamma_H^O} \subset \tau^\gamma|H$.

(iii) By (i)' and (ii)', the required equality is obtained. □

Corollary 3.11 (i) *If $\gamma : \tau \to P(X)$ is a monotone operation on τ such that $\gamma \neq$ "id" and H is γ-open in (X,τ), then we have the following equality:*

- $(\tau|H)^{\gamma_H^O} = \tau^\gamma|H = \tau_H^\gamma$ *hold.*

(ii) *If $\gamma =$ "id", then $\tau^\gamma|H = \tau_H^\gamma = (\tau|H)^{\gamma_H^O} = \tau|H$ hold for any subset H of (X,τ).*

Proof. **(i)** Since γ is monotone with $\gamma \neq$ "id" and H is open in (X,τ), we have $\tau_H^\gamma = (\tau|H)^{\gamma_H^O}$ holds (cf. Theorem 3.9(ii)(ii-2)). And, we recall that every monotone operation is regular. Then, by Theorem 3.10(iii), it is shown that $(\tau|H)^{\gamma_H^O} = \tau^\gamma|H$.

(ii) By using Remark 3.3 and Remark 3.7, it is shown that $\tau^{\text{"id"}}|H = \tau_H^{\text{"id"}} = (\tau|H)^{\text{"id"}_H^O} = \tau|H$ hold. □

4 Operation-closures in subspaces

In the end of the present paper, we investigate some forms of *operation-closures in subspaces*. We recall here that, for a topological space (X,τ), a subset H of X and a subset B of H,

· $\mathrm{Cl}_H(B) = H \cap \mathrm{Cl}(B)$ holds, where $\mathrm{Cl}(B) := \tau\text{-}\mathrm{Cl}(B) = \bigcap\{F | B \subset F, F \text{ is closed in } (X,\tau)\}$ and $\mathrm{Cl}_H(B) := (\tau|H)\text{-}\mathrm{Cl}(B) = \bigcap\{F | B \subset F, F \text{ is closed in } (H,\tau|H)\}$. Moreover, for a point $x \in X$ and a subset E of (X,τ),

· $x \in \mathrm{Cl}(E)$ if and only if $U \cap E \neq \emptyset$ holds for every open set U of (X,τ) such that $x \in U$; and for a point $y \in H$ and a subset B of $(H,\tau|H)$,

· $y \in \mathrm{Cl}_H(B)$ if and only if $V \cap B \neq \emptyset$ holds for every open set V of $(H,\tau|H)$ such that $y \in V$.

By Janković [1], the concept of operation-closures in topological spaces is introduced.

Definition 4.1 (i) (Janković [1]) For a subset A of a topological space (X,τ) and an operation $\gamma : \tau \to P(X)$, the *γ-closure of A*, say $\mathrm{Cl}_\gamma(A)$, is defined as follows:

- $\mathrm{Cl}_\gamma(A) = \{x \in X |\ U^\gamma \cap A \neq \emptyset$ for every open set U of (X,τ) with $x \in U\}$.

(ii) Let $\gamma : \tau \to P(X)$ be a given operation. Let $(H,\tau|H)$ be a subspace of (X,τ) and B a subset of H. Suppose that H is open in (X,τ) if $\gamma \neq$ "*id*". For the restriction $\gamma_H^O : \tau|H \to P(H)$ of γ (cf. Definition 2.1) and a subset B of H, we can define the concept of the *operation-closure of B in a subspace* $(H,\tau|H)$, say $\mathrm{Cl}_{\gamma_H^O}(B)$, as follows:

- $\mathrm{Cl}_{\gamma_H^O}(B) := \{x \in H | \gamma_H^O(U) \cap B \neq \emptyset$ holds for every open set U of $(H,\tau|H)$ with $x \in U\}$.

(Note: $\mathrm{Cl}_{\gamma_H^O}(B) \subset H$ for every subset $B \subset H$).

We need the following concept of the *open operation* defined by Ogata [4, Definition 2.6].

Definition 4.2 (Ogata [4, Definition 2.6]) An operation $\gamma : \tau \to P(X)$ is said to be *open on τ* if for every open neighbourhood U of each point $x \in X$ there exists a γ-open set S such that $x \in S$ and $S \subset U^\gamma$, where $U^\gamma := \gamma(U)$ (the value of γ at U).

Any "Int ∘ Cl"-operation, say $\gamma : \tau \to P(X)$, is open on τ, where $\gamma(U) := \mathrm{Int}(\mathrm{Cl}(U))$ for every set $U \in \tau$ ([4, Example 2.7]). By definition, it is known that every identity operation "*id*" : $\tau \to P(X)$ is open on τ, where τ is a topology of X.

Remark 4.3 (i) For families τ^γ and $(\tau|H)^{\gamma_H^O}$, other operation-closures are defined, respectively (e.g., [4, (3.2), Proposition 3.3]):

- $\tau^\gamma\text{-}\mathrm{Cl}(A) := \bigcap\{F | A \subset F, F \text{ is } \gamma\text{-closed in } (X,\tau)\}$, where $A \subset X$;
- $(\tau|H)^{\gamma_H^O}\text{-}\mathrm{Cl}(B) := \bigcap\{F_1 | B \subset F_1, F_1 \text{ is } \gamma_H^O\text{-closed in } (H,\tau|H)\}$, where $B \subset H$.

(ii) It is well known that $A \subset \text{Cl}(A) \subset \text{Cl}_\gamma(A) \subset \tau^\gamma\text{-Cl}(A)$ holds for any subset $A \subset X$, any topology τ and any operation $\gamma : \tau \to P(X)$ (e.g., [4, (3.4)]). And the example in [4, Remark 3.5] shows that $\text{Cl}_\gamma(A) \neq \tau^\gamma\text{-Cl}(A)$ in general.

(iii) If $\gamma : \tau \to P(X)$ is open on τ (cf. Definition 4.2 above), then $\text{Cl}_\gamma(A) = \tau^\gamma\text{-Cl}(A)$ holds for any subset A of X ([4, Theorem 3.6 (iii)]).

(iv) Moreover, in [4, Proposition 3.3], $x \in \tau^\gamma\text{-Cl}(A)$ if and only if $U \cap A \neq \emptyset$ holds for any γ-open set U of (X, τ) (i.e., $U \in \tau^\gamma$) such that $x \in U$, where γ is a given operation on τ.

We need the following lemma.

Lemma 4.4 (i) *Let $\gamma : \tau \to P(X)$ be a regular operation on τ such that $\gamma \neq$ "id" and H be a γ-open set of (X, τ). If γ is open on τ, then $\gamma_H^O : \tau|H \to P(H)$ is open on $\tau|H$.*

(ii) *If $\gamma =$ "id" and H is a subset of X, then $\gamma_H^O =$ "id"$_H^O : \tau|H \to P(H)$ is open on $\tau|H$.*

Proof. (i) Let $x \in H$ and V be an open set of $(H, \tau|H)$ with $x \in V$. We show that there exists a γ_H^O-open set S in $(H, \tau|H)$ such that $x \in S$ and $S \subset \gamma_H^O(V)$. Indeed, since $H \in \tau$ and $V \in \tau$, by the openness of γ, there exists a γ-open set, say T, in (X, τ) such that $x \in T$ and $T \subset V^\gamma$. We put $S := T \cap H$; then $x \in S$, $S \subset V^\gamma \cap H = \gamma_H^O(V)$ (cf. Definition 2.1) and $S \in \tau^\gamma|H$ (cf. Definition 3.6(i)). We claim that the subset S above is a γ_H^O-open set of $(H, \tau|H)$ (i.e., $S \in (\tau|H)^{\gamma_H^O}$). Indeed, since $\gamma \neq id$, γ is regular and $H \in \tau^\gamma$, we apply Theorem 3.10 (iii) to the present case; and so we have $\tau^\gamma|H \subset (\tau|H)^{\gamma_H^O}$. Thus, we have $S \in (\tau|H)^{\gamma_H^O}$.

Therefore, for the given point $x \in H$ and the given open set V containing x in $(H, \tau|H)$, the subset S is a γ_H^O-open set of $(H, \tau|H)$ such that $x \in S$ and $S \subset \gamma_H^O(V)$. Namely, $\gamma_H^O : \tau|H \to P(H)$ is an open operation on $\tau|H$.

(ii) For $\gamma =$ "*id*", by Definition 2.1 and Remark 2.2, it is known that $\gamma_H^O =$ "*id*"$_H^O$: $\tau|H \to P(H)$ is the identity operation on $\tau|H$. And it is well known that the identity operation on any topology is open on the topology. □

Theorem 4.5 *Let $\gamma : \tau \to P(X)$ be a given operation on τ and $B \subset H \subset X$.*

(I) *Suppose that H is open in (X, τ) and $\gamma \neq$ "id".*

(i) $\text{Cl}_{\gamma_H^O}(B) \supset \text{Cl}_\gamma(B) \cap H$ *holds.*

(ii) *If $\gamma : \tau \to P(X)$ is monotone, then $\text{Cl}_{\gamma_H^O}(B) \subset \text{Cl}_\gamma(B) \cap H$ holds; and so $\text{Cl}_{\gamma_H^O}(B) = \text{Cl}_\gamma(B) \cap H$ holds.*

(iii) *Suppose that H is γ-open in (X, τ). If $\gamma : \tau \to P(X)$ is regular and open on τ, then we have the following properties:*

(iii-1) $\text{Cl}_{\gamma_H^O}(B) \subset \text{Cl}_\gamma(B) \cap H$ *holds; and so* $\text{Cl}_{\gamma_H^O}(B) = \text{Cl}_\gamma(B) \cap H$ *holds;*

(iii-2) $\text{Cl}_{\gamma_H^O}(B) = (\tau|H)^{\gamma_H^O}\text{-Cl}(B)$ *holds for any subset B of H.*

(II) *If $\gamma =$ "id", then* $\text{Cl}_{\text{"}id\text{"}_H^O}(B) = \text{Cl}_{\text{"}id\text{"}}(B) \cap H$ *holds,* i.e., $\text{Cl}_H(B) = \text{Cl}(B) \cap H$ *holds.*

Proof. **(I)** **(i)** Let $x \in \text{Cl}_\gamma(B) \cap H$. In order to prove $x \in \text{Cl}_{\gamma_H^O}(B)$, let U be an open set of $(H, \tau|H)$ with $x \in U$. Since H is open in (X, τ) and $x \in \text{Cl}_\gamma(B)$, we have $U \in \tau$ and so $U^\gamma \cap B \neq \emptyset$. By Definition 2.1(i), it is obtained that $\gamma_H^O(U) \cap B = (U^\gamma \cap H) \cap B = U^\gamma \cap (H \cap B) = U^\gamma \cap B \neq \emptyset$; and so $x \in Cl_{\gamma_H^O}(B)$.

(ii) Let $x \notin \text{Cl}_\gamma(B) \cap H$. We should show $x \notin \text{Cl}_{\gamma_H^O}(B)$. For the point x, we consider the following two cases.

Case 1. $x \notin H$: for this point x, we have $x \notin \text{Cl}_{\gamma_H^O}(B)$ (cf. Note in Definition 4.1).

Case 2. $x \in H$: for this case, we have $x \notin \text{Cl}_\gamma(B)$. Then, there exists a subset $U \in \tau$

such that $x \in U$ and $U^\gamma \cap B = \emptyset$. Since γ is monotone, we have $(U \cap H)^\gamma \subset U^\gamma$ and so $\gamma_H^O(U \cap H) \cap B$=$((U \cap H)^\gamma \cap H) \cap B \subset U^\gamma \cap B = \emptyset$ (indeed, $U \cap H \in \tau$) . Thus, the subset $U \cap H$ is an open set of $(H, \tau|H)$ such that $x \in U \cap H$ and $\gamma_H^O(U \cap H) \cap B = \emptyset$; and so $x \notin \mathrm{Cl}_{\gamma_H^O}(B)$.

Therefore, for both cases we show that $x \notin \mathrm{Cl}_{\gamma_H^O}(B)$.

(iii) First we recall that every γ-open set of (X, τ) is open in (X, τ) for any operation γ and a topology τ.

(iii-1) Let $x \notin \mathrm{Cl}_\gamma(B) \cap H$. We consider the following two cases.

Case 1. $x \notin H$: for this point x, we have $x \notin \mathrm{Cl}_{\gamma_H^O}(B)$, because $\mathrm{Cl}_{\gamma_H^O}(B) \subset H$ (cf. Definition 4.1(ii)).

Case 2. $x \in H$: for this case, we have $x \notin \mathrm{Cl}_\gamma(B)$. There exists a subset $U \in \tau$ such that $x \in U$ and $U^\gamma \cap B = \emptyset$. Since γ is open on τ, there exists a γ-open set S such that $x \in S$ and $S \subset U^\gamma$ and so $S \cap B \subset U^\gamma \cap B = \emptyset$ (i.e., $S \cap B = \emptyset$). Thus we have $S \cap H \in \tau^\gamma|H$ and $x \in S \cap H$. By Theorem 3.10 (i)', it is well known that $\tau^\gamma|H \subset (\tau|H)^{\gamma_H^O}$; and so we have $S \cap H \in (\tau|H)^{\gamma_H^O}$. Namely, the subset $S \cap H$ is a γ_H^O-open set of $(H, \tau|H)$ such that $x \in S \cap H$ and $(S \cap H) \cap B = S \cap B = \emptyset$. This shows that $x \notin (\tau|H)^{\gamma_H^O}\text{-}\mathrm{Cl}(B)$ (cf. Remark 4.3 (iv)). Since $\mathrm{Cl}_{\gamma_H^O}(E) \subset (\tau|H)^{\gamma_H^O}\text{-}\mathrm{Cl}(E)$ holds for any subset E of a topological space $(H, \tau|H)$ (cf. Remark 4.3 (ii)), we have $x \notin \mathrm{Cl}_{\gamma_H^O}(B)$.

Therefore, for both cases, we show that $x \notin \mathrm{Cl}_{\gamma_H^O}(B)$ for any point x with $x \notin \mathrm{Cl}_\gamma(B) \cap H$; and so we have the required implication $\mathrm{Cl}_{\gamma_H^O}(B) \subset \mathrm{Cl}_\gamma(B) \cap H$. Moreover, since any γ-open set of (X, τ) is open in (X, τ), we can apply the result (I)(i) above to the present case; and so we have the required equality.

(iii-2) By Lemma 4.4, $\gamma_H^O : \tau \to P(H)$ is open on $\tau|H$. Using [4, Theorem 3.6] (cf. Remark 4.3 (iii)) for the topological space $(H, \tau|H)$, the subset $B \subset H$ and the operation $\gamma_H^O : \tau|H \to P(H)$, we have the required equality $\mathrm{Cl}_{\gamma_H^O}(B) = (\tau|H)^{\gamma_H^O}\text{-}\mathrm{Cl}(B)$.

(II) Since $\gamma =$ "id", we have "id"$_H^O$ = "id_H" $: \tau|H \to P(H)$, where "id_H" is the identity operation on $\tau|H$, and so $\mathrm{Cl}_{\text{“}id\text{”}_H^O}(B) = \mathrm{Cl}_{\text{“}id_H\text{”}}(B) = \mathrm{Cl}_H(B) = (\tau|H)\text{-}\mathrm{Cl}(B) = \mathrm{Cl}(B) \cap H = \mathrm{Cl}_{\text{“}id\text{”}}(B) \cap H$ hold. □

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Haruo Maki (Corresponding author)
Wakagi-dai 2-10-13; Fukutsu-shi
Fukuoka-ken;811-3221 Japan
e-mail: makih@pop12.odn.ne.jp

G. Sai Sundara Krishnan
Department of Mathematics and Computer Applications PSG college of Technology
Coimbatore -641 004 India
e-mail:g_ssk@yahoo.com

Hayao Ogata
Ijiri 1-15-1; Minami-ku, Fukuoka-shi
Fukuoka-ken; 811-1302 Japan