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COMPUTING K-THEORY GROUPS FOR TENSOR PRODUCTS OF C^*-ALGEBRAS

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Abstract. We compute the K-theory groups for the tensor product of two C^*-algebras, one of which is in the bootstrap category, and whose K-theory groups may have torsion, by using the Künneth theorem in the K-theory for operator algebras.

1 Introduction In this paper, we compute the K-theory groups for the tensor product of two C^*-algebras, one of which is in the bootstrap category, and whose K-theory groups may have torsion, by using the Künneth theorem in the K-theory for operator algebras.

This paper after Introduction is organized of the following sections:

2 Preliminaries;
3 The case with torsion part a product of one cyclic group;
4 The case with torsion part two products of two cyclic groups;
5 The general case with torsion part finite products of cyclic groups.

In Section 2 we recall about the Künneth theorem for K-theory groups of tensor products of C^*-algebras. In Sections 3 to 5, given we are the K-theory groups of two C^*-algebras as in the titles of the sections, and then we compute the K-theory groups of their tensor products by using the Künneth theorem. Since the K-theory groups of two C^*-algebras are given concretely, we can perform the computation by determining the torsion product in the Künneth theorem by using several facts in homology theory.

Our computation results performed and obtained here should be useful for application and be viewed as basic formulae for reference. The case by case results in Sections 3 and 4 are also useful as examples, indeed, from which we could reach to the general results in Section 5.

2 Preliminaries The Künneth theorem for K-theory groups of tensor products of C^*-algebras (recall from Blackadar [2]):

Let $\mathfrak{A}, \mathfrak{B}$ be C^*-algebras. Suppose that $\mathfrak{A}$ is in the bootstrap category $\mathfrak{N}$. Then there is a short exact sequence

$$0 \to K_*(\mathfrak{A}) \otimes K_*(\mathfrak{B}) \xrightarrow{\alpha} K_*(\mathfrak{A} \otimes \mathfrak{B}) \xrightarrow{\sigma} \mathrm{Tor}_1^{\mathbb{Z}}(K_*(\mathfrak{A}), K_*(\mathfrak{B})) \to 0,$$

where $K_*(\cdot) = K_0(\cdot) \oplus K_1(\cdot)$ the direct sum of K-theory groups, and the map α has degree 0 and the map σ has degree 1. The sequence is natural in each variable, and splits unnaturally.

More details are given from Schochet [4] the original as follows. The map α is defined as

$$\alpha : K_p(\mathfrak{A}) \otimes K_q(\mathfrak{B}) \to K_{p+q}(\mathfrak{A} \otimes \mathfrak{B})$$

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where $p, q \in \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ and $\otimes$ means the minimal tensor product of C^*-algebras. The category $\mathfrak{N}$ is the smallest subcategory of the category of separable nuclear C^*-algebras which contains separable type I C^*-algebras and is closed under the operations of taking closed ideals, quotients, extensions, inductive limits, stable isomorphism, and crossed products by the group $\mathbb{Z}$ of integers and by the group $\mathbb{R}$ of reals. The degrees of $K_p(\cdot) \otimes K_q(\cdot)$, $K_p(\cdot) \oplus K_q(\cdot)$, and $\mathrm{Tor}_1^{\mathbb{Z}}(K_p(\cdot), K_q(\cdot))$ the torsion product are given by $p + q \in \mathbb{Z}_2$.

The classical Künneth theorem for topological K-theory groups of products of topological spaces (due to Atiyah [1]) is:

If X, Y are finite CW-complexes, more generally, compact Hausdorff spaces, then

$$0 \to K^*(X) \otimes K^*(Y) \xrightarrow{\alpha} K^*(X \times Y) \xrightarrow{\sigma} \mathrm{Tor}_1^{\mathbb{Z}}(K^*(X), K^*(Y)) \to 0,$$

where $K^*(\cdot) = K^0(\cdot) \oplus K^{-1}(\cdot)$. This is the case where $\mathfrak{A} = C(X)$, $\mathfrak{B} = C(Y)$ the C^*-algebras of all continuous complex-valued functions on X, Y respectively.

Note that $K^0(X) \cong K_0(C(X))$ and $K^{-1}(X) = K^0(X \times \mathbb{R}) = K^0((X \times \mathbb{R})^+, +) \cong K_0(SC(X)) \cong K_1(C(X))$, where $(X \times \mathbb{R})^+$ is the one-point compactification of $X \times \mathbb{R}$ by one point $+$, and $K^0(X, Y)$ means the relative K^0-group for X a locally compact Hausdorff space and Y a closed subspace of X, and $SC(X)$ means the suspension of $C(X)$ (see [2]).

The theorem due to Schochet ([4]) is:

Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras with $\mathfrak{A}$ in the category $\mathfrak{N}$ and $K_*(\mathfrak{B})$ torsion free. Then there is an isomorphism:

$$\alpha : K_*(\mathfrak{A}) \otimes K_*(\mathfrak{B}) \to K_*(\mathfrak{A} \otimes \mathfrak{B}),$$

so that

$$K_0(\mathfrak{A} \otimes \mathfrak{B}) \cong [K_0(\mathfrak{A}) \otimes K_0(\mathfrak{B})] \oplus [K_1(\mathfrak{A}) \otimes K_1(\mathfrak{B})],$$
$$K_1(\mathfrak{A} \otimes \mathfrak{B}) \cong [K_0(\mathfrak{A}) \otimes K_1(\mathfrak{B})] \oplus [K_1(\mathfrak{A}) \otimes K_0(\mathfrak{B})].$$

3 The case with torsion part a product of one cyclic group

Proposition 3.1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras with $\mathfrak{A}$ in the bootstrap category $\mathfrak{N}$. Suppose that $K_j(\mathfrak{A}) = \mathbb{Z}^{n_j} \oplus \mathbb{Z}_p^{m_j}$ and $K_j(\mathfrak{B}) = \mathbb{Z}^{s_j} \oplus \mathbb{Z}_p^{t_j}$ for some positive integers n_j, m_j, s_j, t_j with $j = 0, 1$ and p a prime number. Then*

$$\begin{aligned} K_0(\mathfrak{A} \otimes \mathfrak{B}) &\cong \mathbb{Z}^{n_0 s_0 + n_1 s_1} \oplus \mathbb{Z}_p^{n_0 t_0 + m_0(s_0 + t_0) + n_1 t_1 + m_1(s_1 + t_1)} \\ &\quad \oplus \mathbb{Z}_p^{m_0 t_1 + m_1 t_0}, \\ K_1(\mathfrak{A} \otimes \mathfrak{B}) &\cong \mathbb{Z}^{n_0 s_1 + n_1 s_0} \oplus \mathbb{Z}_p^{n_0 t_1 + m_0(s_1 + t_1) + n_1 t_0 + m_1(s_0 + t_0)} \\ &\quad \oplus \mathbb{Z}_p^{m_0 t_0 + m_1 t_1}, \end{aligned}$$

where the last summands in the K_0 and K_1-groups correspond to the respective torsion products.

Proof. We compute the torsion product in the Künneth theorem for tensor products of C^*-algebras using several facts in homology theory as in [3]:

$$\begin{aligned} \mathrm{Tor}_1^{\mathbb{Z}}(K_j(\mathfrak{A}), K_k(\mathfrak{B})) &= \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}^{n_j} \oplus \mathbb{Z}_p^{m_j}, \mathbb{Z}^{s_k} \oplus \mathbb{Z}_p^{t_k}) \\ &\cong [\oplus^{n_j} \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}, K_k(\mathfrak{B}))] \oplus [\oplus^{m_j} \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_p, K_k(\mathfrak{B})] \\ &\cong [\oplus^{n_j} 0] \oplus [\oplus^{m_j} \oplus^{t_k} \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_p, \mathbb{Z}_p)] \\ &\cong \oplus^{m_j t_k} \mathbb{Z}_p = \mathbb{Z}_p^{m_j t_k} \end{aligned}$$

for $j = 0, 1$ and $k = 0, 1$. Note that $\mathrm{Tor}_1^{\mathbb{Z}}(K_j(\mathfrak{A}), K_k(\mathfrak{B}))$ appears in the split quotient of $K_{j+k+1}(\mathfrak{A} \otimes \mathfrak{B})$. Therefore,

$$\begin{aligned}
&K_0(\mathfrak{A} \otimes \mathfrak{B})/(\mathbb{Z}_p^{m_0 t_1} \oplus \mathbb{Z}_p^{m_1 t_0}) \\
&\cong [K_0(\mathfrak{A}) \otimes K_0(\mathfrak{B})] \oplus [K_1(\mathfrak{A}) \otimes K_1(\mathfrak{B})] \\
&= [(\mathbb{Z}^{n_0} \oplus \mathbb{Z}_p^{m_0}) \otimes (\mathbb{Z}^{s_0} \oplus \mathbb{Z}_p^{t_0})] \oplus [(\mathbb{Z}^{n_1} \oplus \mathbb{Z}_p^{m_1}) \otimes (\mathbb{Z}^{s_1} \oplus \mathbb{Z}_p^{t_1})] \\
&\cong [\mathbb{Z}^{n_0 s_0} \oplus \mathbb{Z}_p^{n_0 t_0 + m_0 s_0 + m_0 t_0}] \oplus [\mathbb{Z}^{n_1 s_1} \oplus \mathbb{Z}_p^{n_1 t_1 + m_1 s_1 + m_1 t_1}] \\
&\cong \mathbb{Z}^{n_0 s_0 + n_1 s_1} \oplus \mathbb{Z}_p^{n_0 t_0 + m_0 s_0 + m_0 t_0 + n_1 t_1 + m_1 s_1 + m_1 t_1},
\end{aligned}$$

where note that $\mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}$, $\mathbb{Z} \otimes \mathbb{Z}_p \cong \mathbb{Z}_p \otimes \mathbb{Z} \cong \mathbb{Z}_p$, and $\mathbb{Z}_p \otimes \mathbb{Z}_p \cong \mathbb{Z}_p$. Indeed, $\sum^p (1 \otimes 1) = 1 \otimes (\sum^p 1) = 1 \otimes 0 = 0$ in $\mathbb{Z} \otimes \mathbb{Z}_p$ and $\mathbb{Z}_p \otimes \mathbb{Z}_p$, where $\sum^p$ means the usual p times summation. Hence we get

$$\begin{aligned}
K_0(\mathfrak{A} \otimes \mathfrak{B}) &\cong \mathbb{Z}^{n_0 s_0 + n_1 s_1} \oplus \mathbb{Z}_p^{n_0 t_0 + m_0 s_0 + m_0 t_0 + n_1 t_1 + m_1 s_1 + m_1 t_1 + m_0 t_1 + m_1 t_0} \\
&= \mathbb{Z}^{n_0 s_0 + n_1 s_1} \oplus \mathbb{Z}_p^{n_0 t_0 + m_0 (s_0 + t_0 + t_1) + n_1 t_1 + m_1 (s_1 + t_1 + t_0)}.
\end{aligned}$$

And also

$$\begin{aligned}
&K_1(\mathfrak{A} \otimes \mathfrak{B})/(\mathbb{Z}_p^{m_0 t_0} \oplus \mathbb{Z}_p^{m_1 t_1}) \\
&\cong [K_0(\mathfrak{A}) \otimes K_1(\mathfrak{B})] \oplus [K_1(\mathfrak{A}) \otimes K_0(\mathfrak{B})] \\
&= [(\mathbb{Z}^{n_0} \oplus \mathbb{Z}_p^{m_0}) \otimes (\mathbb{Z}^{s_1} \oplus \mathbb{Z}_p^{t_1})] \oplus [(\mathbb{Z}^{n_1} \oplus \mathbb{Z}_p^{m_1}) \otimes (\mathbb{Z}^{s_0} \oplus \mathbb{Z}_p^{t_0})] \\
&\cong [\mathbb{Z}^{n_0 s_1} \oplus \mathbb{Z}_p^{n_0 t_1 + m_0 s_1 + m_0 t_1}] \oplus [\mathbb{Z}^{n_1 s_0} \oplus \mathbb{Z}_p^{n_1 t_0 + m_1 s_0 + m_1 t_0}] \\
&\cong \mathbb{Z}^{n_0 s_1 + n_1 s_0} \oplus \mathbb{Z}_p^{n_0 t_1 + m_0 s_1 + m_0 t_1 + n_1 t_0 + m_1 s_0 + m_1 t_0}.
\end{aligned}$$

Hence we get

$$\begin{aligned}
K_1(\mathfrak{A} \otimes \mathfrak{B}) &\cong \mathbb{Z}^{n_0 s_1 + n_1 s_0} \oplus \mathbb{Z}_p^{n_0 t_1 + m_0 s_1 + m_0 t_1 + n_1 t_0 + m_1 s_0 + m_1 t_0 + m_0 t_0 + m_1 t_1} \\
&= \mathbb{Z}^{n_0 s_1 + n_1 s_0} \oplus \mathbb{Z}_p^{n_0 t_1 + m_0 (s_1 + t_0 + t_1) + n_1 t_0 + m_1 (s_0 + t_0 + t_1)}.
\end{aligned}$$

□

Remark. The statement above also holds when p is replaced with the powers p^k of p for positive integers $k \geq 1$.

Proposition 3.2. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras with $\mathfrak{A}$ in the category $\mathfrak{N}$. Suppose that $K_j(\mathfrak{A}) = \mathbb{Z}^{n_j} \oplus \mathbb{Z}_{p^h}^{m_j}$ and $K_j(\mathfrak{B}) = \mathbb{Z}^{s_j} \oplus \mathbb{Z}_{p^l}^{t_j}$ for some positive integers n_j, m_j, s_j, t_j with $j = 0, 1$ and p a prime number with $1 \leq h < l$. Then*

$$\begin{aligned}
K_0(\mathfrak{A} \otimes \mathfrak{B}) &\cong \mathbb{Z}^{n_0 s_0 + n_1 s_1} \oplus \mathbb{Z}_{p^h}^{m_0 (s_0 + t_0) + m_1 (s_1 + t_1)} \oplus \mathbb{Z}_{p^l}^{n_0 t_0 + n_1 t_1} \\
&\quad \oplus \mathbb{Z}_{p^h}^{m_0 t_1 + m_1 t_0}, \\
K_1(\mathfrak{A} \otimes \mathfrak{B}) &\cong \mathbb{Z}^{n_0 s_1 + n_1 s_0} \oplus \mathbb{Z}_{p^h}^{m_0 (s_1 + t_1) + m_1 (s_0 + t_0)} \oplus \mathbb{Z}_{p^l}^{n_0 t_1 + n_1 t_0} \\
&\quad \oplus \mathbb{Z}_{p^h}^{m_0 t_0 + m_1 t_1}
\end{aligned}$$

where the last summands correpond to the respective torsion products.

Proof. Note that $\mathbb{Z}_{p^h}$ is contained in $\mathbb{Z}_{p^l}$. Indeed, check that

$$\mathbb{Z}_{p^h} = \{0, 1, 2, \cdots, p, p+1, \cdots, p^h - 1\}$$

is embedded as

$$\{0, p^{l-h}, 2p^{l-h}, \cdots, p^{l-h+1}, (p+1)p^{l-h}, \ldots, (p^h-1)p^{l-h}\}$$

in $\mathbb{Z}_{p^l}$, where each class $k + (p^h\mathbb{Z}) \in \mathbb{Z}_{p^h} = \mathbb{Z}/p^h\mathbb{Z}$ is identified with the representative $k \in \mathbb{Z}$. Therefore, we have

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_{p^h}, \mathbb{Z}_{p^l}) \cong \mathbb{Z}_{p^h} \cong \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_{p^l}, \mathbb{Z}_{p^h}).$$

We then compute the torsion product as before:

$$\begin{aligned}
&\mathrm{Tor}_1^{\mathbb{Z}}(K_j(\mathfrak{A}), K_k(\mathfrak{B})) = \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}^{n_j} \oplus \mathbb{Z}_{p^h}^{m_j}, \mathbb{Z}^{s_k} \oplus \mathbb{Z}_{p^l}^{t_k}) \\
&\cong \oplus^{m_j} \oplus^{t_k} \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_{p^h}, \mathbb{Z}_{p^l}) \\
&\cong \oplus^{m_j t_k} \mathbb{Z}_{p^h} = \mathbb{Z}_{p^h}^{m_j t_k}
\end{aligned}$$

for $j = 0, 1$ and $k = 0, 1$. Note that $\mathrm{Tor}_1^{\mathbb{Z}}(K_j(\mathfrak{A}), K_k(\mathfrak{B}))$ appears in the split quotient of $K_{j+k+1}(\mathfrak{A} \otimes \mathfrak{B})$. Therefore,

$$\begin{aligned}
&K_0(\mathfrak{A} \otimes \mathfrak{B})/(\mathbb{Z}_{p^h}^{m_0 t_1} \oplus \mathbb{Z}_{p^h}^{m_1 t_0}) \\
&\cong [K_0(\mathfrak{A}) \otimes K_0(\mathfrak{B})] \oplus [K_1(\mathfrak{A}) \otimes K_1(\mathfrak{B})] \\
&= [(\mathbb{Z}^{n_0} \oplus \mathbb{Z}_{p^h}^{m_0}) \otimes (\mathbb{Z}^{s_0} \oplus \mathbb{Z}_{p^l}^{t_0})] \oplus [(\mathbb{Z}^{n_1} \oplus \mathbb{Z}_{p^h}^{m_1}) \otimes (\mathbb{Z}^{s_1} \oplus \mathbb{Z}_{p^l}^{t_1})] \\
&\cong [\mathbb{Z}^{n_0 s_0} \oplus \mathbb{Z}_{p^h}^{m_0 s_0 + m_0 t_0} \oplus \mathbb{Z}_{p^l}^{n_0 t_0}] \oplus [\mathbb{Z}^{n_1 s_1} \oplus \mathbb{Z}_{p^h}^{m_1 s_1 + m_1 t_1} \oplus \mathbb{Z}_{p^l}^{n_1 t_1}] \\
&\cong \mathbb{Z}^{n_0 s_0 + n_1 s_1} \oplus \mathbb{Z}_{p^h}^{m_0 s_0 + m_0 t_0 + m_1 s_1 + m_1 t_1} \oplus \mathbb{Z}_{p^l}^{n_0 t_0 + n_1 t_1},
\end{aligned}$$

where note that $\mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}$, $\mathbb{Z} \otimes \mathbb{Z}_{p^h} \cong \mathbb{Z}_{p^h} \otimes \mathbb{Z} \cong \mathbb{Z}_{p^h}$, and $\mathbb{Z}_{p^h} \otimes \mathbb{Z}_{p^l} \cong \mathbb{Z}_{p^h}$. Indeed, $\sum^{p^h}(1 \otimes 1) = 1 \otimes (\sum^{p^h} 1) = 1 \otimes 0 = 0$ in $\mathbb{Z} \otimes \mathbb{Z}_{p^h}$ and $\mathbb{Z}_{p^h} \otimes \mathbb{Z}_{p^l}$. Hence we get

$$\begin{aligned}
K_0(\mathfrak{A} \otimes \mathfrak{B}) &\cong \mathbb{Z}^{n_0 s_0 + n_1 s_1} \oplus \mathbb{Z}_{p^h}^{m_0 t_1 + m_0 s_0 + m_0 t_0 + m_1 t_0 + m_1 s_1 + m_1 t_1} \oplus \mathbb{Z}_{p^l}^{n_0 t_0 + n_1 t_1}, \\
&\cong \mathbb{Z}^{n_0 s_0 + n_1 s_1} \oplus \mathbb{Z}_{p^h}^{m_0(t_1 + s_0 + t_0) + m_1(t_0 + s_1 + t_1)} \oplus \mathbb{Z}_{p^l}^{n_0 t_0 + n_1 t_1}.
\end{aligned}$$

And also

$$\begin{aligned}
&K_1(\mathfrak{A} \otimes \mathfrak{B})/(\mathbb{Z}_{p^h}^{m_0 t_0} \oplus \mathbb{Z}_{p^h}^{m_1 t_1}) \\
&\cong [K_0(\mathfrak{A}) \otimes K_1(\mathfrak{B})] \oplus [K_1(\mathfrak{A}) \otimes K_0(\mathfrak{B})] \\
&= [(\mathbb{Z}^{n_0} \oplus \mathbb{Z}_{p^h}^{m_0}) \otimes (\mathbb{Z}^{s_1} \oplus \mathbb{Z}_{p^l}^{t_1})] \oplus [(\mathbb{Z}^{n_1} \oplus \mathbb{Z}_{p^h}^{m_1}) \otimes (\mathbb{Z}^{s_0} \oplus \mathbb{Z}_{p^l}^{t_0})] \\
&\cong [\mathbb{Z}^{n_0 s_1} \oplus \mathbb{Z}_{p^h}^{m_0 s_1 + m_0 t_1} \oplus \mathbb{Z}_{p^l}^{n_0 t_1}] \oplus [\mathbb{Z}^{n_1 s_0} \oplus \mathbb{Z}_{p^h}^{m_1 s_0 + m_1 t_0} \oplus \mathbb{Z}_{p^l}^{n_1 t_0}] \\
&\cong \mathbb{Z}^{n_0 s_1 + n_1 s_0} \oplus \mathbb{Z}_{p^h}^{m_0 s_1 + m_0 t_1 + m_1 s_0 + m_1 t_0} \oplus \mathbb{Z}_{p^l}^{n_0 t_1 + n_1 t_0},
\end{aligned}$$

Hence we get

$$\begin{aligned}
K_1(\mathfrak{A} \otimes \mathfrak{B}) &\cong \mathbb{Z}^{n_0 s_1 + n_1 s_0} \oplus \mathbb{Z}_{p^h}^{m_0 t_0 + m_0 s_1 + m_0 t_1 + m_1 t_1 + m_1 s_0 + m_1 t_0} \oplus \mathbb{Z}_{p^l}^{n_0 t_1 + n_1 t_0}, \\
&\cong \mathbb{Z}^{n_0 s_1 + n_1 s_0} \oplus \mathbb{Z}_{p^h}^{m_0(t_0 + s_1 + t_1) + m_1(t_1 + s_0 + t_0)} \oplus \mathbb{Z}_{p^l}^{n_0 t_1 + n_1 t_0}.
\end{aligned}$$

□

More generally,

Proposition 3.3. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras with $\mathfrak{A}$ in the category $\mathfrak{N}$. Suppose that $K_j(\mathfrak{A}) = \mathbb{Z}^{n_j} \oplus \mathbb{Z}^{m_j}_{p^{h_j}}$ and $K_j(\mathfrak{B}) = \mathbb{Z}^{s_j} \oplus \mathbb{Z}^{t_j}_{p^{l_j}}$ for some positive integers n_j, m_j, s_j, t_j and h_j, l_j with $j = 0, 1$ and p a prime number. Then*

$$\begin{aligned}
&K_0(\mathfrak{A} \otimes \mathfrak{B}) \cong \\
&[\mathbb{Z}^{n_0 s_0 + n_1 s_1} \oplus \mathbb{Z}^{m_0 s_0}_{p^{h_0}} \oplus \mathbb{Z}^{n_0 t_0}_{p^{l_0}} \oplus \mathbb{Z}^{m_0 t_0}_{p^{h_0 \wedge l_0}} \oplus \mathbb{Z}^{m_1 s_1}_{p^{h_1}} \oplus \mathbb{Z}^{n_1 t_1}_{p^{l_1}} \oplus \mathbb{Z}^{m_1 t_1}_{p^{h_1 \wedge l_1}}] \\
&\oplus [\mathbb{Z}^{m_0 t_1}_{p^{h_0 \wedge l_1}} \oplus \mathbb{Z}^{m_1 t_0}_{p^{h_1 \wedge l_0}}], \\
&K_1(\mathfrak{A} \otimes \mathfrak{B}) \cong \\
&[\mathbb{Z}^{n_0 s_1 + n_1 s_0} \oplus \mathbb{Z}^{m_0 s_1}_{p^{h_0}} \oplus \mathbb{Z}^{n_0 t_1}_{p^{l_1}} \oplus \mathbb{Z}^{m_0 t_1}_{p^{h_0 \wedge l_1}} \oplus \mathbb{Z}^{m_1 s_0}_{p^{h_1}} \oplus \mathbb{Z}^{n_1 t_0}_{p^{l_0}} \oplus \mathbb{Z}^{m_1 t_0}_{p^{h_1 \wedge l_0}}] \\
&\oplus [\mathbb{Z}^{m_0 t_0}_{p^{h_0 \wedge l_0}} \oplus \mathbb{Z}^{m_1 t_1}_{p^{h_1 \wedge l_1}}],
\end{aligned}$$

where $a \wedge b$ means the minimum $\min\{a, b\}$.

Proof. We have

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_{p^{h_j}}, \mathbb{Z}_{p^{l_k}}) \cong \mathbb{Z}_{p^{h_j \wedge l_k}} \cong \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_{p^{l_k}}, \mathbb{Z}_{p^{h_j}}),$$

where $h_j \wedge l_k$ means the minimum $\min\{h_j, l_k\}$.

We then compute the torsion product as before:

$$\begin{aligned}
&\mathrm{Tor}_1^{\mathbb{Z}}(K_j(\mathfrak{A}), K_k(\mathfrak{B})) = \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}^{n_j} \oplus \mathbb{Z}^{m_j}_{p^{h_j}}, \mathbb{Z}^{s_k} \oplus \mathbb{Z}^{t_k}_{p^{l_k}}) \\
&\cong \oplus^{m_j} \oplus^{t_k} \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_{p^{h_j}}, \mathbb{Z}_{p^{l_k}}) \\
&\cong \oplus^{m_j t_k} \mathbb{Z}_{p^{h_j \wedge l_k}} = \mathbb{Z}^{m_j t_k}_{p^{h_j \wedge l_k}}
\end{aligned}$$

for $j = 0, 1$ and $k = 0, 1$. Note that $\mathrm{Tor}_1^{\mathbb{Z}}(K_j(\mathfrak{A}), K_k(\mathfrak{B}))$ appears in the split quotient of $K_{j+k+1}(\mathfrak{A} \otimes \mathfrak{B})$. Therefore,

$$\begin{aligned}
&K_0(\mathfrak{A} \otimes \mathfrak{B}) / (\mathbb{Z}^{m_0 t_1}_{p^{h_0 \wedge l_1}} \oplus \mathbb{Z}^{m_1 t_0}_{p^{h_1 \wedge l_0}}) \\
&\cong [K_0(\mathfrak{A}) \otimes K_0(\mathfrak{B})] \oplus [K_1(\mathfrak{A}) \otimes K_1(\mathfrak{B})] \\
&= [(\mathbb{Z}^{n_0} \oplus \mathbb{Z}^{m_0}_{p^{h_0}}) \otimes (\mathbb{Z}^{s_0} \oplus \mathbb{Z}^{t_0}_{p^{l_0}})] \oplus [(\mathbb{Z}^{n_1} \oplus \mathbb{Z}^{m_1}_{p^{h_1}}) \otimes (\mathbb{Z}^{s_1} \oplus \mathbb{Z}^{t_1}_{p^{l_1}})] \\
&\cong [\mathbb{Z}^{n_0 s_0} \oplus \mathbb{Z}^{m_0 s_0}_{p^{h_0}} \oplus \mathbb{Z}^{n_0 t_0}_{p^{l_0}} \oplus \mathbb{Z}^{m_0 t_0}_{p^{h_0 \wedge l_0}}] \oplus [\mathbb{Z}^{n_1 s_1} \oplus \mathbb{Z}^{m_1 s_1}_{p^{h_1}} \oplus \mathbb{Z}^{n_1 t_1}_{p^{l_1}} \oplus \mathbb{Z}^{m_1 t_1}_{p^{h_1 \wedge l_1}}].
\end{aligned}$$

Hence we get $K_0(\mathfrak{A} \otimes \mathfrak{B}) \cong$

$$[\mathbb{Z}^{n_0 s_0 + n_1 s_1} \oplus \mathbb{Z}^{m_0 s_0}_{p^{h_0}} \oplus \mathbb{Z}^{n_0 t_0}_{p^{l_0}} \oplus \mathbb{Z}^{m_0 t_0}_{p^{h_0 \wedge l_0}} \oplus \mathbb{Z}^{m_1 s_1}_{p^{h_1}} \oplus \mathbb{Z}^{n_1 t_1}_{p^{l_1}} \oplus \mathbb{Z}^{m_1 t_1}_{p^{h_1 \wedge l_1}}] \oplus [\mathbb{Z}^{m_0 t_1}_{p^{h_0 \wedge l_1}} \oplus \mathbb{Z}^{m_1 t_0}_{p^{h_1 \wedge l_0}}].$$

And also

$$\begin{aligned}
&K_1(\mathfrak{A} \otimes \mathfrak{B}) / (\mathbb{Z}^{m_0 t_0}_{p^{h_0 \wedge l_0}} \oplus \mathbb{Z}^{m_1 t_1}_{p^{h_1 \wedge l_1}}) \\
&\cong [K_0(\mathfrak{A}) \otimes K_1(\mathfrak{B})] \oplus [K_1(\mathfrak{A}) \otimes K_0(\mathfrak{B})] \\
&= [(\mathbb{Z}^{n_0} \oplus \mathbb{Z}^{m_0}_{p^{h_0}}) \otimes (\mathbb{Z}^{s_1} \oplus \mathbb{Z}^{t_1}_{p^{l_1}})] \oplus [(\mathbb{Z}^{n_1} \oplus \mathbb{Z}^{m_1}_{p^{h_1}}) \otimes (\mathbb{Z}^{s_0} \oplus \mathbb{Z}^{t_0}_{p^{l_0}})] \\
&\cong [\mathbb{Z}^{n_0 s_1} \oplus \mathbb{Z}^{m_0 s_1}_{p^{h_0}} \oplus \mathbb{Z}^{n_0 t_1}_{p^{l_1}} \oplus \mathbb{Z}^{m_0 t_1}_{p^{h_0 \wedge l_1}}] \oplus [\mathbb{Z}^{n_1 s_0} \oplus \mathbb{Z}^{m_1 s_0}_{p^{h_1}} \oplus \mathbb{Z}^{n_1 t_0}_{p^{l_0}} \oplus \mathbb{Z}^{m_1 t_0}_{p^{h_1 \wedge l_0}}].
\end{aligned}$$

Hence we get $K_1(\mathfrak{A} \otimes \mathfrak{B}) \cong$

$$[\mathbb{Z}^{n_0 s_1 + n_1 s_0} \oplus \mathbb{Z}^{m_0 s_1}_{p^{h_0}} \oplus \mathbb{Z}^{n_0 t_1}_{p^{l_1}} \oplus \mathbb{Z}^{m_0 t_1}_{p^{h_0 \wedge l_1}} \oplus \mathbb{Z}^{m_1 s_0}_{p^{h_1}} \oplus \mathbb{Z}^{n_1 t_0}_{p^{l_0}} \oplus \mathbb{Z}^{m_1 t_0}_{p^{h_1 \wedge l_0}}] \oplus [\mathbb{Z}^{m_0 t_0}_{p^{h_0 \wedge l_0}} \oplus \mathbb{Z}^{m_1 t_1}_{p^{h_1 \wedge l_1}}].$$

□

On the other hand,

Proposition 3.4. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras with $\mathfrak{A}$ in the category $\mathfrak{N}$. Suppose that $K_j(\mathfrak{A}) = \mathbb{Z}^{n_j} \oplus \mathbb{Z}_p^{m_j}$ and $K_j(\mathfrak{B}) = \mathbb{Z}^{s_j} \oplus \mathbb{Z}_q^{t_j}$ for some positive integers n_j, m_j, s_j, t_j with $j = 0, 1$ and p, q prime numbers relatively prime. Then*

$$K_0(\mathfrak{A} \otimes \mathfrak{B}) \cong \mathbb{Z}^{n_0 s_0 + n_1 s_1} \oplus \mathbb{Z}_p^{m_0 s_0 + m_1 s_1} \oplus \mathbb{Z}_q^{n_0 t_0 + n_1 t_1} \oplus \mathbb{Z}_{p \wedge q}^{m_0 t_0 + m_1 t_1},$$
$$K_1(\mathfrak{A} \otimes \mathfrak{B}) \cong \mathbb{Z}^{n_0 s_1 + n_1 s_0} \oplus \mathbb{Z}_p^{m_0 s_1 + m_1 s_0} \oplus \mathbb{Z}_q^{n_0 t_1 + n_1 t_0} \oplus \mathbb{Z}_{p \wedge q}^{m_0 t_1 + m_1 t_0}$$

where the torsion products are zero.

Proof. We compute

$$\begin{aligned}\mathrm{Tor}_1^{\mathbb{Z}}(K_j(\mathfrak{A}), K_k(\mathfrak{B})) &= \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}^{n_j} \oplus \mathbb{Z}_p^{m_j}, \mathbb{Z}^{s_k} \oplus \mathbb{Z}_q^{t_k}) \\ &\cong \oplus^{m_j t_k} \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_p, \mathbb{Z}_q) \cong 0\end{aligned}$$

for $j = 0, 1$ and $k = 0, 1$. Therefore,

$$\begin{aligned}&K_0(\mathfrak{A} \otimes \mathfrak{B}) \\ &\cong [K_0(\mathfrak{A}) \otimes K_0(\mathfrak{B})] \oplus [K_1(\mathfrak{A}) \otimes K_1(\mathfrak{B})] \\ &= [(\mathbb{Z}^{n_0} \oplus \mathbb{Z}_p^{m_0}) \otimes (\mathbb{Z}^{s_0} \oplus \mathbb{Z}_q^{t_0})] \oplus [(\mathbb{Z}^{n_1} \oplus \mathbb{Z}_p^{m_1}) \otimes (\mathbb{Z}^{s_1} \oplus \mathbb{Z}_q^{t_1})] \\ &\cong [\mathbb{Z}^{n_0 s_0} \oplus \mathbb{Z}_p^{m_0 s_0} \oplus \mathbb{Z}_q^{n_0 t_0} \oplus \mathbb{Z}_{p \wedge q}^{m_0 t_0}] \oplus [\mathbb{Z}^{n_1 s_1} \oplus \mathbb{Z}_p^{m_1 s_1} \oplus \mathbb{Z}_q^{n_1 t_1} \oplus \mathbb{Z}_{p \wedge q}^{m_1 t_1}] \\ &\cong \mathbb{Z}^{n_0 s_0 + n_1 s_1} \oplus \mathbb{Z}_p^{m_0 s_0 + m_1 s_1} \oplus \mathbb{Z}_q^{n_0 t_0 + n_1 t_1} \oplus \mathbb{Z}_{p \wedge q}^{m_0 t_0 + m_1 t_1}.\end{aligned}$$

And also

$$\begin{aligned}&K_1(\mathfrak{A} \otimes \mathfrak{B}) \\ &\cong [K_0(\mathfrak{A}) \otimes K_1(\mathfrak{B})] \oplus [K_1(\mathfrak{A}) \otimes K_0(\mathfrak{B})] \\ &= [(\mathbb{Z}^{n_0} \oplus \mathbb{Z}_p^{m_0}) \otimes (\mathbb{Z}^{s_1} \oplus \mathbb{Z}_q^{t_1})] \oplus [(\mathbb{Z}^{n_1} \oplus \mathbb{Z}_p^{m_1}) \otimes (\mathbb{Z}^{s_0} \oplus \mathbb{Z}_q^{t_0})] \\ &\cong [\mathbb{Z}^{n_0 s_1} \oplus \mathbb{Z}_p^{m_0 s_1} \oplus \mathbb{Z}_q^{n_0 t_1} \oplus \mathbb{Z}_{p \wedge q}^{m_0 t_1}] \oplus [\mathbb{Z}^{n_1 s_0} \oplus \mathbb{Z}_p^{m_1 s_0} \oplus \mathbb{Z}_q^{n_1 t_0} \oplus \mathbb{Z}_{p \wedge q}^{m_1 t_0}] \\ &\cong \mathbb{Z}^{n_0 s_1 + n_1 s_0} \oplus \mathbb{Z}_p^{m_0 s_1 + m_1 s_0} \oplus \mathbb{Z}_q^{n_0 t_1 + n_1 t_0} \oplus \mathbb{Z}_{p \wedge q}^{m_0 t_1 + m_1 t_0}.\end{aligned}$$

□

Remark. The statement above also holds when p and q are replaced with the powers p^k and q^l of p and q for positive integers $k, l \geq 1$, respectively.

Furthermore,

Proposition 3.5. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras with $\mathfrak{A}$ in the category $\mathfrak{N}$. Suppose that $K_j(\mathfrak{A}) = \mathbb{Z}^{n_j} \oplus \mathbb{Z}_{p_j}^{m_j}$ and $K_j(\mathfrak{B}) = \mathbb{Z}^{s_j} \oplus \mathbb{Z}_{q_j}^{t_j}$ for some positive integers n_j, m_j, s_j, t_j with $j = 0, 1$ and p_0, p_1, q_0, q_1 prime numbers which are mutually, relatively prime. Then*

$$K_0(\mathfrak{A} \otimes \mathfrak{B}) \cong \mathbb{Z}^{n_0 s_0 + n_1 s_1} \oplus \mathbb{Z}_{p_0}^{m_0 s_0} \oplus \mathbb{Z}_{q_0}^{n_0 t_0} \oplus \mathbb{Z}_{p_0 \wedge q_0}^{m_0 t_0} \oplus \mathbb{Z}_{p_1}^{m_1 s_1} \oplus \mathbb{Z}_{q_1}^{n_1 t_1} \oplus \mathbb{Z}_{p_1 \wedge q_1}^{m_1 t_1},$$
$$K_1(\mathfrak{A} \otimes \mathfrak{B}) \cong \mathbb{Z}^{n_0 s_1 + n_1 s_0} \oplus \mathbb{Z}_{p_0}^{m_0 s_1} \oplus \mathbb{Z}_{q_1}^{n_0 t_1} \oplus \mathbb{Z}_{p_0 \wedge q_1}^{m_0 t_1} \oplus \mathbb{Z}_{p_1}^{m_1 s_0} \oplus \mathbb{Z}_{q_0}^{n_1 t_0} \oplus \mathbb{Z}_{p_1 \wedge q_0}^{m_1 t_0}.$$

Proof. We compute

$$\mathrm{Tor}_1^{\mathbb{Z}}(K_j(\mathfrak{A}), K_k(\mathfrak{B})) = \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}^{n_j} \oplus \mathbb{Z}_{p_j}^{m_j}, \mathbb{Z}^{s_k} \oplus \mathbb{Z}_{q_k}^{t_k}) \cong \oplus^{m_j t_k} \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_{p_j}, \mathbb{Z}_{q_k}) \cong 0$$

for $j = 0, 1$ and $k = 0, 1$. Therefore,

$$\begin{aligned}
&K_0(\mathfrak{A} \otimes \mathfrak{B}) \\
&\cong [K_0(\mathfrak{A}) \otimes K_0(\mathfrak{B})] \oplus [K_1(\mathfrak{A}) \otimes K_1(\mathfrak{B})] \\
&= [(\mathbb{Z}^{n_0} \oplus \mathbb{Z}_{p_0}^{m_0}) \otimes (\mathbb{Z}^{s_0} \oplus \mathbb{Z}_{q_0}^{t_0})] \oplus [(\mathbb{Z}^{n_1} \oplus \mathbb{Z}_{p_1}^{m_1}) \otimes (\mathbb{Z}^{s_1} \oplus \mathbb{Z}_{q_1}^{t_1})] \\
&\cong [\mathbb{Z}^{n_0 s_0} \oplus \mathbb{Z}_{p_0}^{m_0 s_0} \oplus \mathbb{Z}_{q_0}^{n_0 t_0} \oplus \mathbb{Z}_{p_0 \wedge q_0}^{m_0 t_0}] \oplus [\mathbb{Z}^{n_1 s_1} \oplus \mathbb{Z}_{p_1}^{m_1 s_1} \oplus \mathbb{Z}_{q_1}^{n_1 t_1} \oplus \mathbb{Z}_{p_1 \wedge q_1}^{m_1 t_1}].
\end{aligned}$$

And also

$$\begin{aligned}
&K_1(\mathfrak{A} \otimes \mathfrak{B}) \\
&\cong [K_0(\mathfrak{A}) \otimes K_1(\mathfrak{B})] \oplus [K_1(\mathfrak{A}) \otimes K_0(\mathfrak{B})] \\
&= [(\mathbb{Z}^{n_0} \oplus \mathbb{Z}_{p_0}^{m_0}) \otimes (\mathbb{Z}^{s_1} \oplus \mathbb{Z}_{q_1}^{t_1})] \oplus [(\mathbb{Z}^{n_1} \oplus \mathbb{Z}_{p_1}^{m_1}) \otimes (\mathbb{Z}^{s_0} \oplus \mathbb{Z}_{q_0}^{t_0})] \\
&\cong [\mathbb{Z}^{n_0 s_1} \oplus \mathbb{Z}_{p_0}^{m_0 s_1} \oplus \mathbb{Z}_{q_1}^{n_0 t_1} \oplus \mathbb{Z}_{p_0 \wedge q_1}^{m_0 t_1}] \oplus [\mathbb{Z}^{n_1 s_0} \oplus \mathbb{Z}_{p_1}^{m_1 s_0} \oplus \mathbb{Z}_{q_0}^{n_1 t_0} \oplus \mathbb{Z}_{p_1 \wedge q_0}^{m_1 t_0}].
\end{aligned}$$

□

Remark. The statement above also holds when p_0, p_1 and q_0, q_1 are replaced with their powers $p_0^{k_0}, p_1^{k_0}$ and $q_0^{l_0}, q_1^{l_1}$ of p and q for positive integers $k_0, k_1, l_0, l_1 \geq 1$, respectively.

4 The case with torsion part two products of two cyclic groups

Proposition 4.1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras with $\mathfrak{A}$ in the bootstrap category $\mathfrak{N}$. Suppose that $K_j(\mathfrak{A}) = \mathbb{Z}^{n_j} \oplus \mathbb{Z}_{p^{h_{j1}}}^{m_{j1}} \oplus \mathbb{Z}_{p^{h_{j2}}}^{m_{j2}}$ and $K_j(\mathfrak{B}) = \mathbb{Z}^{s_j} \oplus \mathbb{Z}_{p^{l_{j1}}}^{t_{j1}} \oplus \mathbb{Z}_{p^{l_{j2}}}^{t_{j2}}$ for some positive integers $n_j, m_{j1}, m_{j2}, s_j, t_{j1}, t_{j2}$ with $j = 0, 1$ and p a prime number with $1 \leq h_{j1} < h_{j2}$ and $1 \leq l_{j1} < l_{j2}$. Then*

$$\begin{aligned}
K_0(\mathfrak{A} \otimes \mathfrak{B}) \cong \mathbb{Z}^{n_0 s_0 + n_1 s_1} &\oplus [\mathbb{Z}_{p^{h_{01} \wedge l_{01}}}^{m_{01} t_{01}} \oplus \mathbb{Z}_{p^{h_{01} \wedge l_{02}}}^{m_{01} t_{02}} \oplus \mathbb{Z}_{p^{h_{02} \wedge l_{01}}}^{m_{02} t_{01}} \oplus \mathbb{Z}_{p^{h_{02} \wedge l_{02}}}^{m_{02} t_{02}}] \\
&\oplus [\mathbb{Z}_{p^{h_{11} \wedge l_{11}}}^{m_{11} t_{11}} \oplus \mathbb{Z}_{p^{h_{11} \wedge l_{12}}}^{m_{11} t_{12}} \oplus \mathbb{Z}_{p^{h_{12} \wedge l_{11}}}^{m_{12} t_{11}} \oplus \mathbb{Z}_{p^{h_{12} \wedge l_{12}}}^{m_{11} t_{12}}] \\
&\oplus ([\mathbb{Z}_{p^{h_{01} \wedge l_{11}}}^{m_{01} t_{11}} \oplus \mathbb{Z}_{p^{h_{01} \wedge l_{12}}}^{m_{01} t_{12}} \oplus \mathbb{Z}_{p^{h_{02} \wedge l_{11}}}^{m_{02} t_{11}} \oplus \mathbb{Z}_{p^{h_{02} \wedge l_{12}}}^{m_{02} t_{12}}] \\
&\oplus [\mathbb{Z}_{p^{h_{11} \wedge l_{01}}}^{m_{11} t_{01}} \oplus \mathbb{Z}_{p^{h_{11} \wedge l_{02}}}^{m_{11} t_{02}} \oplus \mathbb{Z}_{p^{h_{12} \wedge l_{01}}}^{m_{12} t_{01}} \oplus \mathbb{Z}_{p^{h_{12} \wedge l_{02}}}^{m_{12} t_{02}}]), \\
K_1(\mathfrak{A} \otimes \mathfrak{B}) \cong \mathbb{Z}^{n_0 s_1 + n_1 s_0} &\oplus [\mathbb{Z}_{p^{h_{01} \wedge l_{11}}}^{m_{01} t_{11}} \oplus \mathbb{Z}_{p^{h_{01} \wedge l_{12}}}^{m_{01} t_{12}} \oplus \mathbb{Z}_{p^{h_{02} \wedge l_{11}}}^{m_{02} t_{11}} \oplus \mathbb{Z}_{p^{h_{02} \wedge l_{12}}}^{m_{02} t_{12}}] \\
&\oplus [\mathbb{Z}_{p^{h_{11} \wedge l_{01}}}^{m_{11} t_{01}} \oplus \mathbb{Z}_{p^{h_{11} \wedge l_{02}}}^{m_{11} t_{02}} \oplus \mathbb{Z}_{p^{h_{12} \wedge l_{01}}}^{m_{12} t_{01}} \oplus \mathbb{Z}_{p^{h_{12} \wedge l_{02}}}^{m_{11} t_{02}}] \\
&\oplus ([\mathbb{Z}_{p^{h_{01} \wedge l_{01}}}^{m_{01} t_{01}} \oplus \mathbb{Z}_{p^{h_{01} \wedge l_{02}}}^{m_{01} t_{02}} \oplus \mathbb{Z}_{p^{h_{02} \wedge l_{01}}}^{m_{02} t_{01}} \oplus \mathbb{Z}_{p^{h_{02} \wedge l_{02}}}^{m_{02} t_{02}}] \\
&\oplus [\mathbb{Z}_{p^{h_{11} \wedge l_{11}}}^{m_{11} t_{11}} \oplus \mathbb{Z}_{p^{h_{11} \wedge l_{12}}}^{m_{11} t_{12}} \oplus \mathbb{Z}_{p^{h_{12} \wedge l_{11}}}^{m_{12} t_{11}} \oplus \mathbb{Z}_{p^{h_{12} \wedge l_{12}}}^{m_{12} t_{12}}]),
\end{aligned}$$

where the last summands $([\cdots] \oplus [\cdots])$ correspond to the respective torsion products.

Proof. We compute the torsion product in the Künneth theorem for tensor products of C^*-algebras using several facts in homology theory as in [3]:

$$\begin{aligned}
&\mathrm{Tor}_1^{\mathbb{Z}}(K_j(\mathfrak{A}), K_k(\mathfrak{B})) \\
&= \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}^{n_j} \oplus \mathbb{Z}_{p^{h_{j1}}}^{m_{j1}} \oplus \mathbb{Z}_{p^{h_{j2}}}^{m_{j2}}, \mathbb{Z}^{s_k} \oplus \mathbb{Z}_{p^{l_{k1}}}^{t_{k1}} \oplus \mathbb{Z}_{p^{l_{k2}}}^{t_{k2}}) \\
&= \mathbb{Z}_{p^{h_{j1} \wedge l_{k1}}}^{m_{j1} t_{k1}} \oplus \mathbb{Z}_{p^{h_{j1} \wedge l_{k2}}}^{m_{j1} t_{k2}} \oplus \mathbb{Z}_{p^{h_{j2} \wedge l_{k1}}}^{m_{j2} t_{k1}} \oplus \mathbb{Z}_{p^{h_{j2} \wedge l_{k2}}}^{m_{j2} t_{k2}}
\end{aligned}$$

for $j = 0, 1$ and $k = 0, 1$. Note that $\mathrm{Tor}_1^{\mathbb{Z}}(K_j(\mathfrak{A}), K_k(\mathfrak{B}))$ appears in the split quotient of $K_{j+k+1}(\mathfrak{A} \otimes \mathfrak{B})$. Therefore,

$$
\begin{aligned}
& K_0(\mathfrak{A} \otimes \mathfrak{B})/([\mathbb{Z}_{p^{h_{01}\wedge l_{11}}}^{m_{01}t_{11}} \oplus \mathbb{Z}_{p^{h_{01}\wedge l_{12}}}^{m_{01}t_{12}} \oplus \mathbb{Z}_{p^{h_{02}\wedge l_{11}}}^{m_{02}t_{11}} \oplus \mathbb{Z}_{p^{h_{02}\wedge l_{12}}}^{m_{02}t_{12}}] \\
& \quad \oplus [\mathbb{Z}_{p^{h_{11}\wedge l_{01}}}^{m_{11}t_{01}} \oplus \mathbb{Z}_{p^{h_{11}\wedge l_{02}}}^{m_{11}t_{02}} \oplus \mathbb{Z}_{p^{h_{12}\wedge l_{01}}}^{m_{12}t_{01}} \oplus \mathbb{Z}_{p^{h_{12}\wedge l_{02}}}^{m_{12}t_{02}}]) \\
& \cong [K_0(\mathfrak{A}) \otimes K_0(\mathfrak{B})] \oplus [K_1(\mathfrak{A}) \otimes K_1(\mathfrak{B})] \\
& = [(\mathbb{Z}^{n_0} \oplus \mathbb{Z}_{p^{h_{01}}}^{m_{01}} \oplus \mathbb{Z}_{p^{h_{02}}}^{m_{02}}) \otimes (\mathbb{Z}^{s_0} \oplus \mathbb{Z}_{p^{l_{01}}}^{t_{01}} \oplus \mathbb{Z}_{p^{l_{02}}}^{t_{02}})] \\
& \quad \oplus [(\mathbb{Z}^{n_1} \oplus \mathbb{Z}_{p^{h_{11}}}^{m_{11}} \oplus \mathbb{Z}_{p^{h_{12}}}^{m_{12}}) \otimes (\mathbb{Z}^{s_1} \oplus \mathbb{Z}_{p^{l_{11}}}^{t_{11}} \oplus \mathbb{Z}_{p^{l_{12}}}^{t_{12}})] \\
& \cong [\mathbb{Z}^{n_0 s_0} \oplus \mathbb{Z}_{p^{h_{01}\wedge l_{01}}}^{m_{01}t_{01}} \oplus \mathbb{Z}_{p^{h_{01}\wedge l_{02}}}^{m_{01}t_{02}} \oplus \mathbb{Z}_{p^{h_{02}\wedge l_{01}}}^{m_{02}t_{01}} \oplus \mathbb{Z}_{p^{h_{02}\wedge l_{02}}}^{m_{02}t_{02}}] \\
& \oplus [\mathbb{Z}^{n_1 s_1} \oplus \mathbb{Z}_{p^{h_{11}\wedge l_{11}}}^{m_{11}t_{11}} \oplus \mathbb{Z}_{p^{h_{11}\wedge l_{12}}}^{m_{11}t_{12}} \oplus \mathbb{Z}_{p^{h_{12}\wedge l_{11}}}^{m_{12}t_{11}} \oplus \mathbb{Z}_{p^{h_{12}\wedge l_{12}}}^{m_{11}t_{12}}].
\end{aligned}
$$

And also

$$
\begin{aligned}
& K_1(\mathfrak{A} \otimes \mathfrak{B})/([\mathbb{Z}_{p^{h_{01}\wedge l_{01}}}^{m_{01}t_{01}} \oplus \mathbb{Z}_{p^{h_{01}\wedge l_{02}}}^{m_{01}t_{02}} \oplus \mathbb{Z}_{p^{h_{02}\wedge l_{01}}}^{m_{02}t_{01}} \oplus \mathbb{Z}_{p^{h_{02}\wedge l_{02}}}^{m_{02}t_{02}}] \\
& \quad \oplus [\mathbb{Z}_{p^{h_{11}\wedge l_{11}}}^{m_{11}t_{11}} \oplus \mathbb{Z}_{p^{h_{11}\wedge l_{12}}}^{m_{11}t_{12}} \oplus \mathbb{Z}_{p^{h_{12}\wedge l_{11}}}^{m_{12}t_{11}} \oplus \mathbb{Z}_{p^{h_{12}\wedge l_{12}}}^{m_{12}t_{12}}]) \\
& \cong [K_0(\mathfrak{A}) \otimes K_1(\mathfrak{B})] \oplus [K_1(\mathfrak{A}) \otimes K_0(\mathfrak{B})] \\
& = [(\mathbb{Z}^{n_0} \oplus \mathbb{Z}_{p^{h_{01}}}^{m_{01}} \oplus \mathbb{Z}_{p^{h_{02}}}^{m_{02}}) \otimes (\mathbb{Z}^{s_1} \oplus \mathbb{Z}_{p^{l_{11}}}^{t_{11}} \oplus \mathbb{Z}_{p^{l_{12}}}^{t_{12}})] \\
& \quad \oplus [(\mathbb{Z}^{n_1} \oplus \mathbb{Z}_{p^{h_{11}}}^{m_{11}} \oplus \mathbb{Z}_{p^{h_{12}}}^{m_{12}}) \otimes (\mathbb{Z}^{s_0} \oplus \mathbb{Z}_{p^{l_{01}}}^{t_{01}} \oplus \mathbb{Z}_{p^{l_{02}}}^{t_{02}})] \\
& \cong [\mathbb{Z}^{n_0 s_1} \oplus \mathbb{Z}_{p^{h_{01}\wedge l_{11}}}^{m_{01}t_{11}} \oplus \mathbb{Z}_{p^{h_{01}\wedge l_{12}}}^{m_{01}t_{12}} \oplus \mathbb{Z}_{p^{h_{02}\wedge l_{11}}}^{m_{02}t_{11}} \oplus \mathbb{Z}_{p^{h_{02}\wedge l_{12}}}^{m_{02}t_{12}}] \\
& \oplus [\mathbb{Z}^{n_1 s_0} \oplus \mathbb{Z}_{p^{h_{11}\wedge l_{01}}}^{m_{11}t_{01}} \oplus \mathbb{Z}_{p^{h_{11}\wedge l_{02}}}^{m_{11}t_{02}} \oplus \mathbb{Z}_{p^{h_{12}\wedge l_{01}}}^{m_{12}t_{01}} \oplus \mathbb{Z}_{p^{h_{12}\wedge l_{02}}}^{m_{11}t_{02}}].
\end{aligned}
$$

□

Proposition 4.2. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras with $\mathfrak{A}$ in the category $\mathfrak{N}$. Suppose that $K_j(\mathfrak{A}) = \mathbb{Z}^{n_j} \oplus \mathbb{Z}_{p^{h_{j1}}}^{m_{j1}} \oplus \mathbb{Z}_{q^{h_{j2}}}^{m_{j2}}$ and $K_j(\mathfrak{B}) = \mathbb{Z}^{s_j} \oplus \mathbb{Z}_{p^{l_{j1}}}^{t_{j1}} \oplus \mathbb{Z}_{q^{l_{j2}}}^{t_{j2}}$ for some positive integers n_j, m_{j1}, m_{j2}, s_j, t_{j1}, t_{j2} with $j = 0, 1$ and p, q prime numbers relatively prime with $h_{j1}, h_{j2} \geq 1$ and $l_{j1}, l_{j2} \geq 1$. Then*

$$
\begin{aligned}
K_0(\mathfrak{A} \otimes \mathfrak{B}) \cong \mathbb{Z}^{n_0 s_0 + n_1 s_1} & \oplus [\mathbb{Z}_{p^{h_{01}\wedge l_{01}}}^{m_{01}t_{01}} \oplus \mathbb{Z}_{p^{h_{01}}\wedge q^{l_{02}}}^{m_{01}t_{02}} \oplus \mathbb{Z}_{q^{h_{02}}\wedge p^{l_{01}}}^{m_{02}t_{01}} \oplus \mathbb{Z}_{q^{h_{02}\wedge l_{02}}}^{m_{02}t_{02}}] \\
& \oplus [\mathbb{Z}_{p^{h_{11}\wedge l_{11}}}^{m_{11}t_{11}} \oplus \mathbb{Z}_{p^{h_{11}}\wedge q^{l_{12}}}^{m_{11}t_{12}} \oplus \mathbb{Z}_{q^{h_{12}}\wedge p^{l_{11}}}^{m_{12}t_{11}} \oplus \mathbb{Z}_{q^{h_{12}\wedge l_{12}}}^{m_{11}t_{12}}] \\
& \oplus ([\mathbb{Z}_{p^{h_{01}\wedge l_{11}}}^{m_{01}t_{11}} \oplus \mathbb{Z}_{q^{h_{02}\wedge l_{12}}}^{m_{02}t_{12}} \oplus \mathbb{Z}_{p^{h_{11}\wedge l_{01}}}^{m_{11}t_{01}} \oplus \mathbb{Z}_{q^{h_{12}\wedge l_{02}}}^{m_{12}t_{02}}]), \\
K_1(\mathfrak{A} \otimes \mathfrak{B}) \cong \mathbb{Z}^{n_0 s_1 + n_1 s_0} & \oplus [\mathbb{Z}_{p^{h_{01}\wedge l_{11}}}^{m_{01}t_{11}} \oplus \mathbb{Z}_{p^{h_{01}}\wedge q^{l_{12}}}^{m_{01}t_{12}} \oplus \mathbb{Z}_{q^{h_{02}}\wedge p^{l_{11}}}^{m_{02}t_{11}} \oplus \mathbb{Z}_{q^{h_{02}\wedge l_{12}}}^{m_{02}t_{12}}] \\
& \oplus [\mathbb{Z}_{p^{h_{11}\wedge l_{01}}}^{m_{11}t_{01}} \oplus \mathbb{Z}_{p^{h_{11}}\wedge q^{l_{02}}}^{m_{11}t_{02}} \oplus \mathbb{Z}_{q^{h_{12}}\wedge p^{l_{01}}}^{m_{12}t_{01}} \oplus \mathbb{Z}_{q^{h_{12}\wedge l_{02}}}^{m_{11}t_{02}}] \\
& \oplus ([\mathbb{Z}_{p^{h_{01}\wedge l_{01}}}^{m_{01}t_{01}} \oplus \mathbb{Z}_{q^{h_{02}\wedge l_{02}}}^{m_{02}t_{02}} \oplus \mathbb{Z}_{p^{h_{11}\wedge l_{11}}}^{m_{11}t_{11}} \oplus \mathbb{Z}_{q^{h_{12}\wedge l_{12}}}^{m_{12}t_{12}}]),
\end{aligned}
$$

where the last summands $([\cdots])$ correspond to the respective torsion products.

Proof. We compute the torsion product in the Künneth theorem for tensor products of C^*-algebras using several facts in homology theory as in [3]:

$$
\begin{aligned}
& \mathrm{Tor}_1^{\mathbb{Z}}(K_j(\mathfrak{A}), K_k(\mathfrak{B})) \\
& = \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}^{n_j} \oplus \mathbb{Z}_{p^{h_{j1}}}^{m_{j1}} \oplus \mathbb{Z}_{q^{h_{j2}}}^{m_{j2}}, \mathbb{Z}^{s_k} \oplus \mathbb{Z}_{p^{l_{k1}}}^{t_{k1}} \oplus \mathbb{Z}_{q^{l_{k2}}}^{t_{k2}}) \\
& = \mathbb{Z}_{p^{h_{j1}\wedge l_{k1}}}^{m_{j1}t_{k1}} \oplus \mathbb{Z}_{q^{h_{j2}\wedge l_{k2}}}^{m_{j2}t_{k2}}
\end{aligned}
$$

for $j = 0, 1$ and $k = 0, 1$. Note that $\mathrm{Tor}_1^{\mathbb{Z}}(K_j(\mathfrak{A}), K_k(\mathfrak{B}))$ appears in the split quotient of $K_{j+k+1}(\mathfrak{A} \otimes \mathfrak{B})$. Therefore,

$$\begin{aligned}
&K_0(\mathfrak{A} \otimes \mathfrak{B})/([\mathbb{Z}^{m_{01}t_{11}}_{p^{h_{01}\wedge l_{11}}} \oplus \mathbb{Z}^{m_{02}t_{12}}_{q^{h_{02}\wedge l_{12}}}] \\
&\quad \oplus [\mathbb{Z}^{m_{11}t_{01}}_{p^{h_{11}\wedge l_{01}}} \oplus \mathbb{Z}^{m_{12}t_{02}}_{q^{h_{12}\wedge l_{02}}}]) \\
&\cong [K_0(\mathfrak{A}) \otimes K_0(\mathfrak{B})] \oplus [K_1(\mathfrak{A}) \otimes K_1(\mathfrak{B})] \\
&= [(\mathbb{Z}^{n_0} \oplus \mathbb{Z}^{m_{01}}_{p^{h_{01}}} \oplus \mathbb{Z}^{m_{02}}_{q^{h_{02}}}) \otimes (\mathbb{Z}^{s_0} \oplus \mathbb{Z}^{t_{01}}_{p^{l_{01}}} \oplus \mathbb{Z}^{t_{02}}_{q^{l_{02}}})] \\
&\quad \oplus [(\mathbb{Z}^{n_1} \oplus \mathbb{Z}^{m_{11}}_{p^{h_{11}}} \oplus \mathbb{Z}^{m_{12}}_{q^{h_{12}}}) \otimes (\mathbb{Z}^{s_1} \oplus \mathbb{Z}^{t_{11}}_{p^{l_{11}}} \oplus \mathbb{Z}^{t_{12}}_{q^{l_{12}}})] \\
&\cong [\mathbb{Z}^{n_0 s_0} \oplus \mathbb{Z}^{m_{01}t_{01}}_{p^{h_{01}\wedge l_{01}}} \oplus \mathbb{Z}^{m_{01}t_{02}}_{p^{h_{01}}\wedge q^{l_{02}}} \oplus \mathbb{Z}^{m_{02}t_{01}}_{q^{h_{02}}\wedge p^{l_{01}}} \oplus \mathbb{Z}^{m_{02}t_{02}}_{q^{h_{02}\wedge l_{02}}}] \\
&\quad \oplus [\mathbb{Z}^{n_1 s_1} \oplus \mathbb{Z}^{m_{11}t_{11}}_{p^{h_{11}\wedge l_{11}}} \oplus \mathbb{Z}^{m_{11}t_{12}}_{p^{h_{11}}\wedge q^{l_{12}}} \oplus \mathbb{Z}^{m_{12}t_{11}}_{q^{h_{12}}\wedge p^{l_{11}}} \oplus \mathbb{Z}^{m_{11}t_{12}}_{q^{h_{12}\wedge l_{12}}}].
\end{aligned}$$

And also

$$\begin{aligned}
&K_1(\mathfrak{A} \otimes \mathfrak{B})/([\mathbb{Z}^{m_{01}t_{01}}_{p^{h_{01}\wedge l_{01}}} \oplus \mathbb{Z}^{m_{02}t_{02}}_{q^{h_{02}\wedge l_{02}}}] \\
&\quad \oplus [\mathbb{Z}^{m_{11}t_{11}}_{p^{h_{11}\wedge l_{11}}} \oplus \mathbb{Z}^{m_{12}t_{12}}_{q^{h_{12}\wedge l_{12}}}]) \\
&\cong [K_0(\mathfrak{A}) \otimes K_1(\mathfrak{B})] \oplus [K_1(\mathfrak{A}) \otimes K_0(\mathfrak{B})] \\
&= [(\mathbb{Z}^{n_0} \oplus \mathbb{Z}^{m_{01}}_{p^{h_{01}}} \oplus \mathbb{Z}^{m_{02}}_{q^{h_{02}}}) \otimes (\mathbb{Z}^{s_1} \oplus \mathbb{Z}^{t_{11}}_{p^{l_{11}}} \oplus \mathbb{Z}^{t_{12}}_{q^{l_{12}}})] \\
&\quad \oplus [(\mathbb{Z}^{n_1} \oplus \mathbb{Z}^{m_{11}}_{p^{h_{11}}} \oplus \mathbb{Z}^{m_{12}}_{q^{h_{12}}}) \otimes (\mathbb{Z}^{s_0} \oplus \mathbb{Z}^{t_{01}}_{p^{l_{01}}} \oplus \mathbb{Z}^{t_{02}}_{q^{l_{02}}})] \\
&\cong [\mathbb{Z}^{n_0 s_1} \oplus \mathbb{Z}^{m_{01}t_{11}}_{p^{h_{01}\wedge l_{11}}} \oplus \mathbb{Z}^{m_{01}t_{12}}_{p^{h_{01}}\wedge q^{l_{12}}} \oplus \mathbb{Z}^{m_{02}t_{11}}_{q^{h_{02}}\wedge p^{l_{11}}} \oplus \mathbb{Z}^{m_{02}t_{12}}_{q^{h_{02}\wedge l_{12}}}] \\
&\quad \oplus [\mathbb{Z}^{n_1 s_0} \oplus \mathbb{Z}^{m_{11}t_{01}}_{p^{h_{11}\wedge l_{01}}} \oplus \mathbb{Z}^{m_{11}t_{02}}_{p^{h_{11}}\wedge q^{l_{02}}} \oplus \mathbb{Z}^{m_{12}t_{01}}_{q^{h_{12}}\wedge p^{l_{01}}} \oplus \mathbb{Z}^{m_{11}t_{02}}_{q^{h_{12}\wedge l_{02}}}].
\end{aligned}$$

□

Proposition 4.3. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras with $\mathfrak{A}$ in the category $\mathfrak{N}$. Suppose that $K_j(\mathfrak{A}) = \mathbb{Z}^{n_j} \oplus \mathbb{Z}^{m_{j1}}_{p_1^{h_{j1}}} \oplus \mathbb{Z}^{m_{j2}}_{p_2^{h_{j2}}}$ and $K_j(\mathfrak{B}) = \mathbb{Z}^{s_j} \oplus \mathbb{Z}^{t_{j1}}_{q_1^{l_{j1}}} \oplus \mathbb{Z}^{t_{j2}}_{q_2^{l_{j2}}}$ for some positive integers n_j, m_{j1}, m_{j2}, s_j, t_{j1}, t_{j2} with $j = 0, 1$ and p_1, p_2, q_1, q_2 prime numbers mutually relatively prime with $h_{j1}, h_{j2} \geq 1$ and $l_{j1}, l_{j2} \geq 1$. Then*

$$\begin{aligned}
K_0(\mathfrak{A} \otimes \mathfrak{B}) &\cong \mathbb{Z}^{n_0 s_0 + n_1 s_1} \oplus [\mathbb{Z}^{m_{01}t_{01}}_{p_1^{h_{01}}\wedge q_1^{l_{01}}} \oplus \mathbb{Z}^{m_{01}t_{02}}_{p_1^{h_{01}}\wedge q_2^{l_{02}}} \oplus \mathbb{Z}^{m_{02}t_{01}}_{p_2^{h_{02}}\wedge q_1^{l_{01}}} \oplus \mathbb{Z}^{m_{02}t_{02}}_{p_2^{h_{02}}\wedge q_2^{l_{02}}}] \\
&\quad \oplus [\mathbb{Z}^{m_{11}t_{11}}_{p_1^{h_{11}}\wedge q_1^{l_{11}}} \oplus \mathbb{Z}^{m_{11}t_{12}}_{p_1^{h_{11}}\wedge q_2^{l_{12}}} \oplus \mathbb{Z}^{m_{12}t_{11}}_{p_2^{h_{12}}\wedge q_1^{l_{11}}} \oplus \mathbb{Z}^{m_{11}t_{12}}_{p_2^{h_{12}}\wedge q_2^{l_{12}}}], \\
K_1(\mathfrak{A} \otimes \mathfrak{B}) &\cong \mathbb{Z}^{n_0 s_1 + n_1 s_0} \oplus [\mathbb{Z}^{m_{01}t_{11}}_{p_1^{h_{01}}\wedge q_1^{l_{11}}} \oplus \mathbb{Z}^{m_{01}t_{12}}_{p_1^{h_{01}}\wedge q_2^{l_{12}}} \oplus \mathbb{Z}^{m_{02}t_{11}}_{p_2^{h_{02}}\wedge q_1^{l_{11}}} \oplus \mathbb{Z}^{m_{02}t_{12}}_{p_2^{h_{02}}\wedge q_2^{l_{12}}}] \\
&\quad \oplus [\mathbb{Z}^{m_{11}t_{01}}_{p_1^{h_{11}}\wedge q_1^{l_{01}}} \oplus \mathbb{Z}^{m_{11}t_{02}}_{p_1^{h_{11}}\wedge q_2^{l_{02}}} \oplus \mathbb{Z}^{m_{12}t_{01}}_{p_2^{h_{12}}\wedge q_1^{l_{01}}} \oplus \mathbb{Z}^{m_{11}t_{02}}_{p_2^{h_{12}}\wedge q_2^{l_{02}}}],
\end{aligned}$$

where the torsion products are zero.

Proof. We compute the torsion product in the Künneth theorem for tensor products of C^*-algebras using several facts in homology theory as in [3]:

$$\begin{aligned}
&\mathrm{Tor}_1^{\mathbb{Z}}(K_j(\mathfrak{A}), K_k(\mathfrak{B})) \\
&= \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}^{n_j} \oplus \mathbb{Z}^{m_{j1}}_{p_1^{h_{j1}}} \oplus \mathbb{Z}^{m_{j2}}_{p_2^{h_{j2}}}, \mathbb{Z}^{s_k} \oplus \mathbb{Z}^{t_{k1}}_{q_1^{l_{k1}}} \oplus \mathbb{Z}^{t_{k2}}_{q_2^{l_{k2}}}) \cong 0
\end{aligned}$$

for $j = 0, 1$ and $k = 0, 1$. Therefore,

$$
\begin{aligned}
&K_0(\mathfrak{A} \otimes \mathfrak{B}) \\
&\cong [K_0(\mathfrak{A}) \otimes K_0(\mathfrak{B})] \oplus [K_1(\mathfrak{A}) \otimes K_1(\mathfrak{B})] \\
&= [(\mathbb{Z}^{n_0} \oplus \mathbb{Z}^{m_{01}}_{p_1^{h_{01}}} \oplus \mathbb{Z}^{m_{02}}_{p_2^{h_{02}}}) \otimes (\mathbb{Z}^{s_0} \oplus \mathbb{Z}^{t_{01}}_{q_1^{l_{01}}} \oplus \mathbb{Z}^{t_{02}}_{q_2^{l_{02}}})] \\
&\quad \oplus [(\mathbb{Z}^{n_1} \oplus \mathbb{Z}^{m_{11}}_{p_1^{h_{11}}} \oplus \mathbb{Z}^{m_{12}}_{p_2^{h_{12}}}) \otimes (\mathbb{Z}^{s_1} \oplus \mathbb{Z}^{t_{11}}_{q_1^{l_{11}}} \oplus \mathbb{Z}^{t_{12}}_{q_2^{l_{12}}})] \\
&\cong [\mathbb{Z}^{n_0 s_0} \oplus \mathbb{Z}^{m_{01}t_{01}}_{p_1^{h_{01}} \wedge q_1^{l_{01}}} \oplus \mathbb{Z}^{m_{01}t_{02}}_{p_1^{h_{01}} \wedge q_2^{l_{02}}} \oplus \mathbb{Z}^{m_{02}t_{01}}_{p_2^{h_{02}} \wedge q_1^{l_{01}}} \oplus \mathbb{Z}^{m_{02}t_{02}}_{p_2^{h_{02}} \wedge q_2^{l_{02}}}] \\
&\quad \oplus [\mathbb{Z}^{n_1 s_1} \oplus \mathbb{Z}^{m_{11}t_{11}}_{p_1^{h_{11}} \wedge q_1^{l_{11}}} \oplus \mathbb{Z}^{m_{11}t_{12}}_{p_1^{h_{11}} \wedge q_2^{l_{12}}} \oplus \mathbb{Z}^{m_{12}t_{11}}_{p_2^{h_{12}} \wedge q_1^{l_{11}}} \oplus \mathbb{Z}^{m_{11}t_{12}}_{p_2^{h_{12}} \wedge q_2^{l_{12}}}].
\end{aligned}
$$

And also

$$
\begin{aligned}
&K_1(\mathfrak{A} \otimes \mathfrak{B}) \\
&\cong [K_0(\mathfrak{A}) \otimes K_1(\mathfrak{B})] \oplus [K_1(\mathfrak{A}) \otimes K_0(\mathfrak{B})] \\
&= [(\mathbb{Z}^{n_0} \oplus \mathbb{Z}^{m_{01}}_{p_1^{h_{01}}} \oplus \mathbb{Z}^{m_{02}}_{p_2^{h_{02}}}) \otimes (\mathbb{Z}^{s_1} \oplus \mathbb{Z}^{t_{11}}_{q_1^{l_{11}}} \oplus \mathbb{Z}^{t_{12}}_{q_2^{l_{12}}})] \\
&\quad \oplus [(\mathbb{Z}^{n_1} \oplus \mathbb{Z}^{m_{11}}_{p_1^{h_{11}}} \oplus \mathbb{Z}^{m_{12}}_{p_2^{h_{12}}}) \otimes (\mathbb{Z}^{s_0} \oplus \mathbb{Z}^{t_{01}}_{q_1^{l_{01}}} \oplus \mathbb{Z}^{t_{02}}_{q_2^{l_{02}}})] \\
&\cong [\mathbb{Z}^{n_0 s_1} \oplus \mathbb{Z}^{m_{01}t_{11}}_{p_1^{h_{01}} \wedge q_1^{l_{11}}} \oplus \mathbb{Z}^{m_{01}t_{12}}_{p_1^{h_{01}} \wedge q_2^{l_{12}}} \oplus \mathbb{Z}^{m_{02}t_{11}}_{p_2^{h_{02}} \wedge q_1^{l_{11}}} \oplus \mathbb{Z}^{m_{02}t_{12}}_{p_2^{h_{02}} \wedge q_2^{l_{12}}}] \\
&\quad \oplus [\mathbb{Z}^{n_1 s_0} \oplus \mathbb{Z}^{m_{11}t_{01}}_{p_1^{h_{11}} \wedge q_1^{l_{01}}} \oplus \mathbb{Z}^{m_{11}t_{02}}_{p_1^{h_{11}} \wedge q_2^{l_{02}}} \oplus \mathbb{Z}^{m_{12}t_{01}}_{p_2^{h_{12}} \wedge q_1^{l_{01}}} \oplus \mathbb{Z}^{m_{11}t_{02}}_{p_2^{h_{12}} \wedge q_2^{l_{02}}}].
\end{aligned}
$$

□

5 The general case with torsion part finite products of cyclic groups

Theorem 5.1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras with $\mathfrak{A}$ in the bootstrap category $\mathfrak{N}$. Suppose that*

$$
\begin{aligned}
K_j(\mathfrak{A}) &= \mathbb{Z}^{n_j} \oplus \mathbb{Z}^{m_{j1}}_{p^{h_{j1}}} \oplus \mathbb{Z}^{m_{j2}}_{p^{h_{j2}}} \oplus \cdots \oplus \mathbb{Z}^{m_{ju}}_{p^{h_{ju}}}, \\
K_j(\mathfrak{B}) &= \mathbb{Z}^{s_j} \oplus \mathbb{Z}^{t_{j1}}_{p^{l_{j1}}} \oplus \mathbb{Z}^{t_{j2}}_{p^{l_{j2}}} \oplus \cdots \oplus \mathbb{Z}^{t_{ju}}_{p^{l_{ju}}}
\end{aligned}
$$

for some positive integers $n_j, m_{j1}, m_{j2}, \cdots, m_{ju}$, $s_j, t_{j1}, t_{j2}, \cdots, t_{ju}$ with $j = 0, 1$ and p a prime number with $1 \leq h_{j1} < h_{j2} < \cdots < h_{ju}$ and $1 \leq l_{j1} < l_{j2} < \cdots < l_{ju}$, and for some integer $u \geq 2$. Then

$$
\begin{aligned}
K_0(\mathfrak{A} \otimes \mathfrak{B}) &\cong \mathbb{Z}^{n_0 s_0 + n_1 s_1} \oplus [\oplus_{x,y=1}^{u} \mathbb{Z}^{m_{0x}t_{0y}}_{p^{h_{0x} \wedge l_{0y}}}] \oplus [\oplus_{x,y=1}^{u} \mathbb{Z}^{m_{1x}t_{1y}}_{p^{h_{1x} \wedge l_{1y}}}] \\
&\quad \oplus ([\oplus_{x,y=1}^{u} \mathbb{Z}^{m_{0x}t_{1y}}_{p^{h_{0x} \wedge l_{1y}}}] \oplus [\oplus_{x,y=1}^{u} \mathbb{Z}^{m_{1x}t_{0y}}_{p^{h_{1x} \wedge l_{0y}}}]), \\
K_1(\mathfrak{A} \otimes \mathfrak{B}) &\cong \mathbb{Z}^{n_0 s_1 + n_1 s_0} \oplus [\oplus_{x,y=1}^{u} \mathbb{Z}^{m_{0x}t_{1y}}_{p^{h_{0x} \wedge l_{1y}}}] \oplus [\oplus_{x,y=1}^{u} \mathbb{Z}^{m_{1x}t_{0y}}_{p^{h_{1x} \wedge l_{0y}}}] \\
&\quad \oplus ([\oplus_{x,y=1}^{u} \mathbb{Z}^{m_{0x}t_{0y}}_{p^{h_{0x} \wedge l_{0y}}}] \oplus [\oplus_{x,y=1}^{u} \mathbb{Z}^{m_{1x}t_{1y}}_{p^{h_{1x} \wedge l_{1y}}}])
\end{aligned}
$$

where the last summands $([\cdots] \oplus [\cdots])$ correspond to the respective torsion products.

Proof. We compute the torsion product in the Künneth theorem for tensor products of C^*-algebras using several facts in homology theory as in [3]:

$$
\begin{aligned}
&\mathrm{Tor}_1^{\mathbb{Z}}(K_j(\mathfrak{A}), K_k(\mathfrak{B})) \\
&= \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}^{n_j} \oplus \mathbb{Z}^{m_{j1}}_{p^{h_{j1}}} \oplus \cdots \oplus \mathbb{Z}^{m_{ju}}_{p^{h_{ju}}}, \mathbb{Z}^{s_k} \oplus \mathbb{Z}^{t_{k1}}_{p^{l_{k1}}} \oplus \cdots \oplus \mathbb{Z}^{t_{ku}}_{p^{l_{ku}}}) \\
&= \oplus_{x,y=1}^{u} \mathbb{Z}^{m_{jx}t_{ky}}_{p^{h_{jx} \wedge l_{ky}}}
\end{aligned}
$$

for $j = 0,1$ and $k = 0,1$. Note that $\mathrm{Tor}_1^{\mathbb{Z}}(K_j(\mathfrak{A}), K_k(\mathfrak{B}))$ appears in the split quotient of $K_{j+k+1}(\mathfrak{A} \otimes \mathfrak{B})$. Therefore,

$$\begin{aligned}
&K_0(\mathfrak{A} \otimes \mathfrak{B})/([\oplus_{x,y=1}^u \mathbb{Z}_{p^{h_{0x} \wedge l_{1y}}}^{m_{0x}t_{1y}}] \oplus [\oplus_{x,y=1}^u \mathbb{Z}_{p^{h_{1x} \wedge l_{0y}}}^{m_{1x}t_{0y}}]) \\
&\cong [K_0(\mathfrak{A}) \otimes K_0(\mathfrak{B})] \oplus [K_1(\mathfrak{A}) \otimes K_1(\mathfrak{B})] \\
&= [(\mathbb{Z}^{n_0} \oplus \mathbb{Z}_{p^{h_{01}}}^{m_{01}} \oplus \cdots \oplus \mathbb{Z}_{p^{h_{0u}}}^{m_{0u}}) \otimes (\mathbb{Z}^{s_0} \oplus \mathbb{Z}_{p^{l_{01}}}^{t_{01}} \oplus \cdots \oplus \mathbb{Z}_{p^{l_{0u}}}^{t_{0u}})] \\
&\quad \oplus [(\mathbb{Z}^{n_1} \oplus \mathbb{Z}_{p^{h_{11}}}^{m_{11}} \oplus \cdots \oplus \mathbb{Z}_{p^{h_{1u}}}^{m_{1u}}) \otimes (\mathbb{Z}^{s_1} \oplus \mathbb{Z}_{p^{l_{11}}}^{t_{11}} \oplus \cdots \oplus \mathbb{Z}_{p^{l_{1u}}}^{t_{1u}})] \\
&\cong \mathbb{Z}^{n_0 s_0} \oplus [\oplus_{x,y=1}^u \mathbb{Z}_{p^{h_{0x} \wedge l_{0y}}}^{m_{0x}t_{0y}}] \\
&\quad \oplus \mathbb{Z}^{n_1 s_1} \oplus [\oplus_{x,y=1}^u \mathbb{Z}_{p^{h_{1x} \wedge l_{1y}}}^{m_{1x}t_{1y}}].
\end{aligned}$$

And also

$$\begin{aligned}
&K_1(\mathfrak{A} \otimes \mathfrak{B})/([\oplus_{x,y=1}^u \mathbb{Z}_{p^{h_{0x} \wedge l_{0y}}}^{m_{0x}t_{0y}}] \oplus [\oplus_{x,y=1}^u \mathbb{Z}_{p^{h_{1x} \wedge l_{1y}}}^{m_{1x}t_{1y}}]) \\
&\cong [K_0(\mathfrak{A}) \otimes K_1(\mathfrak{B})] \oplus [K_1(\mathfrak{A}) \otimes K_0(\mathfrak{B})] \\
&= [(\mathbb{Z}^{n_0} \oplus \mathbb{Z}_{p^{h_{01}}}^{m_{01}} \oplus \cdots \oplus \mathbb{Z}_{p^{h_{0u}}}^{m_{0u}}) \otimes (\mathbb{Z}^{s_1} \oplus \mathbb{Z}_{p^{l_{11}}}^{t_{11}} \oplus \cdots \oplus \mathbb{Z}_{p^{l_{1u}}}^{t_{1u}})] \\
&\quad \oplus [(\mathbb{Z}^{n_1} \oplus \mathbb{Z}_{p^{h_{11}}}^{m_{11}} \oplus \cdots \oplus \mathbb{Z}_{p^{h_{1u}}}^{m_{1u}}) \otimes (\mathbb{Z}^{s_0} \oplus \mathbb{Z}_{p^{l_{01}}}^{t_{01}} \oplus \cdots \oplus \mathbb{Z}_{p^{l_{0u}}}^{t_{0u}})] \\
&\cong \mathbb{Z}^{n_0 s_1} \oplus [\oplus_{x,y=1}^u \mathbb{Z}_{p^{h_{0x} \wedge l_{1y}}}^{m_{0x}t_{1y}}] \\
&\quad \oplus \mathbb{Z}^{n_1 s_0} \oplus [\oplus_{x,y=1}^u \mathbb{Z}_{p^{h_{1x} \wedge l_{0y}}}^{m_{1x}t_{0y}}].
\end{aligned}$$

□

Theorem 5.2. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras with $\mathfrak{A}$ in the category $\mathfrak{N}$. Suppose that*

$$K_j(\mathfrak{A}) = \mathbb{Z}^{n_j} \oplus \mathbb{Z}_{p_1^{h_{j1}}}^{m_{j1}} \oplus \mathbb{Z}_{p_2^{h_{j2}}}^{m_{j2}} \oplus \cdots \oplus \mathbb{Z}_{p_u^{h_{ju}}}^{m_{ju}},$$
$$K_j(\mathfrak{B}) = \mathbb{Z}^{s_j} \oplus \mathbb{Z}_{p_1^{l_{j1}}}^{t_{j1}} \oplus \mathbb{Z}_{p_2^{l_{j2}}}^{t_{j2}} \oplus \cdots \oplus \mathbb{Z}_{p_u^{l_{ju}}}^{t_{ju}}$$

for some positive integers $n_j, m_{j1}, m_{j2}, \cdots, m_{ju}$, $s_j, t_{j1}, t_{j2}, \cdots, t_{ju}$ with $j = 0,1$ and p_1, $p_2, \cdots, p_u$ prime numbers mutually relatively prime with $1 \le h_{j1}, 1 \le h_{j2}, \cdots, 1 \le h_{ju}$ and $1 \le l_{j1}, 1 \le l_{j2}, \cdots, 1 \le l_{ju}$, and for some integer $u \ge 2$. Then

$$\begin{aligned}
K_0(\mathfrak{A} \otimes \mathfrak{B}) &\cong \mathbb{Z}^{n_0 s_0 + n_1 s_1} \oplus [\oplus_{x,y=1}^u \mathbb{Z}_{p_x^{h_{0x}} \wedge p_y^{l_{0y}}}^{m_{0x}t_{0y}}] \oplus [\oplus_{x,y=1}^u \mathbb{Z}_{p_x^{h_{1x}} \wedge p_y^{l_{1y}}}^{m_{1x}t_{1y}}] \\
&\quad \oplus ([\oplus_{x=1}^u \mathbb{Z}_{p_x^{h_{0x} \wedge l_{1x}}}^{m_{0x}t_{1x}}] \oplus [\oplus_{x=1}^u \mathbb{Z}_{p_x^{h_{1x} \wedge l_{0x}}}^{m_{1x}t_{0x}}]), \\
K_1(\mathfrak{A} \otimes \mathfrak{B}) &\cong \mathbb{Z}^{n_0 s_1 + n_1 s_0} \oplus [\oplus_{x,y=1}^u \mathbb{Z}_{p_x^{h_{0x}} \wedge p_y^{l_{1y}}}^{m_{0x}t_{1y}}] \oplus [\oplus_{x,y=1}^u \mathbb{Z}_{p_x^{h_{1x}} \wedge p_y^{l_{0y}}}^{m_{1x}t_{0y}}] \\
&\quad \oplus ([\oplus_{x=1}^u \mathbb{Z}_{p_x^{h_{0x} \wedge l_{0x}}}^{m_{0x}t_{0x}}] \oplus [\oplus_{x=1}^u \mathbb{Z}_{p_x^{h_{1x} \wedge l_{1x}}}^{m_{1x}t_{1x}}]),
\end{aligned}$$

where the last summands $([\cdots] \oplus [\cdots])$ correspond to the respective torsion products.

Proof. We compute the torsion product in the Künneth theorem for tensor products of C^*-algebras using several facts in homology theory as in [3]:

$$\begin{aligned}
&\mathrm{Tor}_1^{\mathbb{Z}}(K_j(\mathfrak{A}), K_k(\mathfrak{B})) \\
&= \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}^{n_j} \oplus \mathbb{Z}_{p_1^{h_{j1}}}^{m_{j1}} \oplus \cdots \oplus \mathbb{Z}_{p_u^{h_{ju}}}^{m_{ju}}, \mathbb{Z}^{s_k} \oplus \mathbb{Z}_{p_1^{l_{k1}}}^{t_{k1}} \oplus \cdots \oplus \mathbb{Z}_{p_u^{l_{ku}}}^{t_{ku}}) \\
&= \oplus_{x=1}^u \mathbb{Z}_{p_x^{h_{jx} \wedge l_{kx}}}^{m_{jx}t_{kx}}
\end{aligned}$$

for $j = 0, 1$ and $k = 0, 1$. Note that $\mathrm{Tor}_1^{\mathbb{Z}}(K_j(\mathfrak{A}), K_k(\mathfrak{B}))$ appears in the split quotient of $K_{j+k+1}(\mathfrak{A} \otimes \mathfrak{B})$. Therefore,

$$\begin{aligned}
&K_0(\mathfrak{A} \otimes \mathfrak{B})/([\oplus_{x=1}^u \mathbb{Z}^{m_{0x}t_{1x}}_{p_x^{h_{0x} \wedge l_{1x}}}] \oplus [\oplus_{x=1}^u \mathbb{Z}^{m_{1x}t_{0x}}_{p_x^{h_{1x} \wedge l_{0x}}}]) \\
&\cong [K_0(\mathfrak{A}) \otimes K_0(\mathfrak{B})] \oplus [K_1(\mathfrak{A}) \otimes K_1(\mathfrak{B})] \\
&= [(\mathbb{Z}^{n_0} \oplus \mathbb{Z}^{m_{01}}_{p_1^{h_{01}}} \oplus \cdots \oplus \mathbb{Z}^{m_{0u}}_{p_u^{h_{0u}}}) \otimes (\mathbb{Z}^{s_0} \oplus \mathbb{Z}^{t_{01}}_{p_1^{l_{01}}} \oplus \cdots \oplus \mathbb{Z}^{t_{0u}}_{p_u^{l_{0u}}})] \\
&\quad \oplus [(\mathbb{Z}^{n_1} \oplus \mathbb{Z}^{m_{11}}_{p_1^{h_{11}}} \oplus \cdots \oplus \mathbb{Z}^{m_{1u}}_{p_u^{h_{1u}}}) \otimes (\mathbb{Z}^{s_1} \oplus \mathbb{Z}^{t_{11}}_{p_1^{l_{11}}} \oplus \cdots \oplus \mathbb{Z}^{t_{1u}}_{p_u^{l_{1u}}})] \\
&\cong \mathbb{Z}^{n_0 s_0} \oplus [\oplus_{x,y=1}^u \mathbb{Z}^{m_{0x}t_{0y}}_{p_x^{h_{0x}} \wedge p_y^{l_{0y}}}] \\
&\quad \oplus \mathbb{Z}^{n_1 s_1} \oplus [\oplus_{x,y=1}^u \mathbb{Z}^{m_{1x}t_{1y}}_{p_x^{h_{1x}} \wedge p_y^{l_{1y}}}].
\end{aligned}$$

And also

$$\begin{aligned}
&K_1(\mathfrak{A} \otimes \mathfrak{B})/([\oplus_{x=1}^u \mathbb{Z}^{m_{0x}t_{0x}}_{p_x^{h_{0x} \wedge l_{0x}}}] \oplus [\oplus_{x=1}^u \mathbb{Z}^{m_{1x}t_{1x}}_{p_x^{h_{1x} \wedge l_{1x}}}]) \\
&\cong [K_0(\mathfrak{A}) \otimes K_1(\mathfrak{B})] \oplus [K_1(\mathfrak{A}) \otimes K_0(\mathfrak{B})] \\
&= [(\mathbb{Z}^{n_0} \oplus \mathbb{Z}^{m_{01}}_{p_1^{h_{01}}} \oplus \cdots \oplus \mathbb{Z}^{m_{0u}}_{p_u^{h_{0u}}}) \otimes (\mathbb{Z}^{s_1} \oplus \mathbb{Z}^{t_{11}}_{p_1^{l_{11}}} \oplus \cdots \oplus \mathbb{Z}^{t_{1u}}_{p_u^{l_{1u}}})] \\
&\quad \oplus [(\mathbb{Z}^{n_1} \oplus \mathbb{Z}^{m_{11}}_{p_1^{h_{11}}} \oplus \cdots \oplus \mathbb{Z}^{m_{1u}}_{p_u^{h_{1u}}}) \otimes (\mathbb{Z}^{s_0} \oplus \mathbb{Z}^{t_{01}}_{p_1^{l_{01}}} \oplus \cdots \oplus \mathbb{Z}^{t_{0u}}_{p_u^{l_{0u}}})] \\
&\cong \mathbb{Z}^{n_0 s_1} \oplus [\oplus_{x,y=1}^u \mathbb{Z}^{m_{0x}t_{1y}}_{p_x^{h_{0x}} \wedge p_y^{l_{1y}}}] \\
&\quad \oplus \mathbb{Z}^{n_1 s_0} \oplus [\oplus_{x,y=1}^u \mathbb{Z}^{m_{1x}t_{0y}}_{p_x^{h_{1x}} \wedge p_y^{l_{0y}}}].
\end{aligned}$$

□

Theorem 5.3. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras with $\mathfrak{A}$ in the category $\mathfrak{N}$. Suppose that*

$$\begin{aligned}
K_j(\mathfrak{A}) &= \mathbb{Z}^{n_j} \oplus \mathbb{Z}^{m_{j1}}_{p_1^{h_{j1}}} \oplus \mathbb{Z}^{m_{j2}}_{p_2^{h_{j2}}} \oplus \cdots \oplus \mathbb{Z}^{m_{ju}}_{p_u^{h_{ju}}}, \\
K_j(\mathfrak{B}) &= \mathbb{Z}^{s_j} \oplus \mathbb{Z}^{t_{j1}}_{q_1^{l_{j1}}} \oplus \mathbb{Z}^{t_{j2}}_{q_2^{l_{j2}}} \oplus \cdots \oplus \mathbb{Z}^{t_{ju}}_{q_u^{l_{ju}}}
\end{aligned}$$

for some positive integers $n_j, m_{j1}, m_{j2}, \cdots, m_{ju}$, $s_j, t_{j1}, t_{j2}, \cdots, t_{ju}$ with $j = 0, 1$ and p_1, $p_2, \cdots, p_u$, q_1, $q_2, \cdots, q_u$ prime numbers mutually relatively prime with $1 \leq h_{j1}, 1 \leq h_{j2}, \cdots, 1 \leq h_{ju}$ and $1 \leq l_{j1}, 1 \leq l_{j2}, \cdots, 1 \leq l_{ju}$, and for some integer $u \geq 2$. Then

$$\begin{aligned}
K_0(\mathfrak{A} \otimes \mathfrak{B}) &\cong \mathbb{Z}^{n_0 s_0 + n_1 s_1} \oplus [\oplus_{x,y=1}^u \mathbb{Z}^{m_{0x}t_{0y}}_{p_x^{h_{0x}} \wedge q_y^{l_{0y}}}] \oplus [\oplus_{x,y=1}^u \mathbb{Z}^{m_{1x}t_{1y}}_{p_x^{h_{1x}} \wedge q_y^{l_{1y}}}], \\
K_1(\mathfrak{A} \otimes \mathfrak{B}) &\cong \mathbb{Z}^{n_0 s_1 + n_1 s_0} \oplus [\oplus_{x,y=1}^u \mathbb{Z}^{m_{0x}t_{1y}}_{p_x^{h_{0x}} \wedge q_y^{l_{1y}}}] \oplus [\oplus_{x,y=1}^u \mathbb{Z}^{m_{1x}t_{0y}}_{p_x^{h_{1x}} \wedge q_y^{l_{0y}}}],
\end{aligned}$$

where the torsion products are zero.

Proof. We compute the torsion product in the Künneth theorem for tensor products of C^*-algebras using several facts in homology theory as in [3]:

$$\begin{aligned}
&\mathrm{Tor}_1^{\mathbb{Z}}(K_j(\mathfrak{A}), K_k(\mathfrak{B})) \\
&= \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}^{n_j} \oplus \mathbb{Z}^{m_{j1}}_{p_1^{h_{j1}}} \oplus \cdots \oplus \mathbb{Z}^{m_{ju}}_{p_u^{h_{ju}}}, \mathbb{Z}^{s_k} \oplus \mathbb{Z}^{t_{k1}}_{q_1^{l_{k1}}} \oplus \cdots \oplus \mathbb{Z}^{t_{ku}}_{q_u^{l_{ku}}}) \\
&= 0
\end{aligned}$$

for $j = 0, 1$ and $k = 0, 1$. Therefore,

$$\begin{aligned}
&K_0(\mathfrak{A} \otimes \mathfrak{B}) \\
&\cong [K_0(\mathfrak{A}) \otimes K_0(\mathfrak{B})] \oplus [K_1(\mathfrak{A}) \otimes K_1(\mathfrak{B})] \\
&= [(\mathbb{Z}^{n_0} \oplus \mathbb{Z}^{m_{01}}_{p_1^{h_{01}}} \oplus \cdots \oplus \mathbb{Z}^{m_{0u}}_{p_u^{h_{0u}}}) \otimes (\mathbb{Z}^{s_0} \oplus \mathbb{Z}^{t_{01}}_{q_1^{l_{01}}} \oplus \cdots \oplus \mathbb{Z}^{t_{0u}}_{q_u^{l_{0u}}})] \\
&\quad \oplus [(\mathbb{Z}^{n_1} \oplus \mathbb{Z}^{m_{11}}_{p_1^{h_{11}}} \oplus \cdots \oplus \mathbb{Z}^{m_{1u}}_{p_u^{h_{1u}}}) \otimes (\mathbb{Z}^{s_1} \oplus \mathbb{Z}^{t_{11}}_{q_1^{l_{11}}} \oplus \cdots \oplus \mathbb{Z}^{t_{1u}}_{q_u^{l_{1u}}})] \\
&\cong \mathbb{Z}^{n_0 s_0} \oplus [\oplus_{x,y=1}^{u} \mathbb{Z}^{m_{0x} t_{0y}}_{p_x^{h_{0x}} \wedge q_y^{l_{0y}}}] \\
&\quad \oplus \mathbb{Z}^{n_1 s_1} \oplus [\oplus_{x,y=1}^{u} \mathbb{Z}^{m_{1x} t_{1y}}_{p_x^{h_{1x}} \wedge q_y^{l_{1y}}}].
\end{aligned}$$

And also

$$\begin{aligned}
&K_1(\mathfrak{A} \otimes \mathfrak{B}) \\
&\cong [K_0(\mathfrak{A}) \otimes K_1(\mathfrak{B})] \oplus [K_1(\mathfrak{A}) \otimes K_0(\mathfrak{B})] \\
&= [(\mathbb{Z}^{n_0} \oplus \mathbb{Z}^{m_{01}}_{p_1^{h_{01}}} \oplus \cdots \oplus \mathbb{Z}^{m_{0u}}_{p_u^{h_{0u}}}) \otimes (\mathbb{Z}^{s_1} \oplus \mathbb{Z}^{t_{11}}_{q_1^{l_{11}}} \oplus \cdots \oplus \mathbb{Z}^{t_{1u}}_{q_u^{l_{1u}}})] \\
&\quad \oplus [(\mathbb{Z}^{n_1} \oplus \mathbb{Z}^{m_{11}}_{p_1^{h_{11}}} \oplus \cdots \oplus \mathbb{Z}^{m_{1u}}_{p_u^{h_{1u}}}) \otimes (\mathbb{Z}^{s_0} \oplus \mathbb{Z}^{t_{01}}_{q_1^{l_{01}}} \oplus \cdots \oplus \mathbb{Z}^{t_{0u}}_{q_u^{l_{0u}}})] \\
&\cong \mathbb{Z}^{n_0 s_1} \oplus [\oplus_{x,y=1}^{u} \mathbb{Z}^{m_{0x} t_{1y}}_{p_x^{h_{0x}} \wedge q_y^{l_{1y}}}] \\
&\quad \oplus \mathbb{Z}^{n_1 s_0} \oplus [\oplus_{x,y=1}^{u} \mathbb{Z}^{m_{1x} t_{0y}}_{p_x^{h_{1x}} \wedge q_y^{l_{0y}}}].
\end{aligned}$$

□

Remark. There are more general cases when both $K_j(\mathfrak{A})$ and $K_j(\mathfrak{B})$ $(j = 0, 1)$ are finitely generated, abelian groups, but one can compute $K_j(\mathfrak{A} \otimes \mathfrak{B})$ combining the results in the cases considered above. Indeed, it is a well known fact in the group theory that any finite abelian group can be written as a finite product of cyclic groups with orders multiples of prime numbers which may be mutually relatively prime or not, as: $\mathbb{Z}_{p_1^{e_1}} \times \mathbb{Z}_{p_2^{e_2}} \times \cdots \mathbb{Z}_{p_k^{e_k}}$ for some integer $k \geq 1$, so that finitely generated abelian groups have the torsion part as this product and the free part as $\mathbb{Z}^l$ for some integer $l \geq 0$.

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