CHARACTERIZATION OF DIAGONALITY FOR OPERATORS

Takashi Itoh and Masaru Nagisa

Received May 11, 2012; revised June 5, 2012

Abstract. Let A be an invertible $n \times n$ matrix over $\mathbb{C}$. If the k-th power A^k of A and the k-th power $A^{\circ k}$ of Schur product of A equals ($k = 1, 2, \dots, n+1$), then A becomes diagonal. In the case that A is an invertible bounded linear operator on an infinite dimensional Hilbert space H, we can also define Schur product of operators, and we can show that A is diagonal, if it satisfies $A^k = A^{\circ k}$ for any $k = 1, 2, \dots$.

1 Introduction We denote by $\mathbb{M}_n(\mathbb{C})$ the set of all $n \times n$ matrices over $\mathbb{C}$. For $A, B \in \mathbb{M}_n(\mathbb{C})$, we define their Schur product (or Hadamard product) $A \circ B$ as follows:

$$A \circ B = (a_{ij} b_{ij})_{i,j=1}^n,$$

where $A = (a_{ij})_{i,j=1}^n$ and $B = (b_{ij})_{i,j=1}^n$. We denote the k-th power of Schur product of A by

$$A^{\circ k} = \overbrace{A \circ A \circ \cdots \circ A}^{k}.$$

By definition, for any diagonal matrix A, we have

$$A^k = A^{\circ k}$$

for all $k = 1, 2, 3, \dots$.

In the field of operator inequality, many results are known related to Schur product ([1],[2]). In other words, Schur product is useful for topics related to self-adjoint or positive operators. For example, if A is self-adjoint, i.e., $A = A^*$, then we can easily check that the property $A^2 = A^{\circ 2}$ implies the diagonality of A. But, without the assumption of self-adjointness of operators, we remark that the property $A^k = A^{\circ k}$ for any k does not imply the diagonality of A. The following matrix A is not diagonal, but A satisfies this property:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in \mathbb{M}_2(\mathbb{C}), \quad A^k = A = A^{\circ k} \quad \text{for any } k = 1, 2, 3, \dots.$$

In this paper, first we show the following fact:

Theorem 1.1. *Let A be an $n \times n$ matrix over $\mathbb{C}$ satisfying*

$$A^k = A^{\circ k}, \qquad k = 1, 2, \dots, n+1.$$

Then we have the followings:

(1) $A^k = A^{\circ k}$ for any positive integer k.

(2) If A is invertible, then A is diagonal.

2010 *Mathematics Subject Classification.* 47A05, 47A06,15A15 .
Key words and phrases. diagonality, invertible matrix, bounded linenar operator, Schur product .

As the infinite dimensional case, we consider a bounded linear operator on a (infinite dimensional) Hilbert space. Let $\mathcal{H}$ be a Hilbert space. We fix the completely orthonormal system $\{\xi_i\}_{i\in I}$ of $\mathcal{H}$. Let A be a bounded linear operator on $\mathcal{H}$ with

$$A\xi_j = \sum_{i\in I} a_{ij}\xi_i, \quad (a_{ij} \in \mathbb{C},\ j \in I).$$

Then we denote $A \in B(\mathcal{H})$ by $(a_{ij})_{i,j\in I}$. For two operators $A = (a_{ij})_{i,j\in I}, B = (b_{ij})_{i.j\in I} \in B(\mathcal{H})$, we can define $A \circ B \in B(\mathcal{H})$ as follows([4]):

$$A \circ B = (a_{ij}b_{ij})_{i,j\in I}.$$

Since A is bounded, we have

$$\sum_{j\in I} |a_{ij}|^2 < \infty, \quad \sum_{i\in I} |a_{ij}|^2 < \infty.$$

We remark that

$$\sum_{k\in I} |a_{ik}a_{kj}| < \infty$$

and the set $\{k \in I \mid a_{ik}a_{kj} \neq 0\}$ is at most countable for any $i, j \in I$. Then we can show the following theorem as infinite dimensional version of Theorem 1.1.

Theorem 1.2. *Let A be a bounded invertible linear operator on $\mathcal{H}$ with*

$$A^n = A^{\circ n} \qquad \text{for any } n = 1, 2, 3, \ldots.$$

Then A is diagonal, i.e., $a_{ij} = 0$ when $i \neq j$.

Let $A \in \mathbb{M}_3(\mathbb{C})$ be as follows:

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then A is invertible, is not diagonal and satisfies

$$A^2 = A^{\circ 2} \text{ and } A^3 \neq A^{\circ 3}.$$

In the last section, we determine the smallest integer m satisfying that, for any invertible $A \in \mathbb{M}_n(\mathbb{C})$,

$$A^k = A^{\circ k} \quad (k = 1, 2, \ldots, m)$$

implies the diagonality of A.

2 Proof of Theorem 1.1

In this section, we give a proof of Theorem 1.1.

Proof. (1) Let $p(t) = \det(tI_n - A)$ be a characteristic polynomial of A. Then we have, by Cayley-Hamilton theorem,

$$p(A) = 0.$$

We define

$$q_1(t) = t^{n+1} - tp(t) = \sum_{k=1}^{n} b_k t^k,$$

then we have $q_1(A) = A^{n+1}$.

We assume that $N \geq n+1$ and it holds

$$A^l = A^{\circ l} \quad l = 1, 2, \dots, N.$$

If we can show that $A^{N+1} = A^{\circ(N+1)}$, then (1) holds by induction. It follows from

$$\begin{aligned} A^{\circ(N+1)} &= A^{\circ(N-n)} \circ (A^{\circ(n+1)}) = A^{\circ(N-n)} \circ (A^{n+1}) = A^{\circ(N-n)} \circ q_1(A) \\ &= A^{\circ(N-n)} \circ (\sum_{k=1}^{n} b_k A^k) = A^{\circ(N-n)} \circ (\sum_{k=1}^{n} b_k A^{\circ k}) \\ &= \sum_{k=1}^{n} b_k A^{\circ(N-n+k)} = \sum_{k=1}^{n} b_k A^{N-n+k} \quad (\text{since } 0 < N-n+k \leq N) \\ &= A^{N-n}(\sum_{k=1}^{n} b_k A^k) = A^{N-n} q_1(A) = A^{N+1}. \end{aligned}$$

(2) Since A is invertible, if we define

$$q_2(t) = \frac{p(t) - (-1)^n \det(A)}{(-1)^{n+1} \det(A)} = \sum_{k=1}^{n} a_k t^k,$$

we can get $q_2(A) = I_n$.

Then we have

$$\begin{aligned} A \circ I_n &= A \circ q_2(A) = A \circ (\sum_{k=1}^{n} a_k A^k) = A \circ (\sum_{k=1}^{n} a_k A^{\circ k}) \\ &= \sum_{k=1}^{n} a_k A^{\circ k+1} = \sum_{k=1}^{n} a_k A^{k+1} \\ &= A(\sum_{k=1}^{n} a_k A^k) = Aq_2(A) = AI_n = A. \end{aligned}$$

Since $A \circ I_n$ is diagonal, so is A. □

3 Proof of Theorem 1.2

Lemma 3.1. *Let $(x_i)_{i=1}^{\infty}$ be a 1-summable sequence of complex numbers, i.e., $\sum_{i=1}^{\infty} |x_i| < \infty$. If it holds that*

$$\sum_{i=1}^{\infty} x_i^j = 0, \qquad \textit{for all } j = 1, 2, 3, \dots,$$

then $x_i = 0$ for all $i = 1, 2, 3, \dots$.

Proof. We set $x_n = r_n e^{2\pi\theta_n\sqrt{-1}}$ $(r_n = |x_n| \geq 0)$. We assume that some of x_i's is not equal to 0. Arranging the sequence, we may assume that

$$1 = r_1 \geq r_2 \geq \cdots \text{ and } \sum_{n=k+1}^{\infty} r_n < \frac{1}{2}$$

for some k. Since $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ is compact, we can choose an infinite subset N_1 of $\mathbb{N}$ such that

$$s, t \in N_1 \Rightarrow |e^{2\pi s\theta_1\sqrt{-1}} - e^{2\pi t\theta_1\sqrt{-1}}| < \frac{1}{3}.$$

By the same method, we can choose an infinite subset N_2 of N_1 such that

$$s, t \in N_2 \Rightarrow |e^{2\pi s\theta_2\sqrt{-1}} - e^{2\pi t\theta_2\sqrt{-1}}| < \frac{1}{3}.$$

Continuing this argument, we can choose numbers $s, t \in \mathbb{N}$ such that

$$|e^{2\pi s\theta_j\sqrt{-1}} - e^{2\pi t\theta_j\sqrt{-1}}| < \frac{1}{3} \quad \text{for all } j = 1, 2, \ldots, k.$$

We set $K = |s - t|$. Then we have

$$|1 - e^{2\pi K\theta_j\sqrt{-1}}| < \frac{1}{3} \quad \text{for all } j = 1, 2, \ldots, k.$$

This means that

$$\mathrm{Re}(e^{2\pi K\theta_j\sqrt{-1}}) > \frac{2}{3} \quad \text{for all } j = 1, 2, \ldots, k.$$

By the assumption, we have

$$|\sum_{n=k+1}^{\infty} x_n^K| \leq \sum_{n=k+1}^{\infty} r_n^K \leq \sum_{n=k+1}^{\infty} r_n < \frac{1}{2}.$$

We also have

$$\begin{aligned} |\sum_{n=1}^{k} x_n^K| \geq \mathrm{Re}(\sum_{n=1}^{k} x_n^K) &= \sum_{n=1}^{k} r_n^K \mathrm{Re}(e^{2\pi K\theta_n\sqrt{-1}}) \\ &> \frac{2}{3}(1 + r_2^K + \cdots + r_k^K) > \frac{1}{2}. \end{aligned}$$

This contradicts to

$$\sum_{n=1}^{\infty} x_n^K = 0.$$

□

Proposition 3.2. *Let $(x_i)_{i=1}^{\infty}$ be a 1-summable sequence of complex numbers. For some $\alpha \in \mathbb{C}$, it holds that*

$$\sum_{i=1}^{\infty} x_i^j = \alpha^j, \qquad \text{for all } j = 1, 2, 3, \ldots.$$

Then there is a number i_0 such that

$$x_i = \begin{cases} \alpha, & i = i_0 \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Put $r_n = |x_n|$. In the case $\alpha = 0$, it follows from the preceding lemma. So we may assume that

$$\alpha = 1, \quad r_1 \geq r_2 \geq \cdots \quad \text{and} \quad \sum_{n=k+1}^{\infty} r_n < \frac{1}{2}$$

for some k. Then we show that $r_1 \geq 1$. Assume that $r_1 < 1$. We can choose a number N_0 satisfying

$$r_1^{N_0} < \frac{1}{2k}.$$

So we have

$$|\sum_{n=1}^{\infty} x_n^{N_0}| \leq \sum_{n=1}^{k} r_n^{N_0} + \sum_{n=k+1}^{\infty} r_n^{N_0} \leq k \cdot \frac{1}{2k} + \sum_{n=k+1}^{\infty} r_n < 1 = \alpha.$$

This is a contradiction.

We set

$$r_1 \geq r_2 \geq \cdots \geq r_l \geq 1 > r_{l+1} \geq r_{l+2} \geq \cdots \geq r_k.$$

Using the same argument in the proof of Lemma 3.1, for any positive integer N, we can choose a positive integer $K(N)$ satisfying that

$$\mathrm{Re}(e^{2\pi K(N) N \theta_j \sqrt{-1}}) > \frac{2}{3} \quad \text{for all } j = 1, 2, \ldots, k.$$

Then we have

$$\begin{aligned} \mathrm{Re}(\sum_{n=1}^{k} x_n^{K(N)N}) &= \sum_{n=1}^{k} r_n^{K(N)N} \mathrm{Re}(e^{2\pi K(N) N \theta_n \sqrt{-1}}) \\ &> \frac{2}{3}(\sum_{n=1}^{l} r_n^{K(N)N} + \sum_{n=l+1}^{k} r_n^{K(N)N}) > \frac{2l}{3} \end{aligned}$$

and

$$|\sum_{n=k+1}^{\infty} x_n^{K(N)N}| \leq \sum_{n=k+1}^{\infty} r_n^{K(N)N} \leq \sum_{n=k+1}^{\infty} r_n r_{k+1}^{K(N)N-1} < \frac{1}{2} r_{k+1}^{K(N)N-1}.$$

For a sufficiently large N, we may assume that

$$\frac{1}{2} r_{k+1}^{K(N)N-1} < \frac{1}{3}.$$

Since

$$|1 - \mathrm{Re}(\sum_{n=1}^{k} x_n^{K(N)N})| = |\mathrm{Re}(\sum_{n=k+1}^{\infty} x_n^{K(N)N})| \leq |\sum_{n=k+1}^{\infty} x_n^{K(N)N}| < \frac{1}{3},$$

we have $l = 1$ and get the relation $r_1 \geq 1 > r_2 \geq \cdots \geq r_k$.

If $r_1 > 1$, then we may also assume that $r_1^{K(N)N} > 2$, i.e., $\mathrm{Re}(\sum_{n=1}^{k} x_n^{K(N)N}) > \frac{4}{3}$. This contradicts to $\sum_{n=1}^{\infty} x_n^{K(N)N} = 1$. So we have $r_1 = 1$.

If $x_1 \neq 1$, we can choose a sequence of integers

$$0 < m(1) < m(2) < \cdots < m(k) < \cdots$$

such that

$$\lim_{k\to\infty} x_1^{m(k)} = e^{\theta\sqrt{-1}} \neq 1$$

for some real θ. For a sufficiently large k, we may assume

$$|1 - x_1^{m(k)}| > \frac{1}{2}|1 - e^{\theta\sqrt{-1}}| \text{ and}$$

$$|\sum_{n=2}^{\infty} x_n^{m(k)}| \leq |x_2|^{m(k)-1}(\sum_{n=2}^{\infty} |x_n|) < \frac{1}{2}|1 - e^{\theta\sqrt{-1}}|.$$

This contradicts to $\sum_{n=1}^{\infty} x_n^{m(k)} = 1$. So we have $x_1 = 1$.

Therefore we have the following relation:

$$\sum_{i=2}^{\infty} x_i^j = 0, \qquad \text{for all } j = 1, 2, 3, \ldots.$$

By Lemma 3.1, we can get $x_2 = x_3 = \cdots = 0$. □

Now we can give the proof for Theorem 1.2 as follows:

Proof. By the assumption, we have

$$A^{\circ n}A^{\circ n} = A^n A^n = A^{2n} = A^{\circ(2n)},$$

that is,

$$\sum_{s\in I} a_{is}^n a_{sj}^n = \sum_{s\in I} (a_{is}a_{sj})^n = a_{ij}^{2n}$$

for all $n = 1, 2, 3, \ldots$ and $i, j \in I$. We fix i. When $i = j$, we can get the relation:

$$\sum_{s\in I\setminus\{i\}} (a_{is}a_{si})^n = 0.$$

By Lemma 3.1, we have $a_{ij}a_{ji} = 0$ $(j \neq i)$.

We set

$$K = \{s \in I \mid a_{is} = 0\} \setminus \{i\}, \quad J = I \setminus K.$$

For $j \in J \setminus \{i\}$, $a_{ij} \neq 0$ implies $a_{ji} = 0$. When $j \in K$, it holds

$$\sum_{s\in I} (a_{is}a_{sj})^n = a_{ij}^{2n} = 0.$$

By Lemma 3.1 we have $a_{is}a_{sj} = 0$ for all s. For $s \in J \setminus \{i\}$, $a_{is} \neq 0$ implies $a_{sj} = 0$. Therefore we have

$$a_{sj} = 0 \qquad (s \in J,\ j \in K),$$
$$a_{si} = 0 \qquad (s \in J \setminus \{i\}).$$

To prove the diagonality of A, it suffices to show the following statement:

(1) $a_{ii} \neq 0$ implies $a_{ij} = a_{ji} = 0$ $(j \neq i)$.

(2) $a_{ii} \neq 0$.

(1) Let $a_{ii} \neq 0$. For any $j \in J$, we have

$$\sum_{s \in I} (a_{is} a_{sj})^n = a_{ij}^{2n} \neq 0.$$

By Proposition 3.2 it holds that there exists $s_0 \in I$ with

$$a_{is_0} a_{s_0 j} = a_{ij}^2, \quad a_{is} a_{sj} = 0 \ (s \neq s_0).$$

The fact $a_{ij} \neq 0$ $(j \in J)$ implies $s_0 = i$. So we have

$$a_{jk} = 0 \qquad (j \in J \setminus \{i\},\ k \in I).$$

This means

$$A\xi \perp \xi_j \qquad (\forall \xi \in \mathcal{H}, j \in J \setminus \{i\}).$$

Since A is invertible, we can get $J = \{i\}$, that is,

$$a_{ij} = 0 \qquad (j \neq i).$$

We remark that A^* is also invertible and satisfies the condition $(A^*)^n = (A^*)^{\circ n}$ for all $n = 1, 2, 3, \ldots$. So we have

$$a_{ij} = a_{ji} = 0 \qquad (j \neq i).$$

(2) Assume that $a_{ii} = 0$. For $i(1) \in J \setminus \{i\}$, we have

$$\sum_{s \in I} (a_{is} a_{s,i(1)})^n = a_{i,i(1)}^{2n} \neq 0.$$

By Proposition 3.2, there exists an $i(2) \in J \setminus \{i\}$ satisfying

$$a_{i(2),i(1)} \neq 0 \quad \text{and } a_{s,i(1)} = 0 \ (s \in J \setminus \{i, i(2)\}).$$

If $i(1) = i(2)$, then $a_{i(1),i(1)} \neq 0$ implies $a_{i,i(1)} = 0$ by (1). This contradicts to $i(1) \in J \setminus \{i\}$. So we have $i(1) \neq i(2)$. Since $a_{i(2),i} = 0$, we have

$$\begin{aligned} 0 \neq a_{i(2),i(1)}^{2n} &= \sum_{s \in I} (a_{i(2),s} a_{s,i(1)})^n \\ &= \sum_{s \in J} (a_{i(2),s} a_{s,i(1)})^n \\ &= (a_{i(2),i(2)} a_{i(2),i(1)})^n. \end{aligned}$$

By the fact $a_{i(2),i(2)} \neq 0$ and (1), it contradicts to $a_{i(2),i(1)} \neq 0$. □

4 Conclusions For any positive integer n, we define $d(n)$ the smallest integer m satisfying that, for any invertible $A \in \mathbb{M}_n(\mathbb{C})$,

$$A^k = A^{\circ k} \qquad (k = 1, 2, \ldots, m)$$

implies the diagonality of A.

Let σ be a permutation on $\{1, 2, 3, \ldots, n\}$. For $A = (a_{ij})_{i,j=1}^n \in \mathbb{M}_n(\mathbb{C})$, we define $A_\sigma \in \mathbb{M}_n(\mathbb{C})$ as follows:

$$A_\sigma = (a_{\sigma(i),\sigma(j)})_{i,j=1}^n.$$

Then we can easily check the following remarks:

(1) A is invertible $\Leftrightarrow A_\sigma$ is invertible.

(2) A is diagonal $\Leftrightarrow A_\sigma$ is diagonal.

(3) For $A, B \in \mathbb{M}_n(\mathbb{C})$, we have

$$A_\sigma B_\sigma = (AB)_\sigma, \qquad A_\sigma \circ B_\sigma = (A \circ B)_\sigma.$$

Proposition 4.1. *(1) $d(n) \leq n+1$.*

(2) $d(2) = 3$.

(3) $d(3) = 3$.

Proof. (1) It follows from Theorem 1.1 .

(2) By (1), $d(2) \leq 3$. Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \in \mathbb{M}_2(\mathbb{C})$. Then A is invertible, not diagonal and satisfying

$$A^k = A^{\circ k} \qquad (k = 1, 2).$$

So we have $d(2) \geq 3$. Therefore $d(2) = 3$.

(3) Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ be invertible and satisfy $A^k = A^{\circ k}$ $(k = 1, 2, 3)$. We compute

$$A^2 = A^{\circ 2}, \quad A \cdot A^{\circ 2} = A \cdot A^2 = A^3 = A^{\circ 3}.$$

From the (i, j)-th componet of above calculation, we have the following relation (i, j):

$$a_{i1}a_{1j}^k + a_{i2}a_{2j}^k + a_{i3}a_{3j}^k = a_{ij}^{k+1} \quad (k = 1, 2).$$

We first show that A is diagonal in the case $a_{12} = a_{13} = 0$. Since A is invertible, the matrix $B = \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}$ is also invertible, and satisfies $B^k = B^{\circ k}$ $(k = 1, 2, 3)$. Because $d(2) = 3$, we have $a_{23} = a_{32} = 0$. Applying the same argument for $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, we can get $a_{21} = 0$. For $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$, considering A_σ instead of A, we have $a_{31} = 0$. So A is diagonal.

Next we show that A is diagonal in the case $a_{12} = 0$. By the relation $(1, 1)$, we have $a_{13} = 0$ or a_{31}=0. In the case $a_{13} = 0$ we heve already shown that A is diagonal. Assume $a_{31} = 0$. By the relation $(1, 2)$, we have $a_{13} = 0$ or $a_{32} = 0$. In the case $a_{32} = 0$, for $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$, considering A_σ instead of A, we can get the diagonality of A.

We consider the case $a_{i_0, j_0} = 0$ for some $i_0, j_0 (i_0 \neq j_0)$. We set $k_0 \in \{1, 2, 3\} \setminus \{i_0, j_0\}$ and

$$\sigma = \begin{pmatrix} i_0 & j_0 & k_0 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ i_0 & j_0 & k_0 \end{pmatrix}^{-1}.$$

Then the $(1, 2)$-th component of A_σ is 0. So A is diagonal.

From the relation $(1, 1)$, we have

$$a_{12}a_{21} + a_{13}a_{31} = 0, \quad a_{12}a_{21}^2 + a_{13}a_{31}^2 = 0.$$

Since $a_{12}a_{21}a_{31} = a_{12}a_{21}^2$, we have $a_{12} = 0$, $a_{21} = 0$ or $a_{21} = a_{31}$. We assume that A is not diagonal. Then $a_{ij} \neq 0$ if $i \neq j$. So we have

$$a_{21} = a_{31} \neq 0 \text{ and } a_{12} = -a_{13}.$$

From the relation $(2, 2)$ and $(3, 3)$, we can get

$$(a_{12} = a_{32} \neq 0 \text{ and } a_{21} = -a_{23}) \text{ and } (a_{13} = a_{23} \neq 0 \text{ and } a_{31} = -a_{32}).$$

This implies the contradiction

$$a_{12} = -a_{13} = -a_{23} = a_{21} = a_{31} = -a_{32} = -a_{12} \neq 0.$$

□

Acknowledgements The works of M. N. was partially supported by Grant-in-Aid for Scientific Research (C)22540220.

References

[1] T. Ando, *Concavity of certain maps on positive definite matrices and applications to Hadamard products*, Linear Algebra Appl. 26(1979), 203–241.

[2] M. Fujii, R. Nakamoto and M. Nakamura, *Conditional expectation and Hadamard product of operators*, Math. Japon. 42(1995), 239–244.

[3] R. A. Horn and C. R. Johnson, Matrix Analysis, Cambridge Univ. Press 1985.

[4] T. Itoh and M. Nagisa, *Schur products and module maps on* $B(\mathcal{H})$, Publ. RIMS, Kyoto Univ. 36(2000), 253–268.

[5] V. I. Paulsen, Completely bounded maps and dilations, Pitman Res. Notes in Math. Ser. 146, 1986.

Communicated by *Moto O'uchi*

Takashi Itoh
Department of Mathematics
Gunma University
4-2 Aramaki, Maebashi, Gunma 371-8510
Japan
e-mail: itoh@edu.gunma-u.ac.jp

Masaru Nagisa
Graduate School of Science
Chiba University
Inage-ku, Chiba 263-8522
Japan
e-mail: nagisa@math.s.chiba-u.ac.jp