UNIQUENESS OF MINIMAL 1−SAFE PETRI NET GENERATING ALL THE BINARY n-VECTORS AS ITS MARKING VECTORS EXACTLY ONCE

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Received November 25, 2011

Abstract. The problem of characterizing 1-safe Petri nets generating all the 2^n binary n-vectors as marking vectors exactly once is an open problem [4]. In this note, we completely settle a part of this problem, viz., to determine minimal such Petri nets, 'minimal' in the sense that the depth of their reachability tree is minimum possible. In fact, we show here that such a 1-safe Petri net has a unique structure.

1. INTRODUCTION For terminology and notation in standard Petri net theory and graph theory, we refer the reader to [5, 2] respectively.

1−safe Petri net generating all the binary $n-$vectors has been recognized as an important class of Petri nets, not only from the theoretical point of view but also due to their applications [1, 3, 4]. While this problem in its full generality is still open, a crucial partial question raised in [4], viz., to determine minimal 1−safe Petri nets that generate all the binary $n-$vectors as marking vectors exactly once, 'minimal' in the sense that the the depth of the reachability tree is kept minimum, where the *depth* of a rooted tree is defined as the maximum distance of any vertex in it from the root, is settled in this note.

1 The Main Result In [4], we established the following result.

Theorem 1. *There exists a* 1−*safe Petri net that generates every binary* $n-$*vector as a marking vector exactly once.*

The 1−safe Petri net N=(P, T, I^-, I^+, μ^0) which generates every binary $n-$vector as a marking vector exactly once, discovered to establish the above existence theorem, has the following structure.

1. The initial marking of N is the 'all-1 n-vector' $\mu_0 = (1, 1, \ldots, 1)$.

2. N has n-places, 2^n-1 transitions $t_1, t_2, t_3, \cdots$ and has the structure as schematically shown in Figure-1, which has the following parameters:

$|p^\bullet| = 2^n - 1, \ \forall \ p \in P,$
$|^\bullet p| = 2^{n-1} - 1, \ \forall \ p \in P,$
$|^\bullet t| = n, \ \forall \ t \in T.$

3. Let ${}^nC_k$ denote the combinatorial function that gives the number of distinct ways in which k objects can be selected out of n objects. Then, the total number of transitions whose

2000 *Mathematics Subject Classification.* 37, 93, 94 .
Key words and phrases. 1-safe Petri net, reachability tree, binary n-vector, marking vector, complete bipartite graph.

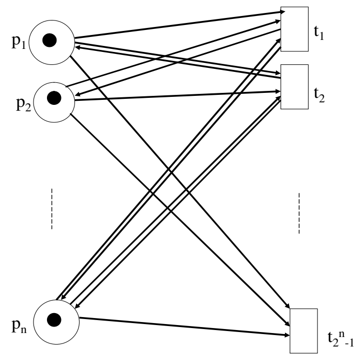

Figure 1: The minimal Petri net generating all the binary $n-$vectors as its marking vectors exacly once.

post-sets have $n-1$ elements $= {}^nC_{n-1} = {}^nC_1 = n$ and these transitions are $t_1, t_2, t_3, \cdots, t_n$.

The total number of transitions whose post-sets have $n-2$ elements $= {}^nC_{n-2} = {}^nC_2 = \frac{n(n-1)}{2}$ and these transitions are $t_{n+1}, t_{n+2}, t_{n+3}, \cdots, t_{\frac{n^2+n}{2}}$.

The total number of transitions whose post-sets have $n-3$ elements $= {}^nC_{n-3} = {}^nC_3 = \frac{n(n-1)(n-2)}{6}$ and these transitions are $t_{\frac{n^2+n+2}{2}}, t_{\frac{n^2+n+4}{2}}, t_{\frac{n^2+n+6}{2}}, \cdots, t_{\frac{n^3+5n}{6}}$.

$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$

The total number of transitions whose post-sets have one element $= {}^nC_1 = n$ and these transitions are

$$t_{2^n-n-1}, t_{2^n-n}, t_{2^n-n+1}, \cdots, t_{2^n-2}$$

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The total number of transitions whose post-sets have no element $= {}^nC_0 = 1$ and this transition is t_{2^n-1}.

Theorem 2. *N=(P, T, I^-, I^+, μ^0) is the only minimal $1-$safe Petri net which generates every binary $n-$vector as a marking vector exactly once and the underlying graph of its reachability tree is isomorphic to the star $\downarrow K_{1,2^n-1}$, where '$\downarrow$' indicates the fact that arcs of the reachability tree of N are oriented downward from its root which is the center of $K_{1,2^n-1}$.*

Proof. The existence of N with the properties described just before the statement of the theorem has already been established in [4]. We will establish here the uniqueness of N. Suppose there exists a Petri net

$$N' = (P', T', {I'}^-, {I'}^+, \mu^0)$$

satisfying the hypothesis of the theorem. This implies, in particular that the reachability set $R(N', \mu^0)$ of N' is isomorphic to the reachability set of $R(N, \mu^0)$, i.e.,

$$R(N', \mu^0) \cong R(N, \mu^0) \cong\downarrow K_{1,2^n-1}.$$

Now, we need to show that $N' \cong N$.

Toward this end, define a map $\psi : P' \cup T' \longrightarrow P \cup T$ satisfying $\psi(p'_i) = p_i$ and $\psi(t'_i) = t_i$. Clearly, ψ is a bijection. We shall now show that it preserve the directed adjacency of N' onto N. For this, consider any isomorphism $\varphi : R(N', \mu^0) \longrightarrow R(N, \mu^0)$ from the reachability tree of N' onto the reachability tree of N; this has the property that

$$(\mu^0, \mu^i) \in \mathcal{A}(R(N', \mu^0)) \Leftrightarrow (\varphi(\mu^0), \varphi(\mu^i)) \in \mathcal{A}(R(N, \mu^0)), \qquad \cdots (1)$$

where $\mathcal{A}(D)$ denotes the set of arcs of any digraph D (in this case, D is the reachability tree of the corresponding Petri net).

Let (p'_i, t'_j) be an arc in N', we will show then that $(\psi(p'_i), \psi(t'_j))$ is an arc in N. Suppose, on the contrary $(\psi(p'_i), \psi(t'_j)) = (p_i, t_j)$ is not an arc in N. This implies in N that the marking vector μ^i whose i^{th} component is zero does not get generated by firing t_j or when $t_j^\bullet = \emptyset$ the marking vector obtained by firing t_j is repeated. The latter case does not arise due to the hypothesis that every marking vector is generated exactly once in N. But, then the former statement implies $\varphi(\mu^0)$ does not form the arc $(\varphi(\mu^0), \varphi(\mu^i))$ in $R(N, \mu^0)$ and hence, from (1), it follows that (μ^0, μ^i) does not form an arc in the reachability tree $R(N', \mu^0)$. This is a contradiction to our assumption that N' generates all the binary $n-$vectors exactly once. Similarly, one can arrive at a contradiction by assuming $(\psi(t'_j), \psi(p'_i))$ is not an arc in N. Thus, $N' \cong N$ follows, because the choice of the arcs (p'_i, t'_j) and (t'_j, p'_i) was arbitrary in each case. $\square$

We illustrate the above theorem by taking $n = 3$.

A $1 - safe$ Petri net N generating all the $2^3 = 8$ binary $3-$vectors exactly once as mentioned in [4] is given in Figure 2.
In N, we have
the total number of transitions $= 2^3 - 1 = 8 - 1 = 7$,
$|p^\bullet| = 2^3 - 1 = 8 - 1 = 7, \ \forall \ p$,
$|^\bullet p| = 2^{3-1} - 1 = 3, \ \forall \ p$,
$|^\bullet t| = 3, \ \forall \ t$.

The total number of transitions whose post-sets have two elements $= {}^3C_2 = 3$ and these three transitions are t_1, t_2, t_3.
Total number of transitions whose post-sets have one element each $= {}^3C_1 = 3$ and these three transitions are t_4, t_5, t_6.
The total number of transitions whose post-sets have no element $= {}^3C_0 = 1$ and this transition is t_7. Together with μ_0, we get $2^3 = 8$ binary 3-vectors as marking vectors, each generated exactly once.

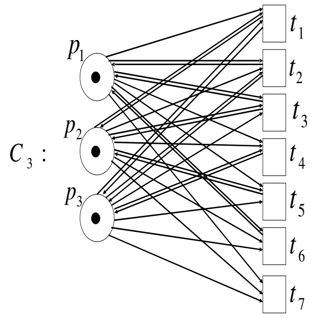

Figure 2: The minimal 1−safe Petri net N that generates all the binary 3−vectors exactly once.

2 Scope for further research

If we relax the condition on the depth of the reachability tree in our original definition of minimality of a 'minimal' 1−safe Petri net generating all the binary n−vectors exactly once and require instead that the number of enabled transitions be kept minimum possible, the reachability trees of such Petri nets may not have their underlying graph structures isomorphic to $K_{1,2^n-1}$, whence they would all be trees of the same order 2^n. Since they would be finite in number, determination of the structures of such Petri nets and their enumeration would be of potential practical interest. It involves orienting trees of order 2^n (in general, for theoretical purposes, trees of any order as such) that admit an orientation of their edges to make them the reachability trees of minimal 1−safe Petri nets generating all the binary n−vectors exactly once.

Acknowledgement.

The authors deeply acknowledge the valuable suggestions made by Dr. B.D. Acharya from time to time in carrying out the work reported in this paper. Especially, we are thankful to him for giving us his perspectives detailed in the last section.

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