THE NON-RUIN PROBABILITY FOR THE RISK RESERVE PROCESS WITH ERLANG TYPE CLAIMS

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Abstract. In this paper, we consider the risk reserve process with Erlang type claims. We make the model that the claim inter-arrival time has an exponential distribution and the claim size has an Erlang distribution. For this model we obtain a general formula that derives the non-ruin probability in finite time. In order to have an easy calculation for the non-ruin probability we reduce the multi-summation to the single-summation.

1 Introduction For the risk reserve process in the steady state, the expected ruin time and the ruin probability have been studied by Doi [1] and [2], respectively. Furthermore the non-ruin probability with exponential type claims in finite time has been studied by A. Kishikawa and M. Doi [3] for the model with single claim. In this paper we discuss the non-ruin probability, in finite time, for the risk reserve process with main and optional claims as the expansion of [3].

Let us denote $U(t)$ the reserve level at time t , where $\{U(t)\}_{t\geq 0}$ is called the risk reserve process. If the reserve level is less than zero, the process ruins. We denote by T_n $n\,th$ claim's arrival time. We assume that the claim inter-arrival time between $(n-1)\,th$ and $n\,th$ claim's arrival times $W_n\,(=T_n - T_{n-1})$ and the $n\,th$ claim size X_n are independent and identically distributed random variables. W_n and X_n are also mutually independent. We also assume that W_n has an exponential distribution with parameter λ and X_n has an Erlang distribution with parameter μ and phase k. Since the claim consists of the sum of main and optional ones, which are mutually independent and exponentially distributed random variables, we consider that the claim size has the Erlang distribution : Erlang type claims.

The risk reserve process is controlled by the claim inter-arrival time, the claim size and the premium rate.

The total claim amount process $\{S(t)\}_{t\geq 0}$ is defined by

$$S(t) = \sum_{n=1}^{N(t)} X_n \quad (t \geq 0),$$

where $\{N(t)\}_{t\geq 0}$ is the claim number process defined by

$$N(t) = \max\{n \geq 1 : T_n \leq t\} \quad (t \geq 0).$$

We also define the premium income by

$$I(t) = ct$$

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where the premium rate c is a constant.
Thus the risk reserve process is expressed by

$$U(t) = u + I(t) - S(t), \quad (t \geq 0),$$

where u is the initial reserve level.

In Proposition 2, we derive the non-ruin probability with muliti-summation. Next we reduce it to the single summation in proposition 3.

2 Mathematical model and analyses In the risk reserve process the ruin can occur at the time $t = T_n$ (some $n \geq 1$). By use of skelton process (Mikosch [4]) the event of ruin is defined by

$$\begin{aligned} \{\text{ruin}\} &= \left\{ \inf_{n\geq 1} [u + I(T_n) - S(T_n)] < 0 \right\} \\ &= \left\{ \inf_{n\geq 1} \left[u - \sum_{i=1}^{n} (X_i - cW_i) \right] < 0 \right\}. \end{aligned} \tag{1}$$

Also, we define the following.

$$\begin{aligned} Z_n &= X_n - cW_n, \quad (n \geq 1), \\ S_n &= Z_1 + \cdots + Z_n, \quad (n \geq 1,\ S_0 = 0). \end{aligned} \tag{2}$$

Now we propose R|Ex|Er model where R means the risk reserve process, Ex means that the claim inter-arrival time W_n has the exponential distribution with rate λ and Er means that the claim size X_n has the Erlang distribution with parameter μ and phase k.

In the next subsection, we omit the subscript n.

2.1 Probability density function of Z for R|Ex|Er model We derive the probability distribution of random variable Z for this model. Since X and cW are independent random variables, we obtain the joint probability density function with respect to Z and $V = cW$

$$f_{ZV}(z, v) = \frac{(k\mu)^k}{(k-1)!}(z+v)^{k-1}e^{-k\mu(z+v)} \cdot \frac{\lambda}{c}e^{-\frac{\lambda}{c}v} \tag{3}$$

where the domain of v is

$$\begin{cases} 0 \leq v < \infty & (z \geq 0) \\ -z < v < \infty & (z < 0). \end{cases}$$

We describe the following two lemmas.

Lemma 1 *For any natural number k, the following relation holds.*

$$\int_0^\infty (y+x)^k e^{-rx} dx = \sum_{i=0}^{k} \frac{k!}{(k-i)!} r^{-i-1} y^{k-i}, \quad r > 0,\ y \geq 0. \tag{4}$$

Lemma 2 *For any natural number k, the following relation holds.*

$$\int_{-y}^\infty (y+x)^k e^{-rx} dx = k! r^{-k-1} e^{ry}, \quad r > 0,\ y < 0. \tag{5}$$

For $z \geq 0$, from Lemma 1, we obtain the probability density function of Z as follows:

(6)
$$\begin{aligned} g_k(z) =& \int_0^\infty \frac{(k\mu)^k}{(k-1)!}(z+v)^{k-1}e^{-k\mu(z+v)} \cdot \frac{\lambda}{c}e^{-\frac{\lambda}{c}v}dv \\ =& \frac{\lambda(k\mu)^k}{c(k-1)!}e^{-k\mu z}\sum_{i=0}^{k-1}\frac{(k-1)!}{(k-i-1)!}\left(k\mu+\frac{\lambda}{c}\right)^{-i-1} \cdot z^{k-i-1} \\ =& \frac{\lambda}{c}(k\mu)^k e^{-k\mu z}\left(k\mu+\frac{\lambda}{c}\right)^{-k}\sum_{i=0}^{k-1}\frac{1}{(k-i-1)!}\left(k\mu+\frac{\lambda}{c}\right)^{k-i-1} \cdot z^{k-i-1} \\ =& \frac{\lambda}{c}(k\mu)^k\left(k\mu+\frac{\lambda}{c}\right)^{-k} \cdot e^{-k\mu z}\sum_{i=0}^{k-1}\frac{1}{i!}\left(k\mu+\frac{\lambda}{c}\right)^i z^i. \end{aligned}$$

For $z < 0$, from Lemma 2,

(7)
$$\begin{aligned} g_k(z) =& \int_{-z}^\infty \frac{(k\mu)^k}{(k-1)!}(z+v)^{k-1}e^{-k\mu(z+v)} \cdot \frac{\lambda}{c}e^{-\frac{\lambda}{c}v}dv \\ =& \frac{\lambda(k\mu)^k}{c(k-1)!}e^{-k\mu z}(k-1)!\left(k\mu+\frac{\lambda}{c}\right)^{-k} \cdot e^{(k\mu+\frac{\lambda}{c})z} \\ =& \frac{\lambda}{c}(k\mu)^k\left(k\mu+\frac{\lambda}{c}\right)^{-k} \cdot e^{\frac{\lambda}{c}z}. \end{aligned}$$

Now, let us set

(8)
$$\begin{cases} \alpha = k\mu + \dfrac{\lambda}{c} \\ \beta = k\mu \\ A = \alpha^{-1}\beta, \end{cases}$$

then

(9)
$$g_k(z) = \begin{cases} \dfrac{\lambda}{c}A^k e^{-\beta z}\displaystyle\sum_{i=0}^{k-1}\frac{\alpha^i}{i!}z^i & (z \geq 0) \\ \dfrac{\lambda}{c}A^k e^{\frac{\lambda}{c}z} & (z < 0). \end{cases}$$

2.2 Propositions for R|Ex|Er model We define the non-ruin probability by $r_n^{(k)}(u,c)$ that the risk reserve process does not ruin till n-th claim arrival time given the initial reserve level u and the premium rate c, that is,

(10) $r_n^{(k)}(u,c)$
$$= P(Z_1 < u, Z_2 < u - S_1, \cdots, Z_n < u - S_{n-1}|U(0) = u, T_1 < T_2 < \cdots < T_n < \infty),$$

where the claim size X_m, $(m = 1, 2, \cdots n)$ has the Erlang distribution with parameter μ and phase k.

By use of $g_k(z)$ above, we formulate $r_n^{(k)}(u,c)$. First, for $r_1^{(k)}(u,c)$, we have the following three lemmas.

Lemma 3 *For any non-negative integer i, the following relation holds.*

(11)
$$\int_0^u z^i e^{-\beta z}dz = \frac{i!}{\beta}\left(\beta^{-i} - e^{-u\beta}\sum_{j=0}^{i}\frac{\beta^{-j}}{(i-j)!}u^{i-j}\right).$$

Lemma 4 *For any natural number k and n, the following relation holds.*

$$\sum_{i=0}^{n}\binom{k+i}{i}=\binom{k+n+1}{n}. \tag{12}$$

We have the following lemma with respect to β and A of (8).

Lemma 5 *For any natural number k, the following relation holds.*

$$A^k+\frac{\lambda}{c\beta}\sum_{i=1}^{k}A^i=1. \tag{13}$$

Proof of Lemma 5.

$$A^k+\frac{\lambda}{c\beta}\sum_{i=1}^{k}A^i=\left(\frac{1}{c\beta+\lambda}\right)^k\left\{(c\beta)^k+\frac{\lambda}{c\beta}\sum_{i=1}^{k}(c\beta)^i(c\beta+\lambda)^{k-i}\right\}.$$

On the second term in parenthesis of right hand side, using Lemma 4,

$$\begin{aligned}\sum_{i=1}^{k}(c\beta)^i(c\beta+\lambda)^{k-i}&=\sum_{i=1}^{k}\sum_{j=0}^{k-i}\binom{k-i}{j}(c\beta)^{i+j}\lambda^{k-i-j}\\&=\sum_{s=1}^{k}\sum_{t=0}^{s-1}\binom{k-(s-t)}{t}(c\beta)^s\lambda^{k-s}\\&\qquad (s=i+j,\ t=j)\\&=\sum_{s=1}^{k}(c\beta)^s\lambda^{k-s}\sum_{t=0}^{s-1}\binom{k-s+t}{t}\\&=\sum_{s=1}^{k}(c\beta)^s\lambda^{k-s}\binom{k}{s-1}.\end{aligned}$$

Therefore, we have the following.

$$\begin{aligned}A^k+\frac{\lambda}{c\beta}\sum_{i=1}^{k}A^i&=\left(\frac{1}{c\beta+\lambda}\right)^k\left\{(c\beta)^k+\sum_{s=0}^{k-1}\binom{k}{s}(c\beta)^s\lambda^{k-s}\right\}\\&=\left(\frac{1}{c\beta+\lambda}\right)^k\sum_{s=0}^{k}\binom{k}{s}(c\beta)^s\lambda^{k-s}\\&=1.\end{aligned}$$

□

Using these Lemmas, we derive the following proposition for $r_1^{(k)}(u,c)$.

Proposition 1 *For $R|Ex|Er$ model, the following relation holds.*

$$r_1^{(k)}(u,c)=1-\frac{\lambda}{c\beta}e^{-u\beta}\sum_{i_1=1}^{k}A^{i_1}\sum_{i_2=0}^{k-i_1}\frac{\beta^{i_2}}{i_2!}u^{i_2}. \tag{14}$$

Proof of Proposition 1. Using *Lemma 3*,

$$\begin{aligned} r_1^{(k)}(u,c) =& P(Z_1 < u | U(0) = u,\ T_1 < \infty) \\ =& \int_{-\infty}^{0} g_k(z_1) dz_1 + \int_0^u g_k(z_1) dz_1 \\ =& \frac{\lambda}{c} A^k \left(\int_{-\infty}^{0} e^{\frac{\lambda}{c} z_1} dz_1 + \sum_{i_1=0}^{k-1} \frac{\alpha^{i_1}}{i_1!} \int_0^u z_1^{i_1} e^{-\beta z_1} dz_1 \right) \\ =& \frac{\lambda}{c} A^k \left\{ \frac{c}{\lambda} + \sum_{i_1=0}^{k-1} \frac{\alpha^{i_1}}{i_1!} \frac{i_1!}{\beta} \left(\beta^{-i_1} - e^{-u\beta} \sum_{i_2=0}^{i_1} \frac{\beta^{-i_2}}{(i_1 - i_2)!} u^{i_1 - i_2} \right) \right\}. \end{aligned}$$

Since $A = \alpha^{-1}\beta$,

$$\begin{aligned} r_1^{(k)}(u,c) =& \frac{\lambda}{c} A^k \left(\frac{c}{\lambda} + \frac{1}{\beta} \sum_{i_1=0}^{k-1} A^{-i_1} - \frac{1}{\beta} e^{-u\beta} \sum_{i_1=0}^{k-1} A^{-i_1} \sum_{i_2=0}^{i_1} \frac{\beta^{i_2}}{i_2!} u^{i_2} \right) \\ =& A^k + \frac{\lambda}{c\beta} \sum_{i_1=0}^{k-1} A^{k-i_1} - \frac{\lambda}{c\beta} e^{-u\beta} \sum_{i_1=0}^{k-1} A^{k-i_1} \sum_{i_2=0}^{i_1} \frac{\beta^{i_2}}{i_2!} u^{i_2} \\ =& A^k + \frac{\lambda}{c\beta} \sum_{i_1=1}^{k} A^{i_1} - \frac{\lambda}{c\beta} e^{-u\beta} \sum_{i_1=1}^{k} A^{i_1} \sum_{i_2=0}^{k-i_1} \frac{\beta^{i_2}}{i_2!} u^{i_2}. \end{aligned}$$

Finally using *Lemma 5*, we obtain (14). □

For the general formula of $r_n^{(k)}(u,c)$, we derive the following proposition.

Proposition 2 *For $R|Ex|Er$ model, we formulate the probability $r_n^{(k)}(u,c)$ as follows:*

$$\begin{aligned} & r_0^{(k)}(u,c) = 1, \\ (15)\qquad & r_n^{(k)}(u,c) = r_{n-1}^{(k)}(u,c) - \left(\frac{\lambda}{c}\right)^n \alpha^{-n} A^{k(n-1)-1} e^{-u\beta} \\ & \qquad \cdot \sum_{i_1=1}^{k} A^{i_1} \sum_{i_2=0}^{k-i_1} A^{i_2} \sum_{i_3=0}^{k+i_2} \sum_{i_4=0}^{k+i_3} \cdots \sum_{i_{n+1}=0}^{k+i_n} \frac{(u\alpha)^{i_{n+1}}}{i_{n+1}!}, \quad n \geq 1, \end{aligned}$$

where

$$(16)\qquad \begin{cases} \alpha = k\mu + \dfrac{\lambda}{c} \\ \beta = k\mu \\ A = \alpha^{-1}\beta. \end{cases}$$

Preparatory to the proof of Proposition 2, we have the following two lemmas.

Lemma 6 *For any non-negative integer i, the following relation holds.*

$$(17)\qquad \int_{-\infty}^{0} e^{\alpha z}(u-z)^i dz = \alpha^{-i-1} i! \sum_{j=0}^{i} \frac{(u\alpha)^j}{j!}, \quad \alpha > 0.$$

Lemma 7 *For any non-negative integer i and j, the following relation holds.*

$$\int_0^u (u-z)^i z^j dz = \frac{i!j!}{(i+j+1)!}u^{i+j+1}. \tag{18}$$

Then we proceed to prove Proposition 2.
Proof of Proposition 2. In the case of $n=1$ we prove, by *Proposition 1*, that (15) is true as follows:

$$\begin{aligned} r_0^{(k)}(u,c) - \frac{\lambda}{c}\alpha^{-1}A^{-1}e^{-u\beta}\sum_{i_1=1}^{k}A^{i_1}\sum_{i_2=0}^{k-i_1}A^{i_2}\frac{(u\alpha)^{i_2}}{i_2!} \\ =1-\frac{\lambda}{c\beta}e^{-u\beta}\sum_{i_1=1}^{k}A^{i_1}\sum_{i_2=0}^{k-i_1}\frac{\beta^{i_2}}{i_2!}u^{i_2} = r_1^{(k)}(u,c). \end{aligned}$$

We assume that (15) is true for $n=m$. Then, for $n=m+1$,

$$\begin{aligned} & r_{m+1}^{(k)}(u,c) \\ &= \int_{-\infty}^{u} g_k(z_1) r_m^{(k)}(u-z_1,c)dz_1 \\ &= \int_{-\infty}^{u} g_k(z_1)\left\{ r_{m-1}^{(k)}(u-z_1,c) - \left(\frac{\lambda}{c}\right)^m \alpha^{-m}A^{k(m-1)-1}\right. \\ &\quad \left. \cdot e^{-(u-z_1)\beta}\sum_{i_1=1}^{k}A^{i_1}\sum_{i_2=0}^{k-i_1}A^{i_2}\sum_{i_3=0}^{k+i_2}\sum_{i_4=0}^{k+i_3}\cdots\sum_{i_{m+1}=0}^{k+i_m}\frac{\alpha^{i_{m+1}}}{i_{m+1}!}(u-z_1)^{i_{m+1}}\right\} dz_1 \\ &= \int_{-\infty}^{u} g_k(z_1) r_{m-1}^{(k)}(u-z_1,c)dz_1 - \left(\frac{\lambda}{c}\right)^m \alpha^{-m}A^{k(m-1)-1}e^{-u\beta}\sum_{i_1=1}^{k}A^{i_1}\sum_{i_2=0}^{k-i_1}A^{i_2} \\ &\quad \cdot \sum_{i_3=0}^{k+i_2}\sum_{i_4=0}^{k+i_3}\cdots\sum_{i_{m+1}=0}^{k+i_m}\frac{\alpha^{i_{m+1}}}{i_{m+1}!}\int_{-\infty}^{u} g_k(z_1)e^{\beta z_1}(u-z_1)^{i_{m+1}}dz_1\,. \end{aligned} \tag{19}$$

By use of *Lemmas 6*, *7* and (9), we obtain

$$\begin{aligned} & r_{m+1}^{(k)}(u,c) \\ &= r_m^{(k)}(u,c) - \left(\frac{\lambda}{c}\right)^{m+1}\alpha^{-m}A^{mk-1}e^{-u\beta} \\ &\quad \cdot \sum_{i_1=1}^{k}A^{i_1}\sum_{i_2=0}^{k-i_1}A^{i_2}\sum_{i_3=0}^{k+i_2}\sum_{i_4=0}^{k+i_3}\cdots\sum_{i_{m+1}=0}^{k+i_m}\frac{\alpha^{i_{m+1}}}{i_{m+1}!} \\ &\quad \cdot \left\{\sum_{i_{m+2}=0}^{k-1}\frac{\alpha^{i_{m+2}}}{i_{m+2}!}\int_0^u z_1^{i_{m+2}}(u-z_1)^{i_{m+1}}dz_1 + \int_{-\infty}^{0} e^{(\frac{\lambda}{c}+\beta)z_1}(u-z_1)^{i_{m+1}}dz_1\right\} \end{aligned}$$

$$= r_m^{(k)}(u,c) - \left(\frac{\lambda}{c}\right)^{m+1} \alpha^{-m} A^{mk-1} e^{-u\beta}$$

$$\cdot \sum_{i_1=1}^{k} A^{i_1} \sum_{i_2=0}^{k-i_1} A^{i_2} \sum_{i_3=0}^{k+i_2} \sum_{i_4=0}^{k+i_3} \cdots \sum_{i_{m+1}=0}^{k+i_m} \frac{\alpha^{i_{m+1}}}{i_{m+1}!}$$

$$\cdot \left\{ \sum_{i_{m+2}=0}^{k-1} \frac{\alpha^{i_{m+2}}}{i_{m+2}!} \frac{i_{m+1}! i_{m+2}!}{(i_{m+1}+i_{m+2}+1)!} u^{i_{m+1}+i_{m+2}+1} + \alpha^{-i_{m+1}-1} i_{m+1}! \sum_{i_{m+2}=0}^{i_{m+1}} \frac{(u\alpha)^{i_{m+2}}}{i_{m+2}!} \right\}$$

$$= r_m^{(k)}(u,c) - \left(\frac{\lambda}{c}\right)^{m+1} \alpha^{-m} A^{mk-1} e^{-u\beta}$$

$$\cdot \sum_{i_1=1}^{k} A^{i_1} \sum_{i_2=0}^{k-i_1} A^{i_2} \sum_{i_3=0}^{k+i_2} \sum_{i_4=0}^{k+i_3} \cdots \sum_{i_{m+1}=0}^{k+i_m} \alpha^{i_{m+1}}$$

$$\cdot \alpha^{-i_{m+1}-1} \left\{ \sum_{i_{m+2}=0}^{k-1} \frac{(u\alpha)^{i_{m+1}+i_{m+2}+1}}{(i_{m+1}+i_{m+2}+1)!} + \sum_{i_{m+2}=0}^{i_{m+1}} \frac{(u\alpha)^{i_{m+2}}}{i_{m+2}!} \right\}$$

$$= r_m^{(k)}(u,c) - \left(\frac{\lambda}{c}\right)^{m+1} \alpha^{-m-1} A^{mk-1} e^{-u\beta}$$

$$\cdot \sum_{i_1=1}^{k} A^{i_1} \sum_{i_2=0}^{k-i_1} A^{i_2} \sum_{i_3=0}^{k+i_2} \sum_{i_4=0}^{k+i_3} \cdots \sum_{i_{m+1}=0}^{k+i_m} \left\{ \sum_{i_{m+2}=i_{m+1}+1}^{k+i_{m+1}} \frac{(u\alpha)^{i_{m+2}}}{i_{m+2}!} + \sum_{i_{m+2}=0}^{i_{m+1}} \frac{(u\alpha)^{i_{m+2}}}{i_{m+2}!} \right\}.$$

Thus, in the case of $n = m+1$, (15) is true as follows:

$$r_{m+1}^{(k)}(u,c) = r_m^{(k)}(u,c) - \left(\frac{\lambda}{c}\right)^{m+1} \alpha^{-(m+1)} A^{mk-1} e^{-u\beta}$$

$$\cdot \sum_{i_1=1}^{k} A^{i_1} \sum_{i_2=0}^{k-i_1} A^{i_2} \sum_{i_3=0}^{k+i_2} \sum_{i_4=0}^{k+i_3} \cdots \sum_{i_{m+1}=0}^{k+i_m} \sum_{i_{m+2}=0}^{k+i_{m+1}} \frac{(u\alpha)^{i_{m+2}}}{i_{m+2}!}.$$

Therefore, Proposition 2 is true for any non-negative integer n. □

2.3 Reduction of multi-summation Although we have formulated the general formula of $r_n^{(k)}(u,c)$, this general formula is not suitable for numerical calculations because of too many summations. In order to reduce the number of summation, let us denote the summation part of (15) by

$$\phi_k(n) = \sum_{i_1=1}^{k} A^{i_1} \sum_{i_2=0}^{k-i_1} A^{i_2} \sum_{i_3=0}^{k+i_2} \sum_{i_4=0}^{k+i_3} \cdots \sum_{i_{n+1}=0}^{k+i_n} \frac{(u\alpha)^{i_{n+1}}}{i_{n+1}!}, \quad n \geq 1. \tag{20}$$

We derive the following proposition.

Proposition 3 *If we define the following, for any natural number k,*

$$
\begin{cases}
K_{n,1}^{(k)} = A^{k-1}, & (n \geq 1) \\
K_{1,m}^{(k)} = \sum_{j=1}^{m} A^{k-j}, & (2 \leq m \leq k) \\
K_{n,kn-j}^{(k)} = K_{n,k(n-1)}^{(k)}, & (n \geq 2,\ 1 \leq j \leq k) \\
K_{n,m}^{(k)} = K_{n-1,m}^{(k)} + K_{n,m-1}^{(k)}, & (n \geq 2,\ 2 \leq m \leq k(n-1)) \\
K_{n,m}^{(k)} = 0, & (\textit{others}),
\end{cases}
\tag{21}
$$

then (20) *is reduced to*

$$
\phi_k(n) = A \sum_{i=0}^{kn-1} K_{n,kn-i}^{(k)} \frac{(u\alpha)^i}{i!}, \quad n \geq 1. \tag{22}
$$

Preliminarily, we derive the following lemma.

Lemma 8 *If k is given, for any natural number m and n ($m \leq kn$), the following relation holds.*

$$
\sum_{i=1}^{m} K_{n,i}^{(k)} = K_{n+1,m}^{(k)}. \tag{23}
$$

Proof of Lemma 8. In the case of $m = 1$, by use of (21), we find the left-hand side is equal to the right-hand side as follows:

$$
\sum_{i=1}^{1} K_{n,i}^{(k)} = K_{n,1}^{(k)} = A^{k-1} = K_{n+1,1}^{(k)}.
$$

In the case of $m = l$, we assume that (23) is true. Next, using (21), we prove that (23) is true if $m = l + 1$.

$$
\sum_{i=1}^{l+1} K_{n,i}^{(k)} = \sum_{i=1}^{l} K_{n,i}^{(k)} + K_{n,l+1}^{(k)} = K_{n+1,l}^{(k)} + K_{n,l+1}^{(k)} = K_{n+1,l+1}^{(k)}.
$$

Therefore, (23) is true for any natural numbers m and n ($m \leq kn$). □

Now we prove Proposition 3.

Proof of Proposition 3. For $n = 1$, using (21), we find (22) is true as follows:

$$
\begin{aligned}
\phi_k(1) &= \sum_{i_1=1}^{k} A^{i_1} \sum_{i_2=0}^{k-i_1} A^{i_2} \frac{(u\alpha)^{i_2}}{i_2!} = \sum_{i_2=0}^{k-1} \sum_{i_1=1}^{k-i_2} A^{i_1+i_2} \frac{(u\alpha)^{i_2}}{i_2!} = \sum_{i_2=0}^{k-1} \frac{(u\alpha)^{i_2}}{i_2!} \sum_{i_1=1}^{k-i_2} A^{k-i_1+1} \\
&= A \sum_{i_2=0}^{k-1} K_{1,k-i_2}^{(k)} \frac{(u\alpha)^{i_2}}{i_2!}.
\end{aligned}
$$

We assume that (22) is true for $n = m$. Then, for $n = m + 1$, from (19)

$$
\begin{aligned}
&r^{(k)}_{m+1}(u,c) \\
&= \int_{-\infty}^{u} g_k(z_1) r^{(k)}_{m-1}(u - z_1, c) dz_1 - \int_{-\infty}^{u} g_k(z_1) \left(\frac{\lambda}{c}\right)^m \alpha^{-m} A^{k(m-1)-1} e^{-(u-z_1)\beta} \\
&\quad \cdot \sum_{i_1=1}^{k} A^{i_1} \sum_{i_2=0}^{k-i_1} A^{i_2} \sum_{i_3=0}^{k+i_2} \sum_{i_4=0}^{k+i_3} \cdots \sum_{i_{m+1}=0}^{k+i_m} \frac{\{(u-z_1)\alpha\}^{i_{m+1}}}{i_{m+1}!} dz_1 \\
&= r^{(k)}_m(u,c) \\
&\quad - \int_{-\infty}^{u} g_k(z_1) \left(\frac{\lambda}{c}\right)^m \alpha^{-m} A^{k(m-1)-1} e^{-(u-z_1)\beta} A \sum_{i=0}^{km-1} K^{(k)}_{m,km-i} \frac{\{(u-z_1)\alpha\}^i}{i!} dz_1 \,.
\end{aligned}
$$

On the second term of right hand side, using *Lemmas 6, 7, 8* and (9), we have the following.

$$
\begin{aligned}
&\left(\frac{\lambda}{c}\right)^m \alpha^{-m} A^{k(m-1)} e^{-u\beta} \sum_{i=0}^{km-1} K^{(k)}_{m,km-i} \frac{\alpha^i}{i!} \int_{-\infty}^{u} g_k(z_1) e^{\beta z_1} (u - z_1)^i dz_1 \\
&\quad = \left(\frac{\lambda}{c}\right)^m \alpha^{-m} A^{k(m-1)} e^{-u\beta} \sum_{i=0}^{km-1} K^{(k)}_{m,km-i} \frac{\alpha^i}{i!} \\
&\qquad \cdot \frac{\lambda}{c} A^k \left\{ \int_{-\infty}^{0} e^{\alpha z_1} (u - z_1)^i dz_1 + \sum_{j=0}^{k-1} \frac{\alpha^j}{j!} \int_0^u (u - z_1)^i z_1^j dz_1 \right\} \\
&\quad = \left(\frac{\lambda}{c}\right)^{m+1} \alpha^{-m} A^{km} e^{-u\beta} \sum_{i=0}^{km-1} K^{(k)}_{m,km-i} \frac{\alpha^i}{i!} \\
&\qquad \cdot \alpha^{-i-1} i! \left\{ \sum_{j=0}^{i} \frac{(u\alpha)^j}{j!} + \sum_{j=0}^{k-1} \frac{(u\alpha)^{i+j+1}}{(i+j+1)!} \right\} \\
&\quad = \left(\frac{\lambda}{c}\right)^{m+1} \alpha^{-m-1} A^{km} e^{-u\beta} \sum_{i=0}^{km-1} K^{(k)}_{m,km-i} \left\{ \sum_{j=0}^{i} \frac{(u\alpha)^j}{j!} + \sum_{j=i+1}^{k+i} \frac{(u\alpha)^j}{j!} \right\} \\
&\quad = \left(\frac{\lambda}{c}\right)^{m+1} \alpha^{-m-1} A^{km} e^{-u\beta} \sum_{i=0}^{km-1} \sum_{j=0}^{k+i} \frac{(u\alpha)^j}{j!} K^{(k)}_{m,km-i} \\
&\quad = \left(\frac{\lambda}{c}\right)^{m+1} \alpha^{-m-1} A^{km} e^{-u\beta} \\
&\qquad \cdot \left\{ \sum_{j=k}^{k(m+1)-1} \sum_{i=j-k}^{km-1} \frac{(u\alpha)^j}{j!} K^{(k)}_{m,km-i} + \sum_{j=0}^{k-1} \sum_{i=0}^{km-1} \frac{(u\alpha)^j}{j!} K^{(k)}_{m,km-i} \right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{\lambda}{c}\right)^{m+1} \alpha^{-m-1} A^{km} e^{-u\beta} \\
&\quad \cdot \left\{ \sum_{j=k}^{k(m+1)-1} \frac{(u\alpha)^j}{j!} K^{(k)}_{m+1,k(m+1)-j} + \sum_{j=0}^{k-1} \frac{(u\alpha)^j}{j!} K^{(k)}_{m+1,km} \right\} \\
&= \left(\frac{\lambda}{c}\right)^{m+1} \alpha^{-m-1} A^{km} e^{-u\beta} \\
&\quad \cdot \left\{ \sum_{j=k}^{k(m+1)-1} K^{(k)}_{m+1,k(m+1)-j} \frac{(u\alpha)^j}{j!} + \sum_{j=0}^{k-1} K^{(k)}_{m+1,k(m+1)-j} \frac{(u\alpha)^j}{j!} \right\}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(24) \quad & r^{(k)}_{m+1}(u,c) \\
&= r^{(k)}_m(u,c) - \left(\frac{\lambda}{c}\right)^{m+1} \alpha^{-m-1} A^{km-1} e^{-u\beta} \cdot A \sum_{j=0}^{k(m+1)-1} K^{(k)}_{m+1,k(m+1)-j} \frac{(u\alpha)^j}{j!}.
\end{aligned}
$$

On the other hand, from *Proposition 2*, we obtain

$$
\begin{aligned}
(25) \quad r^{(k)}_{m+1}(u,c) &= r^{(k)}_{(m+1)-1}(u,c) - \left(\frac{\lambda}{c}\right)^{m+1} \alpha^{-(m+1)} A^{k\{(m+1)-1\}-1} e^{-u\beta} \\
&\quad \cdot \sum_{i_1=1}^{k} A^{i_1} \sum_{i_2=0}^{k-i_1} A^{i_2} \sum_{i_3=0}^{k+i_2} \sum_{i_4=0}^{k+i_3} \cdots \sum_{i_{m+2}=0}^{k+i_{m+1}} \frac{(u\alpha)^{i_{(m+1)+1}}}{i_{(m+1)+1}!} \\
&= r^{(k)}_m(u,c) - \left(\frac{\lambda}{c}\right)^{m+1} \alpha^{-m-1} A^{km-1} e^{-u\beta} \cdot \phi_k(m+1).
\end{aligned}
$$

By comparing (24) and (25), (22) is true for $n = m+1$. Hence (22) is true for any natural number n. □

2.4 Non-ruin probability in finite time for R|Ex|Er model

From *Proposition 2* and *3*, we derive the following theorem.

Theorem 1 *For R|Ex|Er model, we formulate the non-ruin probability in finite time* $r^{(k)}_n(u,c)$ *as follows:*

$$
\begin{aligned}
& r^{(k)}_0(u,c) = 1, \\
(26) \quad & r^{(k)}_n(u,c) = r^{(k)}_{n-1}(u,c) - \left(\frac{\lambda}{c}\right)^n \alpha^{-n} A^{k(n-1)} e^{-u\beta} \sum_{i=0}^{kn-1} K^{(k)}_{n,kn-i} \frac{(u\alpha)^i}{i!}, \quad n \geq 1,
\end{aligned}
$$

where

$$
(27) \quad \begin{cases} \alpha = k\mu + \dfrac{\lambda}{c} \\ \beta = k\mu \\ A = \alpha^{-1}\beta, \end{cases}
$$

and

$$(28) \qquad \begin{cases} K_{n,1}^{(k)} = A^{k-1}, & (n \geq 1) \\ K_{1,m}^{(k)} = \sum_{j=1}^{m} A^{k-j}, & (2 \leq m \leq k) \\ K_{n,kn-j}^{(k)} = K_{n,k(n-1)}^{(k)}, & (n \geq 2,\ 1 \leq j \leq k) \\ K_{n,m}^{(k)} = K_{n-1,m}^{(k)} + K_{n,m-1}^{(k)}, & (n \geq 2,\ 2 \leq m \leq k(n-1)) \\ K_{n,m}^{(k)} = 0, & (\textit{others}). \end{cases}$$

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