AN EXTERNAL VERSION OF THE JENSEN OPERATOR INEQUALITY

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Abstract. Convexity or concavity for a real function is expressed by the relation between internally dividing points in the domain and in the range and so is Jensen's inequality. Observing that it is also expressed by externally dividing points, we discuss Jensen's operator inequality from this point of view.

Usually, Jensen's inequality is expressed by internally dividing points (that is, convex sum): For $\alpha_i \geqq 0$ with $\sum_i \alpha_i = 1$, a concave function f on an (open) interval $\mathcal{I}$ assures

$$f\left(\sum_i \alpha_i x_i\right) \geqq \sum_i \alpha_i f(x_i)$$

for all $x_i \in \mathcal{I}$, which is the same for the definition of concavity. Along this line, it is extended to various operator inequalities (cf. [3, 4]). For example, Hansen-Pedersen [5, 2] showed that if f is operator concave on $\mathcal{I}$, then

$$C^* f(X)C + D^* f(Y)D \leq f(C^* X C + D^* Y D)$$

holds for selfadjoint operators X and Y with $\sigma(X), \sigma(Y) \in \mathcal{I}$ and contractions C and D with $C^* C + D^* D = I$ (Recall that f is *operator concave* on $\mathcal{I}$ if

$$f\left(\frac{X+Y}{2}\right) \geqq \frac{f(X) + f(Y)}{2}$$

holds for all selfadjoint operators X and Y with $\sigma(X), \sigma(Y) \in \mathcal{I}$).

On the other hand, the concavity is also expressed by externally dividing points:

$$r > 0, \ v, w, (1+r)v - rw \in \mathcal{I} \implies f((1+r)v - rw) \leqq (1+r)f(v) - rf(w).$$

Then, putting $v = (1-t)x + ty$, $w = y$, $r = \frac{t}{1-t}$, we have $\frac{1}{1+r} = 1 - t$, $\frac{r}{1+r} = t$ and hence

$$(1+r)v - rw = \frac{(1-t)x + ty - ty}{1-t} = x.$$

It follows that

$$f((1-t)x + ty) = f(v) \geqq \frac{1}{1+r} f((1+r)v - rw) + \frac{r}{1+r} f(w) = (1-t)f(x) + tf(y),$$

which implies the concavity of f.

Thus the purpose of this paper is to express what is an external version of the Jensen operator inequality. First we observe an external version of the classical Jensen inequality (An equivalent inequality was shown by Pečarić et. al. [6, p.83], [1, Theorem B]):

Theorem P (Pečarić-Proschan-Tong). *A function f on $\mathcal{I}$ is concave if and only if*

$$f\left(\left(1+\sum_k r_k\right)x - \sum_k r_k y_k\right) \leqq \left(1+\sum_k r_k\right)f(x) - \sum_k r_k f(y_k) \tag{1}$$

holds for all x, y_k, $z = (1+\sum_k r_k)x - \sum_k r_k y_k \in \mathcal{I}$ and nonnegative numbers r_k.

Remark 1. If $x, y_k \in \mathcal{I}$ with $x \leqq y_k$ (resp. $x \geqq y_k$), then z is indeed an external point for (x, y_k) by $z \leqq x \leqq y_k$ (resp. for (y_k, x) by $z \geqq x \geqq y_k$). Though the above result includes the inequality not only for external points, we also call such an inequality an 'external' version for the sake of convenience.

So we have an external version of Jensen's operator inequality:

Theorem 1. *Let H, K and L be Hilbert spaces. A function f is operator concave on $\mathcal{I}$ if and only if*

$$f(C^*XC - D^*YD) \leq |C|f(V^*XV)|C| - D^*f(Y)D \tag{2}$$

holds for all selfadjoint operators $X \in B(K)$, $Y \in B(L)$ with $\sigma(X)$, $\sigma(Y) \in \mathcal{I}$ and operators $C \in B(H, K)$, $D \in B(H, L)$ with $C^*C - D^*D = I_H$ and $\sigma(C^*XC - D^*YD) \in \mathcal{I}$, where V is the partial isometry in the polar decomposition $C = V|C|$.

In particular, if C is invertible, then (2) is expressed by

$$f(C^*XC - D^*YD) \leq C^*f(X)C - D^*f(Y)D. \tag{2'}$$

Proof. Note that $|C| = \sqrt{C^*C}$ is invertible and $\tilde{C}^*\tilde{C} + \tilde{D}^*\tilde{D} = I_H$ holds for $\tilde{C} = |C|^{-1} = \sqrt{C^*C}^{-1}$ and $\tilde{D} = D|C|^{-1} = D\tilde{C}$. Suppose f is operator concave. Then the Hansen-Pedersen-Jensen operator inequality shows

$$f(\tilde{C}^*A\tilde{C} + \tilde{D}^*B\tilde{D}) \geq \tilde{C}^*f(A)\tilde{C} + \tilde{D}^*f(B)\tilde{D}.$$

It follows that

$$\begin{aligned}
|C|f(V^*XV)|C| &= |C|f\left(\tilde{C}(|C|V^*XV|C| - D^*YD)\tilde{C} + \tilde{D}^*Y\tilde{D}\right)|C| \\
&\geq |C|\tilde{C}f(C^*XC - D^*YD)\tilde{C}|C| + |C|\tilde{D}^*f(Y)\tilde{D}|C| \\
&= f(C^*XC - D^*YD) + D^*f(Y)D.
\end{aligned}$$

So we have

$$f(C^*XC - D^*YD) \leq |C|f(V^*XV)|C| - D^*f(Y)D.$$

Conversely suppose (2). Let X, Y be selfadjoint operators with $\sigma(X)$, $\sigma(Y) \in \mathcal{I}$ and $0 < t < 1$. Putting $V = (1-t)X + tY$, $W = Y$ and $r = \frac{t}{1-t} > 0$, we have $t = \frac{r}{1+r}$ and $(1+r)V - rW = \frac{(1-t)X + tY - tY}{1-t} = X$. Thus V and W satisfy $\sigma(V)$, $\sigma(W)$, $\sigma((1+r)V - rW) \in \mathcal{I}$. Then as a special case of the above inequality,

$$f((1+r)V - rW) \leq (1+r)f(V) - rf(W).$$

It follows that

$$f((1-t)X + tY) = f(V) \geq \frac{1}{1+r}f((1+r)V - rW) + \frac{r}{1+r}f(W) = (1-t)f(X) + tf(Y),$$

which implies the operator concavity.

If C is invertible, then V is unitary and hence $|C|f(V^*XV)|C| = |C|V^*f(X)V|C| = C^*f(X)C$, so that we have (2'). $\square$

Considering $X = \begin{pmatrix} X_1 & & \\ & \ddots & \\ & & X_k \end{pmatrix}$, $Y = \begin{pmatrix} Y_1 & & \\ & \ddots & \\ & & Y_m \end{pmatrix}$, $C = \begin{pmatrix} C_1 \\ \vdots \\ C_k \end{pmatrix}$ and $D = \begin{pmatrix} D_1 \\ \vdots \\ D_m \end{pmatrix}$ according to the decompositions $K = K_1 \oplus \cdots \oplus K_k$ and $L = L_1 \oplus \cdots \oplus L_m$, we have

Corollary 2. *Let f be operator concave on an open interval $\mathcal{I}$. For Hilbert spaces H, $K = \oplus_i K_i$ and $L = \oplus_j L_j$, let $X_i = X_i^* \in B(K_i)$, $Y_j = Y_j^* \in B(L_j)$, $C_i \in B(H, K_i)$ and $D_j \in B(H, L_j)$ with $\sigma(X_i), \sigma(Y_j) \in \mathcal{I}$. Let $C_i = V_i|C_i|$ be the polar decompositions. If $\sum_{i=1}^k C_i^* C_i - \sum_{j=1}^m D_j^* D_j = I_H$ and $\sigma(\sum_{i=1}^k C_i^* X_i C_i - \sum_{j=1}^m D_j^* Y_j D_j) \in \mathcal{I}$, then*

$$f\left(\sum_{i=1}^k C_i^* X_i C_i - \sum_{j=1}^m D_j^* Y_j D_j\right)$$

$$\leq \sqrt{\sum_{i=1}^k |C_i|^2} f\left(\sum_{i=1}^k V_i^* X_i V_i\right) \sqrt{\sum_{i=1}^k |C_i|^2} - \sum_{j=1}^m D_j^* f(Y_j) D_j.$$

In particular, if $C = {}^t(C_1, \cdots, C_k)$ is invertible, then

$$f\left(\sum_{i=1}^k C_i^* X_i C_i - \sum_{j=1}^m D_j^* Y_j D_j\right) \leq \sum_{i=1}^k C_i^* f(X_i) C_i - \sum_{j=1}^m D_j^* f(Y_j) D_j.$$

Remark 2. If $C = \xi = \begin{pmatrix} \sqrt{\alpha_1} \\ \vdots \\ \sqrt{\alpha_k} \end{pmatrix}$ and $D = \eta = \begin{pmatrix} \sqrt{\beta_1} \\ \vdots \\ \sqrt{\beta_m} \end{pmatrix}$ are vectors with nonnegative entries and $\|\xi\|^2 - \|\eta\|^2 = 1$, then ξ cannot be invertible for $k > 1$. So, putting $X = \operatorname{diag}(x_1, \cdots, x_k)$ and $Y = \operatorname{diag}(y_1, \cdots, y_m)$, we only have the following numerical inequality;

$$f\left(\sum_{i=1}^k \alpha_i x_i - \sum_{j=1}^m \beta_j y_j\right) \leqq \|\xi\|^2 f\left(\sum_{i=1}^k \frac{\alpha_i x_i}{\|\xi\|^2}\right) - \sum_{j=1}^m \beta_j f(y_j)$$

holds for a operator concave function f. This also holds for a concave function f and it is equivalent to Theorem 1 by putting $\alpha = \sum_i \alpha_i$ and $x = \sum_i \alpha_i x_i$.

Finally we show an external version for Jensen's operator inequality for projections:

Theorem 3. *Let f be a function on an open interval $\mathcal{I}$ including 0. A function f is operator concave on $\mathcal{I}$ if and only if*

$$f(\sqrt{I+P}X\sqrt{I+P} - PYP) \leq \sqrt{I+P}f(X)\sqrt{I+P} - Pf(Y)P \tag{3}$$

holds for all projections P and selfadjoint operators X, Y such that

$$\sigma(X), \ \sigma(Y), \ \sigma(\sqrt{I+P}X\sqrt{I+P} - PYP) \in \mathcal{I}.$$

Proof. The operator concavity of f implies (3) by Theorem 2. Conversely let $P = I$, the identity operator. For selfadjoint operators A and B with $\sigma(A), \sigma(B) \in \mathcal{I}$. Putting $X = (A + B)/2$ and $Y = B$. Then

$$\sigma(X),\, \sigma(Y) \in \mathcal{I} \quad \text{and} \quad \sigma(2X - Y) = \sigma(A) \in \mathcal{I}.$$

So (3) implies

$$f(A) = f(2X - Y) \leq 2f(X) - f(Y) = 2f\left(\frac{A + B}{2}\right) - f(B),$$

which shows that f is operator concave. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Thus the multi-variable inequality is:

Corollary 4. *Let f be operator concave on an open interval $\mathcal{I}$ including 0. Suppose that P_i are projections with $\sum_i P_i = I$ and $X_i,\, Y_i$ are selfadjoint operators. If*

$$\sigma(X_i),\; \sigma(Y_i),\; \sigma\left(\sum_i \sqrt{I + P_i}X\sqrt{I + P_i} - P_i Y P_i\right) \in \mathcal{I},$$

then

$$f\left(\sum_{i=1}^{k} \sqrt{I + P_i}X_i\sqrt{I + P_i} - P_i^* Y_i P_i\right) \leq \sum_{i=1}^{k} \sqrt{I + P_i}f(X_i)\sqrt{I + P_i} - P_i f(Y_i) P_i.$$

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