STABLE RANK OF THE C^*-ALGEBRAS OF ALMOST COMMUTING OPERATORS

Takahiro Sudo

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Abstract. We estimate the stable rank of the soft tori of Exel and their isometric versions. Especially, it is found that there exists a connection between stable rank and no continuity of C^*-algebra fields.

Introduction

The stable rank for C^*-algebras is introduced and studied by Rieffel [9] as a noncommutative counterpart to the covering dimension for topological spaces. In particular, it is shown among many things that the full group C^*-algebra of the free group with n generators has stable rank infinity, while AF algebras that are inductive limits of finite dimensional C^*-algebras as well as AT algebras that are inductive limits of finite direct sums of matrix algebras over the C^*-algebra of all continuous functions on the 1-torus have stable rank one. In particular, the irrational rotation algebras, that are shown to be AT by [3], have stable rank one. On the other hand, Exel introduced the soft tori of almost commuting unitaries and studied their K-theory, and obtained their continuity as continuous fields of C^*-algebras (see [4] and [5] respectively).

In this paper it is shown in Sections 1 and 2 that there are lower estimates of the stable rank as well as the real rank (of Brown-Pedersen [1]) for the soft tori and their isometric versions (that we call soft Toeplitz tensor products) using universality argument and some known facts on those ranks. Those estimates imply that those soft C^*-algebras are neither AF nor AT. In Section 3 we discover a remarkable connection between stable rank and continuous fields of C^*-algebras over the unit interval, which says that the stable rank of fibers in the continuous fields can not always take any value in general, i.e., there is a reasonable restriction coming from its being bounded, having a gap, or being infinite at fibers. Consequently, it is shown that those soft C^*-algebras have stable rank infinity. This suggests that the C^*-algebras are out of the reach of the classification program (see [6]). The independent arguments in those sections should be of some interest.

Now recall from [9] that a unital C^*-algebra $\mathfrak{A}$ has stable rank $\leq n$ if $L_n(\mathfrak{A})$ (an open subset) is dense in $\mathfrak{A}^n$ (n-direct sum), where $(a_j) \in L_n(\mathfrak{A})$ means that there exists an element $(b_j) \in \mathfrak{A}^n$ such that $\sum_{j=1}^n b_j a_j = 1$ (or invertible). We denote by $\mathrm{sr}(\mathfrak{A})$ the stable rank of $\mathfrak{A}$, that is defined to be the least positive integer of such n. If no such n, set $\mathrm{sr}(\mathfrak{A}) = \infty$. Similarly, the real rank $\mathrm{RR}(\mathfrak{A})$ of $\mathfrak{A}$ is defined, where those $(a_j), (b_j) \in L_n(\mathfrak{A})$ and $\mathfrak{A}^n$ are replaced by self-adjoint $(a_j), (b_j) \in L_{n+1}(\mathfrak{A})$ and $\mathfrak{A}^{n+1}$, so that $\mathrm{RR}(\mathfrak{A}) \geq 0$ ([1]).

1 The soft tori and soft Toeplitz tensor products

Recall that for $\varepsilon \geq 0$, the soft torus A_ε of Exel is defined to be the universal C^*-algebra generated by two unitaries $u_{\varepsilon,1}$, $u_{\varepsilon,2}$ such that $\|u_{\varepsilon,2}u_{\varepsilon,1} - u_{\varepsilon,1}u_{\varepsilon,2}\| \leq \varepsilon$. By definition, A_0

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is isomorphic to $C(\mathbb{T}^2)$ the C^*-algebra of all continuous functions on the 2-torus $\mathbb{T}^2$, and for $\varepsilon \geq 2$, A_ε is isomorphic to $C^*(F_2)$ the full group C^*-algebra of the free group F_2 with two generators.

Proposition 1.1 *For the soft torus A_ε for $\varepsilon \geq 0$, we have* $\mathrm{sr}(A_\varepsilon) \geq 2$.
Moreover, we obtain $\mathrm{RR}(A_\varepsilon) \geq 2$.

Proof. By universality, there exists a quotient map from A_ε to $A_0 = C(\mathbb{T}^2)$. It follows by [9, Theorem 4.3] that $\mathrm{sr}(A_\varepsilon) \geq \mathrm{sr}(A_0)$. Also, $\mathrm{sr}(C(\mathbb{T}^2)) = 2$ by [9, Proposition 1.7].

Similarly, we have $\mathrm{RR}(A_\varepsilon) \geq \mathrm{RR}(A_0)$. Also, $\mathrm{RR}(C(\mathbb{T}^2)) = 2$ by [1]. □

Recall that for $\varepsilon \geq 0$, the soft Toeplitz tensor product $\mathfrak{D}_\varepsilon$ is defined to be the universal C^*-algebra generated by two isometries $s_{\varepsilon,1}$, $s_{\varepsilon,2}$ such that $\|s_{\varepsilon,2}s_{\varepsilon,1} - s_{\varepsilon,1}s_{\varepsilon,2}\| \leq \varepsilon$ (cf. [10]). By definition, $\mathfrak{D}_0$ is isomorphic to $\mathfrak{F} \otimes \mathfrak{F}$ the C^*-tensor product of the Toeplitz algebra $\mathfrak{F}$ defined by the universal C^*-algebra generated by an isometry, and for $\varepsilon \geq 2$, $\mathfrak{D}_\varepsilon$ is isomorphic to $C^*(\mathbb{N} * \mathbb{N})$ the full semigroup C^*-algebra of the free semigroup $\mathbb{N} * \mathbb{N}$ with two generators (or the free product of the semigroup $\mathbb{N}$ of natural numbers).

Proposition 1.2 *For the soft Toeplitz tensor product $\mathfrak{D}_\varepsilon$ for $\varepsilon \geq 0$, we have* $\mathrm{sr}(\mathfrak{D}_\varepsilon) \geq 2$.
Furthermore, we obtain $\mathrm{RR}(\mathfrak{D}_\varepsilon) \geq 2$ *and* $\mathrm{sr}(\mathfrak{D}_2) = \infty$.

Proof. By universality, there exists a quotient map from $\mathfrak{D}_\varepsilon$ to $\mathfrak{D}_0$. It follows by [9, Theorem 4.3] that $\mathrm{sr}(\mathfrak{D}_\varepsilon) \geq \mathrm{sr}(\mathfrak{D}_0)$. Also, there exists a quotient map from $\mathfrak{D}_0$ to $C(\mathbb{T}^2)$ that is deduced from the short exact sequence: $0 \to \mathbb{K} \to \mathfrak{F} \to C(\mathbb{T}) \to 0$, where $\mathbb{K}$ is the C^*-algebra of compact operators. Hence $\mathrm{sr}(\mathfrak{D}_0) \geq \mathrm{sr}(C(\mathbb{T}^2))$. Similarly, the real rank case is done. Since there exists a canonical quotient map from $C^*(\mathbb{N}*\mathbb{N})$ to $C^*(\mathbb{Z}*\mathbb{Z}) = C^*(F_2)$, it follows that $\mathrm{sr}(C^*(\mathbb{N}*\mathbb{N})) = \infty$ by $\mathrm{sr}(C^*(F_2)) = \infty$ by [9, Theorem 6.7]. □

Corollary 1.3 *The C^*-algebras A_ε and $\mathfrak{D}_\varepsilon$ for $\varepsilon \geq 0$ are neither AF nor AT.*

Proof. It is shown by [9, Proposition 3.5] that AF algebras have stable rank one. Furthermore, AT algebras have stable rank one by [9, Theorems 5.1 and 6.1]. □

2 Soft n-tori and soft Toeplitz n-tensor products

For any $\varepsilon \geq 0$ and an integer $n \geq 2$, the (generalized) soft n-torus A_ε^n (by the author) is defined to be the universal C^*-algebra generated by n unitaries $u_{\varepsilon,j}$ $(1 \leq j \leq n)$ such that $\|u_{\varepsilon,k}u_{\varepsilon,j} - u_{\varepsilon,j}u_{\varepsilon,k}\| \leq \varepsilon$ for $1 \leq j,k \leq n$. By definition, A_0^n is isomorphic to $C(\mathbb{T}^n)$ the C^*-algebra of all continuous functions on the n-torus $\mathbb{T}^n$, and for $\varepsilon \geq 2$, A_ε^n is isomorphic to $C^*(F_n)$ the full group C^*-algebra of the free group F_n with n generators.

For $x \in \mathbb{R}$, let $\lfloor x \rfloor$ denote the maximum integer $\leq x$.

Proposition 2.1 *For the soft n-torus A_ε^n for $\varepsilon \geq 0$, we have* $\mathrm{sr}(A_\varepsilon^n) \geq \lfloor n/2 \rfloor + 1$.
Moreover, we obtain $\mathrm{RR}(A_\varepsilon^n) \geq n$.

Proof. By universality, there exists a quotient map from A_ε^n to $A_0^n = C(\mathbb{T}^n)$. It follows by [9, Theorem 4.3] that $\mathrm{sr}(A_\varepsilon^n) \geq \mathrm{sr}(A_0^n)$. Also, $\mathrm{sr}(C(\mathbb{T}^n)) = \lfloor n/2 \rfloor + 1$ by [9, Proposition 1.7].

Similarly, we have $\mathrm{RR}(A_\varepsilon^n) \geq \mathrm{RR}(A_0^n)$. Also, $\mathrm{RR}(C(\mathbb{T}^n)) = n$ by [1]. □

For $\varepsilon \geq 0$, the soft Toeplitz n-tensor product $\mathfrak{D}_\varepsilon^n$ is defined to be the universal C^*-algebra generated by n isometries $s_{\varepsilon,j}$ $(1 \leq j \leq n)$ such that $\|s_{\varepsilon,k}s_{\varepsilon,j} - s_{\varepsilon,j}s_{\varepsilon,k}\| \leq \varepsilon$. By definition, $\mathfrak{D}_0^n$ is isomorphic to $\otimes^n\mathfrak{F}$ the n-fold C^*-tensor product of the Toeplitz algebra $\mathfrak{F}$, and for $\varepsilon \geq 2$, $\mathfrak{D}_\varepsilon^n$ is isomorphic to $C^*(*^n\mathbb{N})$ the full semigroup C^*-algebra of the free semigroup $*^n\mathbb{N}$ with n generators (or the n-fold free product of $\mathbb{N}$).

Proposition 2.2 *For the soft Toeplitz n-tensor product $\mathfrak{D}_\varepsilon^n$ for $\varepsilon \geq 0$, we have* $\mathrm{sr}(\mathfrak{D}_\varepsilon^n) \geq \lfloor n/2 \rfloor + 1$. *Furthermore, we obtain* $\mathrm{RR}(\mathfrak{D}_\varepsilon^n) \geq n$ *and* $\mathrm{sr}(\mathfrak{D}_2^n) = \infty$.

Proof. By universality, there exists a quotient map from $\mathfrak{D}_\varepsilon^n$ to $\mathfrak{D}_0^n$. It follows by [9, Theorem 4.3] that $\mathrm{sr}(\mathfrak{D}_\varepsilon^n) \geq \mathrm{sr}(\mathfrak{D}_0^n)$. Also, there exists a quotient map from $\mathfrak{D}_0^n$ to $C(\mathbb{T}^n)$ that is deduced from the short exact sequence: $0 \to \mathbb{K} \to \mathfrak{F} \to C(\mathbb{T}) \to 0$. Hence $\mathrm{sr}(\mathfrak{D}_0^n) \geq \mathrm{sr}(C(\mathbb{T}^n))$. Similarly, the real rank case is done. Since there exists a canonical quotient map from $C^*(*^n\mathbb{N})$ to $C^*(*^n\mathbb{Z}) = C^*(F_n)$, it follows that $\mathrm{sr}(C^*(*^n\mathbb{N})) = \infty$ by $\mathrm{sr}(C^*(F_n)) = \infty$ by [9, Theorem 6.7]. □

Corollary 2.3 *The C^*-algebras A_ε^n and $\mathfrak{D}_\varepsilon^n$ for $\varepsilon \geq 0$ are neither AF nor AT.*

Proof. See the proof of Corollary 1.3 above. □

Remark. See also the last corollary and its remark below.

3 Stable rank and no continuity

Given a continuous field of C^*-algebras $\mathfrak{A}_t$ for $t \in [0,1]$ the closed interval, we can define the C^*-algebra of (certain) continuous operator fields (or sections) on $[0,1]$, denoted by $\Gamma([0,1], \{\mathfrak{A}_t\}_{t\in[0,1]})$, with point-wise operations and the supremum norm (see [2]).

Theorem 3.1 *For $t \in [0,1]$, let $\mathfrak{A}_t$ be unital C^*-algebras. Suppose that for $t \in [0,1)$ the half open interval, $\mathfrak{A}_t$ have stable rank 1 but $\mathfrak{A}_1$ has stable rank ≥ 3. Then there exist no continuous fields of C^*-algebras on $[0,1]$ with fibers $\mathfrak{A}_t$.*

Proof. Suppose that we had a continuous field of C^*-algebras on $[0,1]$ with fibers $\mathfrak{A}_t$, denoted by $\mathfrak{A} = \Gamma([0,1], \{\mathfrak{A}_t\}_{t\in[0,1]})$. For any positive interger n, let $\mathfrak{B}_n = \Gamma([0,1-n^{-1}], \{\mathfrak{A}_t\}_{t\in[0,1-n^{-1}]})$ be the restriction of $\mathfrak{A}$ to the closed interval $[0,1-n^{-1}]$. By using [8], we obtain

$$\mathrm{sr}(\mathfrak{B}_n) \leq \sup_{t\in[0,1-n^{-1}]} \mathrm{sr}(C([0,1-n^{-1}]) \otimes \mathfrak{A}_t).$$

Furthermore, by [7], we have $\mathrm{sr}(C([0,1-n^{-1}]) \otimes \mathfrak{A}_t) \leq 1 + 1 = 2$. Therefore, $\mathrm{sr}(\mathfrak{B}_n) \leq 2$. It follows that $L_2(\mathfrak{B}_n)$ is dense in $\mathfrak{B}_n^2$, where $(b_{n1}, b_{n2}) \in L_2(\mathfrak{B}_n)$ means that there exists $(c_{n1}, c_{n2}) \in \mathfrak{B}_n^2$ such that $c_{n1}b_{n1} + c_{n2}b_{n2} = 1 \in \mathfrak{B}_n$.

On the other hand, $\mathrm{sr}(\mathfrak{A}_1) \geq 3$ which implies that $L_2(\mathfrak{A}_1)$ is not dense in $\mathfrak{A}_1^2$. Take $(a_{11}, a_{12}) \in \mathfrak{A}_1^2$ not contained in the closure of $L_2(\mathfrak{A}_1)$, that is, for any $(d_{11}, d_{12}) \in \mathfrak{A}_1^2$, we have $d_{11}a_{11} + d_{12}a_{12} - 1 \neq 0$. Then there exists $(f_1, f_2) \in \mathfrak{A}^2$ such that $f_1(1) = a_{11}$ and $f_2(1) = a_{12}$. For any n, the restriction of (f_1, f_2) to $[0, 1-n^{-1}]$ can be assumed in $L_2(\mathfrak{B}_n)$. Namely, there exist $(g_{1n}, g_{2n}) \in \mathfrak{B}_n^2$ such that $g_{1n}f_1 + g_{2n}f_2 - 1 = 0$ on $[0, 1-n^{-1}]$. Taking $n \to \infty$ implies the contradiction to such an existence of (f_1, f_2) (from the gap between being zero and non-zero). □

The same proof as above implies

Theorem 3.2 *For $t \in [0,1]$, let $\mathfrak{A}_t$ be unital C^*-algebras. Suppose that for $t \in [0,1)$, $\mathfrak{A}_t$ have stable rank bounded by M a positive integer but $\mathfrak{A}_1$ has stable rank $\geq M+2$. Then there exist no continuous fields of C^*-algebras on $[0,1]$ with fibers $\mathfrak{A}_t$.*

As applications,

Theorem 3.3 *The soft n-torus A_ε^n and the soft Toeplitz n-tensor product $\mathfrak{D}_\varepsilon^n$ for $0 < \varepsilon \leq 2$ have stable rank ∞.*

Proof. There exists a continuous field of C^*-algebras with fibers A_ε^n (cf. [5]). In fact, A_ε^n can be viewed as the crossed product of a C^*-algebra by (a multi-shift of) the group of integers as given in [4] so that their continuity with respect to ε as a continuous field of C^*-algebras is deduced from the argument as given in [5]. Now suppose that for a positive integer M, we had $\mathrm{sr}(A_\varepsilon^n) \leq M$ for $0 < \varepsilon < 2$. But $\mathrm{sr}(A_2^n) = \infty$ is true. By the theorem above, there exists $0 < \delta < 2$ such that $\mathrm{sr}(A_\delta^n) = \infty$. It follows that $\mathrm{sr}(A_\varepsilon^n) = \infty$ for $\delta \leq \varepsilon < 2$ since there exists an onto $*$-homomorphism from A_ε^n to A_δ^n. Continuing this process, we obtain the conclusion.

The proof for $\mathfrak{D}_\varepsilon^n$ is the same as this. Indeed, there exists a continuous field of C^*-algebras with fibers $\mathfrak{D}_\varepsilon^n$, which is obtained similarly as shown in [5] (cf. [10]). In fact, $\mathfrak{D}_\varepsilon^n$ can be viewed as the crossed product of a C^*-algebra by (a multi-shift of) the semigroup of natural numbers as given in [4] (cf. [10]) so that their continuity with respect to ε as a continuous field of C^*-algebras is deduced from the argument as given in [5]. □

Corollary 3.4 *The C^*-algebras in the theorem above are not isomorphic to a C^*-algebra with stable rank finite. In particular, they are neither isomorphic to inductive limits of homogeneous C^*-algebras, with stable rank finite, nor to inductive limits of finite direct sums of matrix algebras over Toeplitz tensor products, with stable rank finite.*

Remark. An inductive limit of homogeneous C^*-algebras, with slow dimension growth, i.e., the limit of the dimensions of their spectrums divided by the degrees of their representations is zero (cf. [6]), has stable rank ≤ 2 by using [9, Theorems 5.1 and 6.1]. Also, an inductive limit of finite direct sums of matrix algebras over $\otimes^n \mathfrak{F}$ for n fixed with the sizes of the matrix algebras going to infinity has stable rank ≤ 2. As a note, the infinite tensor product $\otimes^\infty \mathfrak{F}$ is an inductive limit of $\otimes^n \mathfrak{F}$ with n going to infinity, which has stable rank infinity.

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Department of Mathematical Sciences, Faculty of Science, University of the Ryukyus, Nishihara, Okinawa 903-0213, Japan.

Email: sudo@math.u-ryukyu.ac.jp