STOPPING GAME ON TWO STOCKS DRIVEN BY LÉVY PROCESSES

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Received November 23, 2007

Abstract. In this note, we treat a two-player zero-sum Dynkin game on two stocks driven by geometric Lévy processes, for a given terminal reward cost. Explicit forms for the optimal stopping times and the value of the game are both sought for, under certain conditions. The present note extends a recent result of the author to include a wider class of diffusion processes with jumps. The main result is derived following a decomposition of a stopping game into two standard optimal stopping problems which is due to Yasuda for a standard Brownian motion.

1. Statement of the Dynkin game Let $Q_t = (x_t, y_t)$ be a non-degenerate, two-dimensional geometric Lévy process given by

$$
\begin{aligned}
dx_t &= x_{t-}[-b_1 dt + a_{11} dB_t^1 + a_{12} dB_t^2 + \int_{-1}^{\infty} \theta_1(u)\widehat{N}_1(dt,du)]; \quad x(0-) = x, \\
dy_t &= y_{t-}[-b_2 dt + a_{21} dB_t^1 + a_{22} dB_t^2 + \int_{-1}^{\infty} \theta_2(v)\widehat{N}_2(dt,dv)]; \quad y(0-) = y,
\end{aligned} \tag{1.1}
$$

where x_t, y_t denote the evolution of stock prices at any instant time t initially starting at $(x,y) \in \mathbb{R}_+^2$, b_i, a_{ij} are some fixed positive market coefficients for $i,j = 1,2$, $B_t = (B_t^1, B_t^2)$ is a two-dimensional, possibly correlated, Brownian motion independent of Poisson random martingale measures $\widehat{N}_i$ $(i = 1,2)$ both defined on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$, $\theta_i(.)$ $(i = 1,2)$ are real-valued functions such that $1 + \theta_i(.) > 0$, and $\widehat{N}_i$ $(i = 1,2)$ is given by

$$
\widehat{N}_i(s,Z) = N_i(s,Z) - s\Pi_i(Z); \quad s \geq 0, \quad Z \in \mathbb{B}(-1,\infty), \tag{1.2}
$$

where

$$
\mathbb{E}N_i(s,Z) = s\Pi_i(Z). \tag{1.3}
$$

Here, $\mathbb{B}$ is a Borel sigma-algebra on $(-1,\infty)$, $\Pi_i(.)$ is a Lévy measure concentrated on $(-1,\infty)$ associated with N_i, satisfying the integrability condition

$$
\int_{-1}^{\infty} (1 \wedge Z^2)\Pi_i(dZ) < \infty. \tag{1.4}
$$

This note concerns a stopping game in a financial market consisting of only two stocks available to investors I (the seller) and II (the buyer), where the first investor I strives to minimize a given payoff

$$
\phi_{x,y}(\tau;\sigma) = \mathbb{E}^{x,y}\left[\frac{y_\tau}{x_\tau}\chi_{\tau<\sigma} + (x_\sigma - y_\sigma)\chi_{\sigma\leq\tau}\right], \tag{1.5}
$$

2000 *Mathematics Subject Classification.* 91A05, 91A15, 91A60.

Key words and phrases. Geometric Lévy processes; Dynkin game; Decomposition of a stopping game; Ito's formula.

over the stopping times $\tau \in \mathcal{X}^Q$, contrary to the second investor II who seeks to maximize the same payoff over the stopping times $\sigma \in \mathcal{X}^Q$, where $\mathcal{X}^Q$ denotes the class of all admissible stopping times with respect to the process Q_t.

A two-player zero-sum Dynkin game for the present problem is to find a pair of (τ_*, σ_*) $(\mathbb{F}_t)$-stopping times and the value of the game $\Phi(x, y)$, if they exist, such that

$$\Phi(x,y) = \inf_{\tau\in\mathcal{X}^Q} \sup_{\sigma\in\mathcal{X}^Q} \phi_{x,y}(\tau;\sigma) = \sup_{\sigma\in\mathcal{X}^Q} \inf_{\tau\in\mathcal{X}^Q} \phi_{x,y}(\tau;\sigma), \tag{1.6}$$

where $\phi_{x,y}(.;.)$ is the expected reward assumed to be of the form in (1.5), and $\mathbb{E}^{x,y}$ is the mathematical expectation for the process (x_t, y_t) corresponding to non-zero initial states (x, y).

We first note that, there are several papers dealing with a characterization on Dynkin games, precisely the existence of optimal values for such type of games (see for instance [1],[4],[6], [8],[9],[12],[15],[16],[23],[24],[27]). However, there are a very few papers, as far as we know, which addresses the question on explicit forms for the optimal values in question. The present note can be regarded as one of those few papers, which treats a particular Dynkin game described by geometric Lévy processes, for which the optimal strategies are made explicit. Our main result extends and supplements our recent result in [13]. The original motivation of the present note is a decomposition result of Yasuda [26], in a stopping game for a standard Brownian motion.

2. Main result Our main result is stated in the next assertion:

PROPOSITION 2.1. Let $Q_t = (x_t, y_t)$ be a non-degenerate geometric Lévy process given by (1.1), and let $\phi_{x,y}(.;.)$ be given by (1.5). Let $\beta = \exp(-1)$ be a positive constant such that there exists $\alpha \in (\beta, 1)$. Suppose that the conditions

$$a_{11}^2 + a_{12}^2 - a_{21}^2 - a_{22}^2 = 0, \tag{2.1}$$

and

$$b_1 - b_2 + \int_{-1}^{\infty} \left[\log\left(\frac{1+\theta_2(v)}{1+\theta_1(u)}\right) + \theta_1(u) - \theta_2(v)\right] (\Pi_1(du) + \Pi_2(dv)) = 0, \tag{2.2}$$

hold.

Then, the two-player zero-sum Dynkin game

$$\Phi(x,y) = \inf_{\tau\in\mathcal{X}^Q} \sup_{\sigma\in\mathcal{X}^Q} \phi_{x,y}(\tau;\sigma) = \sup_{\sigma\in\mathcal{X}^Q} \inf_{\tau\in\mathcal{X}^Q} \phi_{x,y}(\tau;\sigma), \tag{2.3}$$

is solved explicitly by the optimal strategies:

(a) the value of the game $\Phi(x, y)$ is given by

$$\Phi(x,y) = \begin{cases} \varphi(x,y;K); & \text{if } x < \alpha y \\ \psi(x,y;K); & \text{if } x \geq \alpha y \end{cases} \tag{2.4}$$

where $K > 0$ is a positive constant, given by

$$K = -\gamma \log \alpha, \tag{2.5}$$

and φ, ψ are respectively the optimal values for the standard optimal stopping problems

$$\varphi(r,s;K) = \inf_{\tau\in\mathcal{X}^Z} \mathbb{E}^{r,s}\left[\frac{s_\tau}{r_\tau}\chi_{\tau<\sigma_0} + K\chi_{\sigma_0\leq\tau}\right], \tag{2.6}$$

and

$$\psi(p,q;K) = \sup_{\sigma\in\mathcal{X}^M} \mathbb{E}^{p,q}\Big[\,(p_\sigma - q_\sigma)\chi_{\sigma<\tau_0} + K\chi_{\tau_0\le\sigma}\,\Big], \tag{2.7}$$

for which σ_0, τ_0 are fixed stopping times, given by

$$\begin{aligned}\sigma_0 &= \inf\{t>0 : r_t \le \alpha s_t\},\\ \tau_0 &= \inf\{t>0 : p_t \ge \alpha q_t\}.\end{aligned}$$

(b) the pair of optimal stopping times (τ_*,σ_*) respectively given by

$$\tau_* = \inf\{t>0 : x_t \le \beta y_t\}, \tag{2.8}$$

and

$$\sigma_* = \inf\{t>0 : x_t \ge y_t\}, \tag{2.9}$$

is a saddle-point for the stopping game (1.6).

REMARK 2.1. The result states that if initially in the upper cone $C_U = \{(x,y)\in\mathbb{R}^2_+ : x<\beta y\}$, then the optimal strategy for the first investor I is to sell stocks, otherwise if initially in the lower cone $C_L = \{(x,y)\in\mathbb{R}^2_+ : x>y\}$ then it is optimal for the second investor II to buy stocks. Indeed, it is optimal for both investors to neither sell nor buy stocks in the interior of the wedge $C_W = \{(x,y)\in\mathbb{R}^2_+ : \beta y < x < y\}$.

REMARK 2.2. In the above result, observe that the optimal stopping problem (2.6) is subject to a two-dimensional, non-degenerate geometric Brownian motion $V_t = (r_t, s_t)$, whereas (2.7) is with respect to a pure jump-process $M_t = (p_t, q_t)$. The details will follow in the next section.

REMARK 2.3. Notice that as a consequence of our decomposition result, it is possible to formulate a set of optimality conditions based on the optimal stopping problems (2.6) and (2.7), in form of verification theorems which are quite similar to those in [17] and [18] respectively. It is a nice exercise to verify that the optimal values for these optimal stopping problems, which are made explicit in the next section, indeed satisfy the conditions of the above kind of verification theorems under our assumptions. Hence, we shall omit the details.

We remark that the proof of the above result is a consequence of the assertions stated and proved in the next section, where the optimal values for the stopping problems (2.6) and (2.7) are also made explicit.

3. Optimal stopping of geometric Lévy processes In this section, following a separation result of Yasuda [26], we shall decompose the two-player zero-sum Dynkin game (1.6) into two standard independent optimal stopping problems under certain conditions. We note that the present problem is a game variant of the well-known McDonald-Siegel's optimal stopping problem (see [10],[14],[19]), which we further extend to include a wider class of diffusion processes of the jump-type.

In what follows, we shall now resolve explicitly the stopping game (1.6) in the state space $\mathcal{M} = (0,\infty)\times(0,\infty)$. We first assume that the conditions (2.1) and (2.2) hold, and furthermore, we impose the following restrictions:

$$b_2 < b_1\alpha \text{ for } x > \alpha y, \tag{3.1}$$

and

$$a_{11}^2 + a_{12}^2 > (a_{11}a_{21} + a_{12}a_{22}) \text{ for } x < \alpha y. \tag{3.2}$$

Observe that, under the above hypotheses, the state space $\mathcal{M}$ splits into two regions separated by the line $x = \alpha y$. We shall now decompose the stopping game (1.6) into two standard optimal stopping problems, one of minimization subject to a two-dimensional geometric Brownian motion without drift and that of optimal stopping (maximization) for pure jump-processes. This is one of the essential differences with our recent result in [13], where in that case, the processes are respectively geometric Brownian motions. For notational convenience, we shall let $V_t = (r_t, s_t)$ denote the dynamics for the geometric Brownian motions and let $M_t = (p_t, q_t)$ represent the dynamics for the pure jump-processes describing the evolution of stock prices.

Consider the following optimal stopping problems:

PROBLEM 3.1. Let $M_t = (p_t, q_t)$ be a pure jump-process given by,

$$\begin{aligned} dp_t &= p_{t-}[-b_1 dt + \int_{-1}^{\infty} \theta_1(u)\widehat{N}_1(dt, du)]; \quad p(0-) = p, \\ dq_t &= q_{t-}[-b_2 dt + \int_{-1}^{\infty} \theta_2(v)\widehat{N}_2(dt, dv)]; \quad q(0-) = q. \end{aligned} \tag{3.3}$$

Find a stopping time σ_* and a maximal value $\psi(p, q; K)$, if they exist, such that

$$\psi(p, q; K) = \sup_{\sigma \in \mathcal{X}^M} \mathbb{E}^{p,q} \Big[(p_\sigma - q_\sigma)\chi_{\sigma < \tau_0} + K\chi_{\tau_0 \le \sigma} \Big], \tag{3.4}$$

where $\tau_0 = \inf\{t > 0 : p_t \ge \alpha q_t\}$ is a fixed stopping time and K is some constant yet to be made explicit.

PROBLEM 3.2. Find a stopping time τ_* and a minimal value $\varphi(r, s; K)$, if they exist, such that

$$\varphi(r, s; K) = \inf_{\tau \in \mathcal{X}^Z} \mathbb{E}^{r,s} \left[\frac{s_\tau}{r_\tau}\chi_{\tau < \sigma_0} + K\chi_{\sigma_0 \le \tau} \right], \tag{3.5}$$

where $\sigma_0 = \inf\{t > 0 : r_t \le \alpha s_t\}$ is a fixed stopping time, subject to the following non-degenerate, two-dimensional geometric Brownian motion $V_t = (r_t, s_t)$ described by

$$\begin{aligned} dr_t &= a_{11} r_t dB_t^1 + a_{12} r_t dB_t^2; \quad r(0) = r, \\ ds_t &= a_{21} s_t dB_t^1 + a_{22} s_t dB_t^2; \quad s(0) = s. \end{aligned} \tag{3.6}$$

In this section, we have proved the following assertions:

LEMMA 3.1. Suppose that the conditions (2.1), (2.2), (3.1) and (3.2) hold. Then, the optimal stopping problem (3.3) and (3.4) is explicitly solved by:

(a) the maximal value ψ is given by

$$\psi(p, q; K) = \begin{cases} \gamma \log\left(\dfrac{q}{p}\right); & \text{for } \alpha q < p < q \\ p - q; & \text{for } p \ge q \end{cases} \tag{3.7}$$

and the optimal stopping time σ_* is of the form

$$\sigma_* = \inf\{t > 0 : p_t \geq q_t\}, \tag{3.8}$$

Similarly, the optimal stopping problem (3.5) and (3.6) is explicitly solved by the following optimal strategies:

(b) the minimal value φ admits the form

$$\varphi(r, s; K) = \begin{cases} \gamma \log\left(\frac{s}{r}\right); & \text{for } s/\gamma < r < \alpha s \\ \frac{s}{r}; & \text{for } r \leq s/\gamma \end{cases} \tag{3.9}$$

and the optimal stopping time τ_* is of the form

$$\tau_* = \inf\{t > 0 : r_t \leq s_t/\gamma\}. \tag{3.10}$$

where

$$\gamma = \exp(1).$$

Proof. It is well-known in the theory of optimal stopping (see [5],[18],[21],[25]) that the problem (3.4) solves the integro-differential equation

$$\begin{aligned} &\int_{-1}^{\infty} \Big[\psi(p + p\theta_1(u), q + q\theta_2(v)) - \psi(p, q) - p\theta_1(u)\psi_p(p, q) - q\theta_2(v)\psi_q(p, q) \Big] \\ &\times(\Pi_1(du) + \Pi_2(dv)) - b_1 p\psi_p(p, q) - b_2 q\psi_q(p, q) = 0, \end{aligned} \tag{3.11}$$

for all p, q in the continuation region $D_1 = \{(p, q) \in \mathbb{R}^2_+ : \alpha q < p < g(q)\}$, subject to the condition

$$\psi(p, q) = p - q \text{ on } \mathcal{S}_1, \tag{3.12}$$

where $\mathcal{S}_1$ is the stopping region (complement of the set D_1), $\psi(.,)$ and $g(.)$ are both sought for.

Assuming that condition (2.2) holds, consequently we have (3.7). Similarly, assuming that (2.1) holds, the result in (3.9) follows from

$$L_V\varphi(r, s) = 0 \text{ for } r, s \in D_0 = \{(r, s) \in \mathbb{R}^2_+ : f(s) < r < \alpha s\},, \tag{3.13}$$

under the continuity condition and smooth-fit principle respectively,

$$\begin{aligned} \varphi(r, s) &= \frac{s}{r} \text{ on } \mathcal{S}_0, \\ \varphi_s(r, s) &= \frac{1}{r}, \quad \varphi_r(r, s) = -\frac{s}{r^2} \text{ on } \partial D^1_0 = \{(r, s) \in \mathbb{R}^2_+ : r = f(s)\}, \end{aligned} \tag{3.14}$$

where φ, f are both sought for, L_V is the infinitesimal generator associated with the geometric Brownian motion $V_t = (r_t, s_t)$, and $\mathcal{S}_0$ is the stopping set.

Finally, using the compatibility condition $\psi(\alpha q, q; K) = K = \varphi(\alpha s, s; K)$, then the constant K follows immediately. This completes the proof of our assertion. □

REMARK 3.1. Notice that the optimal stopping boundary $p = q$ in the optimal stopping problem (3.3) and (3.4) is derived without using the smooth-fit principle. For the reason that, excess over the stopping boundary is via a discontinuous motion. In this case, the smooth-fit principle breaks down. One should also observe that the solution meant for the McDonald-Siegel's optimal stopping problem ([10],[14],[19]) cannot be extended to the stopping game (1.6).

Finally, we have:

LEMMA 3.2. Assume that $\phi_{x,y}(.,.)$ is of the form (1.5), and Q_t is a non-degenerate geometric Lévy process given by (1.1). Let τ_* and σ_* be the optimal stopping times given by (3.10) and (3.8) respectively, then

$$\phi_{x,y}(\tau_*, \sigma) \le \phi_{x,y}(\tau_*, \sigma_*) \le \phi_{x,y}(\tau, \sigma_*), \tag{3.15}$$

for all stopping times $\tau, \sigma \in \mathcal{X}^Q$.

Proof. The proof follows as a consequence of using a generalized version of Ito's formula (see [20]) and using $\Phi(x, y)$ given in (2.4) for $\beta y < x < y$, assuming that the conditions (2.1) and (2.2) hold. □

4. **Concluding Remarks** This note treats a two-player zero-sum Dynkin game on two stocks driven by geometric Lévy processes, for which the optimal stopping times and the value of the game are both made explicit. The result is derived following a decomposition result of Yasuda [26], extended to a two-dimensional stopping game described by a class of diffusion processes of the jump-type, under certain conditions. The present result generalizes our recent result in [13] to include a wider class of diffusion processes of the jump-type.

Acknowledgements We acknowledge with many thanks to the anonymous referee and the Editor, Professor Krzysztof Szajowski for their constructive and useful comments, which led to an improved version of our earlier submission. We also thank the referee for calling our attention to reference [3], which treats a particular one-dimensional Dynkin game driven by a spectrally negative Lévy process, for a different payoff than ours. Partial results of this note were obtained when the author was holding a postdoc grant PRO12/1003 at the University of Oslo (Norway).

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