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SUBALGEBRAS AND CLOSED IDEALS OF BCH-ALGEBRAS BASED ON BIPOLAR-VALUED FUZZY SETS

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Abstract. The notions of bipolar fuzzy subalgebras and bipolar fuzzy closed ideals in BCH-algebras are introduced, and related properties are investigated. Relations between a bipolar fuzzy subalgebra and a bipolar fuzzy closed ideal are considered. Conditions for a bipolar fuzzy subalgebra to be a bipolar fuzzy closed ideal are provided. Characterizations of a bipolar fuzzy closed ideal are discussed.

1. **Introduction.** In the traditional fuzzy sets, the membership degrees of elements range over the interval $[0,1]$. The membership degree expresses the degree of belongingness of elements to a fuzzy set. The membership degree 1 indicates that an element completely belongs to its corresponding fuzzy set, and the membership degree 0 indicates that an element does not belong to the fuzzy set. The membership degrees on the interval $(0,1)$ indicate the partial membership to the fuzzy set. Sometimes, the membership degree means the satisfaction degree of elements to some property or constraint corresponding to a fuzzy set (see [3, 8]). In the viewpoint of satisfaction degree, the membership degree 0 is assigned to elements which do not satisfy some property. The elements with membership degree 0 are usually regarded as having the same characteristics in the fuzzy set representation. By the way, among such elements, some have irrelevant characteristics to the property corresponding to a fuzzy set and the others have contrary characteristics to the property. The traditional fuzzy set representation cannot tell apart contrary elements from irrelevant elements. Consider a fuzzy set "*young*" defined on the *age* domain $[0,100]$ (see Fig. 1 in [5]).

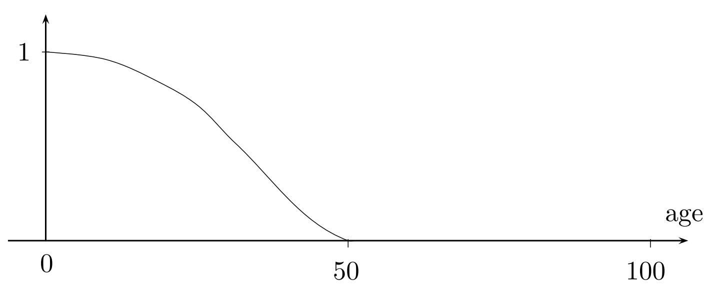

Figure 1. A fuzzy set "*young*"

Now consider two ages 50 and 95 with membership degree 0. Although both of them do not satisfy the property "*young*", we may say that age 95 is more apart from the property

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rather than age 50 (see [5]). Only with the membership degrees ranged on the interval [0, 1], it is difficult to express the difference of the irrelevant elements from the contrary elements in fuzzy sets. If a set representation could express this kind of difference, it would be more informative than the traditional fuzzy set representation. Based on these observations, Lee [5] introduced an extension of fuzzy sets named bipolar-valued fuzzy sets. He gave two kinds of representations of the notion of bipolar-valued fuzzy sets.

In this paper, we apply the notion of bipolar-valued fuzzy set to BCH-algebras. We introduce the concept of bipolar fuzzy subalgebras/ideals of a BCH-algebra, and investigate several properties. We give relations between a bipolar fuzzy subalgebra and a bipolar fuzzy closed ideal. We provide conditions for a bipolar fuzzy subalgebra to be a bipolar fuzzy closed ideal. We also give characterizations of a bipolar fuzzy closed ideal.

2. **Preliminaries.**

2.1. **Basic results on BCH-algebras.** Let $K(\tau)$ be the class of all algebras of type $\tau = (2, 0)$. By a *BCH-algebra* we mean a system $(X; *, 0) \in K(\tau)$ in which the following axioms hold:

$$(\forall x \in X)(x * x = 0), \tag{2.1}$$

$$(\forall x, y \in X)(x * y = 0 \ \& \ y * x = 0 \ \Rightarrow \ x = y), \tag{2.2}$$

$$(\forall x, y, z \in X)((x * y) * z = (x * z) * y). \tag{2.3}$$

Any BCH-algebra X satisfies the following axioms:

$$(\forall x \in X)(x * 0 = x), \tag{2.4}$$

$$(\forall x \in X)(x * 0 = 0 \ \Rightarrow \ x = 0), \tag{2.5}$$

$$(\forall x, y \in X)(0 * (x * y) = (0 * x) * (0 * y)), \tag{2.6}$$

$$(\forall x \in X)(0 * (0 * (0 * x)) = 0 * x). \tag{2.7}$$

A subset S of a BCH-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A subset I of a BCH-algebra X is called a *closed ideal* of X if it satisfies:

$$(\forall x \in X)(x \in I \ \Rightarrow \ 0 * x \in I), \tag{2.8}$$

$$(\forall x \in I)(\forall y \in X)(y * x \in I \ \Rightarrow \ y \in I). \tag{2.9}$$

2.2. **Basic results on bipolar-valued fuzzy set.** As an extension of fuzzy sets, Lee [5] introduced the notion of bipolar-valued fuzzy sets. So, this subsection is based on his paper (see [5] [6]). Fuzzy sets are a kind of useful mathematical structure to represent a collection of objects whose boundary is vague. There are several kinds of fuzzy set extensions in the fuzzy set theory, for example, intuitionistic fuzzy sets, interval-valued fuzzy sets, vague sets etc. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval $[0, 1]$ to $[-1, 1]$. Bipolar-valued fuzzy sets have membership degrees that represent the degree of satisfaction to the property corresponding to a fuzzy set and its counter-property. In a bipolar-valued fuzzy set, the membership degree 0 means that elements are irrelevant to the corresponding property, the membership degrees on $(0, 1]$ indicate that elements somewhat satisfy the property, and the membership degrees on $[-1, 0)$ indicate that elements somewhat satisfy the implicit counter-property (see [5]). Figure 2 shows a bipolar-valued fuzzy set redefined for the fuzzy set "*young*" of Figure 1. The negative membership degrees indicate the satisfaction extent of elements to an implicit counter-property (e.g., old against the property young). This kind of bipolar-valued fuzzy set representation enables the elements with membership degree 0 in traditional fuzzy sets, to be expressed into the elements with membership degree 0 (irrelevant elements) and the elements with negative membership degrees (contrary elements). The age elements 50 and

95, with membership degree 0 in the fuzzy sets of Figure 1, have 0 and a negative membership degree in the bipolar-valued fuzzy set of Figure 2, respectively. Now it is manifested that 50 is an irrelevant age to the property young and 95 is more apart from the property young than 50, i.e., 95 is a contrary age to the property young (see [5]).

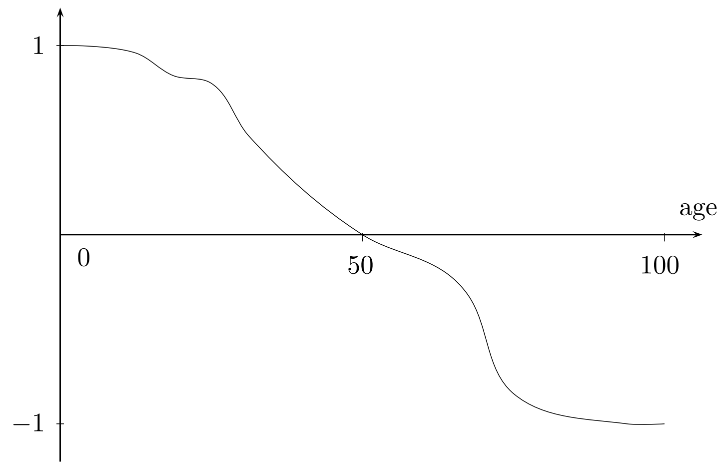

Figure 2. A bipolar fuzzy set "*young*"

In the definition of bipolar-valued fuzzy sets, there are two kinds of representations, so called canonical representation and reduced representation. In this paper, we use the canonical representation of a bipolar-valued fuzzy sets. Let X be the universe of discourse. A *bipolar-valued fuzzy set* Φ in X is an object having the form

$$\Phi = \{(x, \mu_\Phi^P(x), \mu_\Phi^N(x) \mid x \in X\}$$

where $\mu_\Phi^P : X \to [0,1]$ and $\mu_\Phi^N : X \to [-1,0]$ are mappings. The positive membership degree $\mu_\Phi^P(x)$ denoted the satisfaction degree of an element x to the property corresponding to a bipolar-valued fuzzy set $\Phi = \{(x, \mu_\Phi^P(x), \mu_\Phi^N(x) \mid x \in X\}$, and the negative membership degree $\mu_\Phi^N(x)$ denotes the satisfaction degree of x to some implicit counter-property of $\Phi = \{(x, \mu_\Phi^P(x), \mu_\Phi^N(x) \mid x \in X\}$. If $\mu_\Phi^P(x) \neq 0$ and $\mu_\Phi^N(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for $\Phi = \{(x, \mu_\Phi^P(x), \mu_\Phi^N(x) \mid x \in X\}$. If $\mu_\Phi^P(x) = 0$ and $\mu_\Phi^N(x) \neq 0$, it is the situation that x does not satisfy the property of $\Phi = \{(x, \mu_\Phi^P(x), \mu_\Phi^N(x) \mid x \in X\}$ but somewhat satisfies the counter-property of $\Phi = \{(x, \mu_\Phi^P(x), \mu_\Phi^N(x) \mid x \in X\}$. It is possible for an element x to be $\mu_\Phi^P(x) \neq 0$ and $\mu_\Phi^N(x) \neq 0$ when the membership function of the property overlaps that of its counter-property over some portion of the domain (see [6]). For the sake of simplicity, we shall use the symbol $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ for the bipolar-valued fuzzy set $\Phi = \{(x, \mu_\Phi^P(x), \mu_\Phi^N(x) \mid x \in X\}$, and use the notion of bipolar fuzzy sets instead of the notion of bipolar-valued fuzzy sets.

3. **Bipolar fuzzy subalgebras and bipolar fuzzy closed ideals.** In what follows, let X be a BCH-algebra unless otherwise specified.

Definition 3.1. A bipolar fuzzy set $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ in X is called a *bipolar fuzzy subalgebra* of X if it satisfies the following assertions:

$$\begin{aligned} &\mu_\Phi^P(x * y) \geq \min\{\mu_\Phi^P(x), \mu_\Phi^P(y)\}, \\ &\mu_\Phi^N(x * y) \leq \max\{\mu_\Phi^N(x), \mu_\Phi^N(y)\} \end{aligned} \tag{3.1}$$

for all $x, y \in X$.

For a bipolar fuzzy set $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ and $(\beta, \alpha) \in [-1, 0] \times [0, 1]$, we define

$$\Phi_\alpha^P := \{x \in X \mid \mu_\Phi^P(x) \geq \alpha\},$$
$$\Phi_\beta^N := \{x \in X \mid \mu_\Phi^N(x) \leq \beta\} \tag{3.2}$$

which are called the *positive α-cut* of $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ and the *negative β-cut* of $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$, respectively. For every $\delta \in [0, 1]$, the set

$$\Phi_\delta := \Phi_\delta^P \cap \Phi_{-\delta}^N$$

is called the *δ-cut* of $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$.

Theorem 3.2. *Let $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ be a bipolar fuzzy subalgebra of X. Then the following assertions are valid.*

(i) $(\forall \alpha \in [0, 1])$ $(\Phi_\alpha^P \neq \emptyset \Rightarrow \Phi_\alpha^P$ is a subalgebra of $X)$.
(ii) $(\forall \beta \in [-1, 0])$ $(\Phi_\beta^N \neq \emptyset \Rightarrow \Phi_\beta^N$ is a subalgebra of $X)$.

Proof. (i) Let $\alpha \in [0, 1]$ be such that $\Phi_\alpha^P \neq \emptyset$. If $x, y \in \Phi_\alpha^P$, then $\mu_\Phi^P(x) \geq \alpha$ and $\mu_\Phi^P(y) \geq \alpha$. It follows from (3.1) that

$$\mu_\Phi^P(x * y) \geq \min\{\mu_\Phi^P(x), \mu_\Phi^P(y)\} \geq \alpha$$

so that $x * y \in \Phi_\alpha^P$. Therefore Φ_α^P is a subalgebra of X. Now let $\beta \in [-1, 0]$ be such that $\Phi_\beta^N \neq \emptyset$. If $x, y \in \Phi_\beta^N$, then

$$\mu_\Phi^N(x * y) \leq \max\{\mu_\Phi^N(x), \mu_\Phi^N(y)\} \leq \beta$$

by (3.1), and so $x * y \in \Phi_\beta^N$. Hence Φ_β^N is a subalgebra of X. □

Corollary 3.3. *If $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy subalgebra of X, then the sets $\Phi^P_{\mu_\Phi^P(0)}$ and $\Phi^N_{\mu_\Phi^N(0)}$ are subalgebras of X.*

Proof. Straightforward. □

We give conditions for a bipolar fuzzy set to be a bipolar fuzzy subalgebra.

Theorem 3.4. *Let $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ be a bipolar fuzzy set in X defined by*

$$\mu_\Phi^P : X \to [0, 1], \quad x \mapsto \begin{cases} \alpha_1 & \text{if } 0 * (0 * x) = x, \\ \alpha_2 & \text{otherwise,} \end{cases}$$

$$\mu_\Phi^N : X \to [-1, 0], \quad x \mapsto \begin{cases} \beta_1 & \text{if } 0 * (0 * x) = x, \\ \beta_2 & \text{otherwise} \end{cases}$$

for all $x \in X$ where $\alpha_1 > \alpha_2$ in $[0, 1]$ and $\beta_1 < \beta_2$ in $[-1, 0]$. Then $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy subalgebra of X.

Proof. Let $x \in X$ be such that $0 * (0 * x) \neq x$. Then $\mu_\Phi^P(x) = \alpha_2$ and $\mu_\Phi^N(x) = \beta_2$. It follows that

$$\mu_\Phi^P(x * y) \geq \alpha_2 = \min\{\mu_\Phi^P(x), \mu_\Phi^P(y)\},$$
$$\mu_\Phi^N(x * y) \leq \beta_2 = \max\{\mu_\Phi^N(x), \mu_\Phi^N(y)\}.$$

Similarly, if $y \in X$ does not satisfy the equality $0 * (0 * y) = y$, then we have

$$\mu_\Phi^P(x * y) \geq \alpha_2 = \min\{\mu_\Phi^P(x), \mu_\Phi^P(y)\},$$
$$\mu_\Phi^N(x * y) \leq \beta_2 = \max\{\mu_\Phi^N(x), \mu_\Phi^N(y)\}.$$

Let $x, y \in X$ be such that $0 * (0 * x) = x$ and $0 * (0 * y) = y$. Then

$$\begin{aligned}
(x * y) * (0 * (0 * (x * y))) &= ((0 * (0 * x)) * y) * (0 * (0 * (x * y))) \\
&= ((0 * (0 * x)) * (0 * (0 * (x * y)))) * y \\
&= ((0 * (0 * (0 * (x * y)))) * (0 * x)) * y \\
&= ((0 * (x * y)) * (0 * x)) * y \\
&= ((0 * (0 * x)) * (x * y)) * y \\
&= (x * (x * y)) * y \;=\; 0.
\end{aligned}$$

Since $(0*(0*(x*y)))*(x*y) = 0$ by (2.1) and (2.3), it follows from (2.2) that $0*(0*(x*y)) = x*y$. Hence $\mu_\Phi^P(x*y) = \alpha_1 = \min\{\mu_\Phi^P(x), \mu_\Phi^P(y)\}$ and $\mu_\Phi^N(x*y) = \beta_1 = \max\{\mu_\Phi^N(x), \mu_\Phi^N(y)\}$. Therefore $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy subalgebra of X. $\square$

Theorem 3.5. *Let G be a nonempty subset of X and let $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ be a bipolar fuzzy set in X defined by*

$$\mu_\Phi^P : X \to [0,1], \quad x \mapsto \begin{cases} \alpha_1 & \text{if } x * a = (0 * a) * (0 * x),\ a \in G \\ \alpha_2 & \text{otherwise,} \end{cases}$$

$$\mu_\Phi^N : X \to [-1,0], \quad x \mapsto \begin{cases} \beta_1 & \text{if } x * a = (0 * a) * (0 * x),\ a \in G, \\ \beta_2 & \text{otherwise} \end{cases}$$

for all $x \in X$ where $\alpha_1 > \alpha_2$ in $[0,1]$ and $\beta_1 < \beta_2$ in $[-1,0]$. Then $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy subalgebra of X.

Proof. Let $x, y \in X$. If there exists $a \in G$ such that either $x * a \neq (0 * a) * (0 * x)$ or $y * a \neq (0 * a) * (0 * y)$, then either $\mu_\Phi^P(x) = \alpha_2$ or $\mu_\Phi^P(y) = \alpha_2$, and either $\mu_\Phi^N(x) = \beta_2$ or $\mu_\Phi^N(y) = \beta_2$. It follows that $\mu_\Phi^P(x * y) \geq \alpha_2 = \min\{\mu_\Phi^P(x), \mu_\Phi^P(y)\}$ and $\mu_\Phi^N(x * y) \leq \beta_2 = \max\{\mu_\Phi^N(x), \mu_\Phi^N(y)\}$. Assume that x and y satisfy the equalities: $x * a = (0 * a) * (0 * x)$ and $y * a = (0 * a) * (0 * y)$ for all $a \in G$. Then

$$\begin{aligned}
(x * y) * a &= (x * a) * y \;=\; ((0 * a) * (0 * x)) * y \;=\; ((0 * (0 * x)) * a) * y \\
&= ((0 * (0 * x)) * y) * a \;=\; ((0 * y) * (0 * x)) * a \\
&= ((0 * (0 * (0 * y))) * (0 * (0 * (0 * x)))) * a \\
&= ((0 * (0 * (0 * (0 * x)))) * (0 * (0 * y))) * a \\
&= ((0 * (0 * x)) * (0 * (0 * y))) * a \\
&= (0 * ((0 * x) * (0 * y))) * a \;=\; (0 * (0 * (x * y))) * a \\
&= (0 * a) * (0 * (x * y)).
\end{aligned}$$

Hence $\mu_\Phi^P(x * y) = \alpha_1 = \min\{\mu_\Phi^P(x), \mu_\Phi^P(y)\}$ and $\mu_\Phi^N(x * y) = \beta_1 = \max\{\mu_\Phi^N(x), \mu_\Phi^N(y)\}$. Therefore $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy subalgebra of X. $\square$

Definition 3.6. A bipolar fuzzy set $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ in X is called a *bipolar fuzzy closed ideal* of X if it satisfies the following assertions:

$$\mu_\Phi^P(0 * x) \geq \mu_\Phi^P(x) \ \ \& \ \ \mu_\Phi^N(0 * x) \leq \mu_\Phi^N(x), \tag{3.3}$$

$$\begin{aligned} &\mu_\Phi^P(y) \geq \min\{\mu_\Phi^P(y * x), \mu_\Phi^P(x)\}, \\ &\mu_\Phi^N(y) \leq \max\{\mu_\Phi^N(y * x), \mu_\Phi^N(x)\} \end{aligned} \tag{3.4}$$

for all $x, y \in X$.

Proposition 3.7. *Every bipolar fuzzy closed ideal $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ of X satisfies the following assertion:*

$$(\forall x \in X)\,(\mu_\Phi^P(0) \geq \mu_\Phi^P(x) \;\;\&\;\; \mu_\Phi^N(0) \leq \mu_\Phi^N(x)). \tag{3.5}$$

Proof. Straightforward. □

Proposition 3.8. *Every bipolar fuzzy subalgebra $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ of X satisfies the following assertion:*

$$(\forall x \in X)\,(\mu_\Phi^P(0 * x) \geq \mu_\Phi^P(x) \;\;\&\;\; \mu_\Phi^N(0 * x) \leq \mu_\Phi^N(x)). \tag{3.6}$$

Proof. Let $x \in X$. Using (3.1), we have

$$\begin{aligned}
\mu_\Phi^P(0 * x) \;&\geq\; \min\{\mu_\Phi^P(0), \mu_\Phi^P(x)\} \;=\; \min\{\mu_\Phi^P(x * x), \mu_\Phi^P(x)\} \\
&\geq \min\{\min\{\mu_\Phi^P(x), \mu_\Phi^P(x)\}, \mu_\Phi^P(x)\} \;=\; \mu_\Phi^P(x), \\
\mu_\Phi^N(0 * x) \;&\leq\; \max\{\mu_\Phi^N(0), \mu_\Phi^N(x)\} \;=\; \max\{\mu_\Phi^N(x * x), \mu_\Phi^N(x)\} \\
&\leq \max\{\max\{\mu_\Phi^N(x), \mu_\Phi^N(x)\}, \mu_\Phi^N(x)\} \;=\; \mu_\Phi^N(x).
\end{aligned}$$

This completes the proof. □

Corollary 3.9. *Every bipolar fuzzy subalgebra satisfying* (3.4) *is a bipolar fuzzy closed ideal.*

Proposition 3.10. *Let $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ be a bipolar fuzzy closed ideal of X that satisfies:*

$$(\forall x \in X)\,(\mu_\Phi^P(x) \geq \mu_\Phi^P(0 * x) \;\;\&\;\; \mu_\Phi^N(x) \leq \mu_\Phi^N(0 * x)). \tag{3.7}$$

The we have

$$(\forall x, y \in X)\,(\mu_\Phi^P(y * x) \geq \mu_\Phi^P(x * y) \;\;\&\;\; \mu_\Phi^N(y * x) \leq \mu_\Phi^N(x * y)). \tag{3.8}$$

Proof. Let $x, y \in X$. Then

$$\begin{aligned}
\mu_\Phi^P(y * x) \;&\geq\; \mu_\Phi^P(0 * (y * x)) \;\geq\; \min\{\mu_\Phi^P((0 * (y * x)) * (x * y)), \mu_\Phi^P(x * y)\} \\
&=\; \min\{\mu_\Phi^P(((0 * y) * (0 * x)) * (x * y)), \mu_\Phi^P(x * y)\} \\
&=\; \min\{\mu_\Phi^P(((0 * y) * (x * y)) * (0 * y)), \mu_\Phi^P(x * y)\} \\
&=\; \min\{\mu_\Phi^P(((0 * (x * y)) * y) * (0 * x)), \mu_\Phi^P(x * y)\} \\
&=\; \min\{\mu_\Phi^P((((0 * x) * (0 * y)) * (0 * x)) * y), \mu_\Phi^P(x * y)\} \\
&=\; \min\{\mu_\Phi^P((0 * (0 * y)) * y), \mu_\Phi^P(x * y)\} \\
&=\; \min\{\mu_\Phi^P(0), \mu_\Phi^P(x * y)\} \;=\; \mu_\Phi^P(x * y),
\end{aligned}$$

and

$$\begin{aligned}
\mu_\Phi^N(y * x) \;&\leq\; \mu_\Phi^N(0 * (y * x)) \;\leq\; \max\{\mu_\Phi^N((0 * (y * x)) * (x * y)), \mu_\Phi^N(x * y)\} \\
&=\; \max\{\mu_\Phi^N(((0 * y) * (0 * x)) * (x * y)), \mu_\Phi^N(x * y)\} \\
&=\; \max\{\mu_\Phi^N(((0 * y) * (x * y)) * (0 * y)), \mu_\Phi^N(x * y)\} \\
&=\; \max\{\mu_\Phi^N(((0 * (x * y)) * y) * (0 * x)), \mu_\Phi^N(x * y)\} \\
&=\; \max\{\mu_\Phi^N((((0 * x) * (0 * y)) * (0 * x)) * y), \mu_\Phi^N(x * y)\} \\
&=\; \max\{\mu_\Phi^N((0 * (0 * y)) * y), \mu_\Phi^N(x * y)\} \\
&=\; \max\{\mu_\Phi^N(0), \mu_\Phi^N(x * y)\} \;=\; \mu_\Phi^N(x * y)
\end{aligned}$$

This completes the proof. □

Theorem 3.11. *Every bipolar fuzzy closed ideal is a bipolar fuzzy subalgebra.*

Proof. Let $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ be a bipolar fuzzy closed ideal of X. For every $x, y \in X$, we have

$$\begin{aligned}\mu_\Phi^P(x*y) &\geq \min\{\mu_\Phi^P((x*y)*x), \mu_\Phi^P(x)\} = \min\{\mu_\Phi^P((x*x)*y), \mu_\Phi^P(x)\} \\ &= \min\{\mu_\Phi^P(0*y), \mu_\Phi^P(x)\} \geq \min\{\mu_\Phi^P(y), \mu_\Phi^P(x)\},\end{aligned}$$

and

$$\begin{aligned}\mu_\Phi^N(x*y) &\leq \max\{\mu_\Phi^N((x*y)*x), \mu_\Phi^N(x)\} = \max\{\mu_\Phi^N((x*x)*y), \mu_\Phi^N(x)\} \\ &= \max\{\mu_\Phi^N(0*y), \mu_\Phi^N(x)\} \leq \max\{\mu_\Phi^N(y), \mu_\Phi^N(x)\},\end{aligned}$$

Hence $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy subalgebra of X. □

The converse of Theorem 3.11 is not true in general as seen in the following example.

Example 3.12. Consider a BCH-algebra $X = \{0, a, b, c, d\}$ with the following Cayley table (see [1]).

$*$	0	a	b	c	d
0	0	0	0	0	d
a	a	0	0	a	d
b	b	b	0	0	d
c	c	c	c	0	d
d	d	d	d	d	0

Let $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ be a bipolar fuzzy set in X defined by

	0	a	b	c	d
μ_Φ^P	0.7	0.2	0.2	0.2	0.7
μ_Φ^N	-0.8	-0.1	-0.1	-0.1	-0.8

Then $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy subalgebra of X, but it is not a bipolar fuzzy closed ideal of X because

$$\mu_\Phi^P(c) = 0.2 < \min\{\mu_\Phi^P(c*d), \mu_\Phi^P(d)\}$$

and/or

$$\mu_\Phi^N(c) = -0.1 > \max\{\mu_\Phi^N(c*d), \mu_\Phi^N(d)\}.$$

We give conditions to make a bipolar fuzzy closed ideal from a bipolar fuzzy set.

Theorem 3.13. *Let $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ be a bipolar fuzzy set in X defined by*

$$\mu_\Phi^P : X \to [0,1], \quad x \mapsto \begin{cases} \alpha_1 & \text{if } 0*x = 0, \\ \alpha_2 & \text{otherwise,} \end{cases}$$

$$\mu_\Phi^N : X \to [-1,0], \quad x \mapsto \begin{cases} \beta_1 & \text{if } 0*x = 0, \\ \beta_2 & \text{otherwise} \end{cases}$$

for all $x \in X$ where $\alpha_1 > \alpha_2$ in $[0,1]$ and $\beta_1 < \beta_2$ in $[-1,0]$. Then $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy closed ideal of X.

Proof. Let $x \in X$ be such that $0*x \neq 0$. Then $\mu_\Phi^P(x) = \alpha_2 \leq \mu_\Phi^P(0*x)$, $\mu_\Phi^N(x) = \beta_2 \geq \mu_\Phi^N(0*x)$, $\mu_\Phi^P(y) \geq \alpha_2 = \min\{\mu_\Phi^P(y*x), \mu_\Phi^P(x)\}$, and $\mu_\Phi^N(y) \leq \beta_2 = \max\{\mu_\Phi^N(y*x), \mu_\Phi^N(x)\}$ for all $y \in X$. Assume that $x \in X$ satisfies the equality $0*x = 0$. Then $\mu_\Phi^P(0*x) = \mu_\Phi^P(0) = \alpha_1 = \mu_\Phi^P(x)$ and $\mu_\Phi^N(0*x) = \mu_\Phi^N(0) = \beta_1 = \mu_\Phi^N(x)$. Let $y \in X$ satisfy the equality $0*y = 0$. Then $\mu_\Phi^P(y) = \alpha_1 \geq \min\{\mu_\Phi^P(y*x), \mu_\Phi^P(x)\}$ and $\mu_\Phi^N(y) = \beta_1 \leq \max\{\mu_\Phi^N(y*x), \mu_\Phi^N(x)\}$. If

$y \in X$ does not satisfy $0 * y = 0$, then $y * x$ also does not satisfy the equality $0 * (y * x) = 0$. In fact, if $0 * (y * x) = 0$ then

$$\begin{aligned} 0 * y &= ((y * x) * (y * x)) * y \ = \ ((y * x) * y) * (y * x) \\ &= ((y * y) * x) * (y * x) \ = \ (0 * x) * (y * x) \\ &= 0 * (y * x) \ = \ 0, \end{aligned}$$

which is a contradiction. Hence we have $\mu_\Phi^P(y) = \alpha_2 = \min\{\mu_\Phi^P(y * x), \mu_\Phi^P(x)\}$ and $\mu_\Phi^N(y) = \beta_2 = \max\{\mu_\Phi^N(y * x), \mu_\Phi^N(x)\}$. Therefore $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy closed ideal of X. □

We provide a condition for a bipolar fuzzy subalgebra to be a bipolar fuzzy closed ideal.

Theorem 3.14. *Let $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ be a bipolar fuzzy subalgebra of X that satisfies the following assertions:*

$$(\forall x, y \in X)\,(\mu_\Phi^P(x * y) \geq \mu_\Phi^P(y * x) \ \ \& \ \ \mu_\Phi^N(x * y) \leq \mu_\Phi^N(y * x)). \tag{3.9}$$

Then $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy closed ideal of X.

Proof. By means of Proposition 3.8, $\mu_\Phi^P(0 * x) \geq \mu_\Phi^P(x)$ and $\mu_\Phi^N(0 * x) \leq \mu_\Phi^N(x)$ for all $x \in X$. Using (2.4), (2.1), (2.3), (3.1) and (3.9), we have

$$\begin{aligned} \mu_\Phi^P(y) &= \mu_\Phi^P(y * 0) \ \geq \ \mu_\Phi^P(0 * y) \ = \ \mu_\Phi^P((x * x) * y) \ = \ \mu_\Phi^P((x * y) * x) \\ &\geq \min\{\mu_\Phi^P(x * y), \mu_\Phi^P(x)\} \ \geq \ \min\{\mu_\Phi^P(y * x), \mu_\Phi^P(x)\}, \end{aligned}$$

$$\begin{aligned} \mu_\Phi^N(y) &= \mu_\Phi^N(y * 0) \ \leq \ \mu_\Phi^N(0 * y) \ = \ \mu_\Phi^N((x * x) * y) \ = \ \mu_\Phi^N((x * y) * x) \\ &\leq \max\{\mu_\Phi^N(x * y), \mu_\Phi^N(x)\} \ \leq \ \max\{\mu_\Phi^N(y * x), \mu_\Phi^N(x)\} \end{aligned}$$

for all $x, y \in X$. Hence $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy closed ideal of X. □

Theorem 3.15. *Let $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ be a bipolar fuzzy set in X. Then $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy closed ideal of X if and only if it satisfies the following assertions:*

$$\begin{aligned} &(\forall \alpha \in [0,1])(\Phi_\alpha^P \neq \emptyset \ \Rightarrow \ \Phi_\alpha^P \text{ is a closed ideal of } X), \\ &(\forall \beta \in [-1,0])(\Phi_\beta^N \neq \emptyset \ \Rightarrow \ \Phi_\beta^N \text{ is a closed ideal of } X). \end{aligned} \tag{3.10}$$

Proof. Assume that $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy closed ideal of X. Let $(\beta, \alpha) \in [-1,0] \times [0,1]$ be such that $\Phi_\alpha^P \neq \emptyset$ and $\Phi_\beta^N \neq \emptyset$. Obviously, $0 \in \Phi_\alpha^P \cap \Phi_\beta^N$. If $x \in \Phi_\alpha^P$ and $y \in \Phi_\beta^N$, then $\mu_\Phi^P(0 * x) \geq \mu_\Phi^P(x) \geq \alpha$ and $\mu_\Phi^N(0 * y) \leq \mu_\Phi^N(y) \leq \beta$ by (3.3). Hence $0 * x \in \Phi_\alpha^P$ and $0 * y \in \Phi_\beta^N$. Let $x, y \in X$ be such that $x * y \in \Phi_\alpha^P$ and $y \in \Phi_\alpha^P$, and let $a, b \in X$ be such that $a * b \in \Phi_\beta^N$ and $b \in \Phi_\beta^N$. Then $\mu_\Phi^P(x * y) \geq \alpha$, $\mu_\Phi^P(y) \geq \alpha$, $\mu_\Phi^N(a * b) \leq \beta$, and $\mu_\Phi^N(b) \leq \beta$. It follows from (3.4) that

$$\mu_\Phi^P(x) \geq \min\{\mu_\Phi^P(x * y), \mu_\Phi^P(y)\} \geq \alpha,$$
$$\mu_\Phi^N(a) \leq \max\{\mu_\Phi^N(a * b), \mu_\Phi^N(b)\} \leq \beta$$

so that $x \in \Phi_\alpha^P$ and $a \in \Phi_\beta^N$. Therefore Φ_α^P and Φ_β^N are closed ideals of X. Conversely, suppose that the condition (3.10) is valid. For any $x \in X$, let $\mu_\Phi^P(x) = \alpha$ and $\mu_\Phi^N(x) = \beta$. Then $x \in \Phi_\alpha^P \cap \Phi_\beta^N$, and so Φ_α^P and Φ_β^N are non-empty. Since Φ_α^P and Φ_β^N are closed ideals of X, $0 * x \in \Phi_\alpha^P \cap \Phi_\beta^N$. Hence $\mu_\Phi^P(0 * x) \geq \alpha = \mu_\Phi^P(x)$ and $\mu_\Phi^N(0 * x) \leq \beta = \mu_\Phi^N(x)$ for all $x \in X$. If there exist $x', y', a', b' \in X$ such that

$$\mu_\Phi^P(x') < \min\{\mu_\Phi^P(x' * y'), \mu_\Phi^P(y')\},$$
$$\mu_\Phi^N(a') > \max\{\mu_\Phi^N(a' * b'), \mu_\Phi^N(b')\},$$

then by taking

$$\alpha_0 := \tfrac{1}{2}(\mu_\Phi^P(x') + \min\{\mu_\Phi^P(x' * y'), \mu_\Phi^P(y')\}),$$
$$\beta_0 := \tfrac{1}{2}(\mu_\Phi^N(a') + \max\{\mu_\Phi^N(a' * b'), \mu_\Phi^N(b')\}),$$

we have

$$\mu_\Phi^P(x') < \alpha_0 < \min\{\mu_\Phi^P(x' * y'), \mu_\Phi^P(y')\},$$
$$\mu_\Phi^N(a') > \beta_0 > \max\{\mu_\Phi^N(a' * b'), \mu_\Phi^N(b')\}.$$

Hence $x' \notin \Phi_{\alpha_0}^P$, $x' * y' \in \Phi_{\alpha_0}^P$, $y' \in \Phi_{\alpha_0}^P$, $a' \notin \Phi_{\beta_0}^N$, $a' * b' \in \Phi_{\beta_0}^N$, and $b' \in \Phi_{\beta_0}^N$. This is a contradiction, and thus $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy closed ideal of X. $\square$

Corollary 3.16. *If $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy closed ideal of X, then the intersection of a nonempty positive α-cut and a nonempty negative β-cut of $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a closed ideal of X for all $(\beta, \alpha) \in [-1, 0] \times [0, 1]$. In particular, the nonempty δ-cut of $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a closed ideal of X for all $\delta \in [0, 1]$.*

The following example shows that there exists $(\beta, \alpha) \in [-1, 0] \times [0, 1]$ such that if $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy closed ideal of X, then the union of a nonempty positive α-cut and a nonempty negative β-cut of $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is not a closed ideal of X in general.

Example 3.17. Consider a BCH-algebra $X = \{0, 1, 2, a, b\}$ with the following Cayley table:

$*$	0	1	2	a	b
0	0	0	0	a	a
1	1	0	1	b	a
2	2	2	0	a	a
a	a	a	a	0	0
b	b	a	b	1	0

Let $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ be a bipolar fuzzy set in X defined by

	0	1	2	a	b
μ_Φ^P	0.9	0.8	0.7	0.4	0.4
μ_Φ^N	-0.8	-0.3	-0.6	-0.5	-0.3

Then

$$\Phi_\alpha^P = \begin{cases} \emptyset & \text{if } 0.9 < \alpha \le 1, \\ \{0\} & \text{if } 0.8 < \alpha \le 0.9, \\ \{0,1\} & \text{if } 0.7 < \alpha \le 0.8, \\ \{0,1,2\} & \text{if } 0.4 < \alpha \le 0.7, \\ X & \text{if } 0 \le \alpha \le 0.4, \end{cases}$$

$$\Phi_\beta^N = \begin{cases} \emptyset & \text{if } -1 \le \beta < -0.8, \\ \{0\} & \text{if } -0.8 \le \beta < -0.6, \\ \{0,2\} & \text{if } -0.6 \le \beta < -0.5, \\ \{0,2,a\} & \text{if } -0.5 \le \beta < -0.3, \\ X & \text{if } -0.3 \le \beta \le 0. \end{cases}$$

It follows from Theorem 3.15 that $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy closed ideal of X. But $\Phi_{0.7}^P \cup \Phi_{-0.5}^N = \{0,1,2\} \cup \{0,2,a\} = \{0,1,2,a\}$ is not a closed ideal of X since $b * a = 1 \in \{0,1,2,a\}$, but $b \notin \{0,1,2,a\}$.

The following example shows that there exists $\delta \in [0,1]$ such that if $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy closed ideal of X, then the union of a nonempty positive δ-cut and a nonempty negative $(-\delta)$-cut of $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is not a closed ideal of X in general.

Example 3.18. Consider a BCH-algebra $X = \{0, a, b, c, d\}$ with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	b	c	d
a	a	0	b	c	d
b	b	b	0	d	c
c	c	c	d	0	b
d	d	d	c	b	0

Let $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ be a bipolar fuzzy set in X defined by

	0	a	b	c	d
μ_Φ^P	0.7	0.5	0.4	0.2	0.2
μ_Φ^N	-0.8	-0.8	-0.3	-0.5	-0.3

Then

$$\Phi_\alpha^P = \begin{cases} \emptyset & \text{if } 0.7 < \alpha \le 1, \\ \{0\} & \text{if } 0.5 < \alpha \le 0.7, \\ \{0, a\} & \text{if } 0.4 < \alpha \le 0.5, \\ \{0, a, b\} & \text{if } 0.2 < \alpha \le 0.4, \\ X & \text{if } 0 \le \alpha \le 0.2, \end{cases}$$

$$\Phi_\beta^N = \begin{cases} \emptyset & \text{if } -1 \le \beta < -0.8, \\ \{0, a\} & \text{if } -0.8 \le \beta < -0.5, \\ \{0, a, c\} & \text{if } -0.5 \le \beta < -0.3, \\ X & \text{if } -0.3 \le \beta \le 0. \end{cases}$$

It follows from Theorem 3.15 that $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy closed ideal of X. But $\Phi_{0.35}^P \cup \Phi_{-0.35}^N = \{0, a, b\} \cup \{0, a, c\} = \{0, a, b, c\}$ is not a closed ideal of X since $d * b = c \in \{0, a, b, c\}$, but $d \notin \{0, a, b, c\}$.

We provide a condition for the union of a nonempty positive δ-cut and a nonempty negative $(-\delta)$-cut of $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ to be a closed ideal of X.

Theorem 3.19. *If $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy closed ideal of X such that*

$$(\forall x \in X)\, (\mu_\Phi^P(x) + \mu_\Phi^N(x) \ge 0), \tag{3.11}$$

then the union of a nonempty positive δ-cut and a nonempty negative $(-\delta)$-cut of $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a closed ideal of X for all $\delta \in [0,1]$.

Proof. Let $\delta \in [0,1]$. Since $\Phi_\delta^P \neq \emptyset$ and $\Phi_{-\delta}^N \neq \emptyset$, they are closed ideals of X by Theorem 3.15. Hence $0 * x \in \Phi_\delta^P \cup \Phi_{-\delta}^N$ for all $x \in \Phi_\delta^P \cup \Phi_{-\delta}^N$. Let $x, y \in X$ be such that $x * y \in \Phi_\delta^P \cup \Phi_{-\delta}^N$ and $y \in \Phi_\delta^P \cup \Phi_{-\delta}^N$. We can consider the following four cases:

(i) $x * y \in \Phi_\delta^P$ and $y \in \Phi_\delta^P$.
(ii) $x * y \in \Phi_\delta^P$ and $y \in \Phi_{-\delta}^N$.
(iii) $x * y \in \Phi_{-\delta}^N$ and $y \in \Phi_\delta^P$.
(iv) $x * y \in \Phi_{-\delta}^N$ and $y \in \Phi_{-\delta}^N$.

Case (i) implies that $x \in \Phi_\delta^P \subseteq \Phi_\delta^P \cup \Phi_{-\delta}^N$. Case (iv) implies that $x \in \Phi_{-\delta}^N \subseteq \Phi_\delta^P \cup \Phi_{-\delta}^N$. For the case (ii), we have $\mu_\Phi^P(x * y) \geq \delta$ and $\mu_\Phi^N(y) \leq -\delta$. It follows from (3.4) and (3.11) that

$$\mu_\Phi^P(x) \geq \min\{\mu_\Phi^P(x * y), \mu_\Phi^P(y)\} \geq \min\{\mu_\Phi^P(x * y), -\mu_\Phi^N(y)\} \geq \delta$$

so that $x \in \Phi_\delta^P \subseteq \Phi_\delta^P \cup \Phi_{-\delta}^N$. Case (iii) implies that $\mu_\Phi^N(x * y) \leq -\delta$ and $\mu_\Phi^P(y) \geq \delta$. It follows from (3.4) and (3.11) that

$$\mu_\Phi^P(x) \geq \min\{\mu_\Phi^P(x * y), \mu_\Phi^P(y)\} \geq \min\{-\mu_\Phi^N(x * y), \mu_\Phi^P(y)\} \geq \delta$$

so that $x \in \Phi_\delta^P \subseteq \Phi_\delta^P \cup \Phi_{-\delta}^N$. Hence $\Phi_\delta^P \cup \Phi_{-\delta}^N$ is a closed ideal of X. $\square$

Theorem 3.20. *Let G be a nonempty subset of X and let $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ be a bipolar fuzzy set in X defined by*

$$\mu_\Phi^P : X \to [0,1], \quad x \mapsto \begin{cases} \alpha_1 & \text{if } x \in G \\ \alpha_2 & \text{otherwise,} \end{cases}$$

$$\mu_\Phi^N : X \to [-1,0], \quad x \mapsto \begin{cases} \beta_1 & \text{if } x \in G, \\ \beta_2 & \text{otherwise} \end{cases}$$

for all $x \in X$ where $\alpha_1 > \alpha_2$ in $[0,1]$ and $\beta_1 < \beta_2$ in $[-1,0]$. Then $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy closed ideal of X if and only if G is a closed ideal of X. In this case, $\Phi_{\alpha_1}^P = G = \Phi_{\beta_1}^N$.

Proof. Assume that G is a closed ideal of X. Obviously, $\Phi_{\alpha_1}^P = G = \Phi_{\beta_1}^N$ and $\Phi_{\alpha_2}^P = X = \Phi_{\beta_2}^N$, which are closed ideals of X. Using Theorem 3.15, we know that $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy closed ideal of X. Conversely, suppose that $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy closed ideal of X. It is clear that G is a closed ideal of X. $\square$

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