HOEHNKE CATEGORIES

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Received June 24, 2008

In memoriam

Dr. rer. nat. habil. Hans-Jürgen Hoehnke

27.10.1925 - 09.06.2007

Category-theoretical investigations play an important role among the widespread areas of algebraic research and fields of interest of Hans-Jürgen Hoehnke. His aim in this direction was to find an element-free language for an axiomatic characterization of many-sorted partial algebras. Many-sorted algebras (heterogeneous algebras) are, as it is well-known, algebraic systems consisting of a family of carrier sets and a family of functions (operations) such that their definition domains are cartesian products of certain carrier sets and their values are elements of a distinguished carrier set. Therefore, the idea seemed very likely, the carrier sets of a heterogeneous algebra to consider as objects of a certain category and the operations as morphisms in it.
The concept of such algebraic systems was independently introduced and investigated by G.Birkhoff / J. D.Lipson and P. J.Higgins, ([3], [9]).
Since 1967 P. Burmeister considered partial algebras in several papers.
The interest in partial algebras grew in the seventieth of the last century because of the necessity of a theoretical foundation of computer science.

Against the background of this development, Hoehnke published in 1972 a paper on the superposition of partial functions in contrast to the superposition of total functions ([10]) as a first step of his own research in this direction.
During the same time Hoehnke wrote the manuscript of Part A of a common monograph with Lothar Budach, published in 1975 ([4]). This part deals with several types of so-called Kronecker-categories (categories with a monoidal structure), enriched by different additional structures, structure-preserving functors, free theories and circuits, Kronecker-double-categories, and loop-operations.
Hoehnke's next step was the categorical desription of partial algebras in the important paper " On partial algebras ", written by Hoehnke already in 1976, but published only in 1981 ([11], [12]). The study of this paper in the research seminar " Universal Algebra ", Institute of Mathematics, Potsdam Teacher's Training's College, under the leadership of Lugowski and, parallel to it, in the research group of Michler and Schreckenberger at the Teacher's Training's College in Köthen, was the beginning of the author's cooperation with the remarkable mathematician Hans-Jürgen Hoehnke until 2007.
The paper " On partial algebras " gave important stimulations for Schreckenberger's papers [19], [20], [21], the author's paper [22], the author's paper of university lecturing qualification [23], and a number of papers in 1980, 1982, 1984, 1989, 1990, 1997, 2001, and 2005.

Moreover, Hoehnke's publications and contributions in several conferences stimulated other mathematicians, for instance Börner, Denecke, Lugowski, Pöschel, Schepull, SEIBT, STRECKER, WELKE, and others.

The categorical investigation of partial algebras needs an axiomatic characterization of the fundamental properties of the category *Par* of all partial functions between arbitrary sets. The base for an axiomatic characterization of the categorical properties of the category *Par* in view of partial algebras was already given by publications of EILENBERG, KELLY, and MAC LANE on so-called " symmetric monoidal categories ".

Using the structure of a symmetric monoidal category one is able to describe the " composition " of functions in the usual sense by the morphism composition and the " parallel composition " by the monoidal product in such a category.

The crucial point in HOEHNKE's approach was the introduction of an additional structure into a symmetric monoidal category in the sense of EILENBERG-KELLY, namely
- a uniquely determined zero object O (in *Par* this is the empty set $\emptyset$),
- a morphism $o \in K[I,O]$ (in *Par* this is the empty function from the one element set $I = \{\emptyset\}$ into the empty set $O = \emptyset$),
- a family $(d_A \in K[A, A \otimes A] \mid A \in \text{ob } K)$ of " diagonal morphisms ", in the category *Par* given as the (total) functions $d_A : A \to A \otimes A$ according to $a \mapsto (a,a)$ for all $a \in A$,
and
- a family $(t_A \in K[A,I] \mid A \in \text{ob } K)$ of " terminal morphisms ", in *Par* given as the (total) functions $t_A : A \to I = \{\emptyset\}$ according to $a \mapsto \emptyset$ for all $a \in A$.

All properties of this additional structure elements are describable by axioms. In the first representation of this structure by HOEHNKE (cf. [11]), not all characterizing axioms had the form of identities, but SCHRECKENBERGER proved in 1980 ([20]), that there is an equivalent characterizing system of axioms consisting of identities only.

The fact, that the basic conditions as above induce the existence of zero-morphisms $o_{AB} \in K[A,B]$ for all objects A and B in such a category is especially important. The existence of zero-morphisms guarantees the " partiality " of certain morphisms of the category.
Since only the family of diagonal morphisms forms a " natural transformation ", i.e.

$$\forall A, B \in |K| \; \forall \varphi \in K[A,B] \; (\varphi d_B = d_A(\varphi \otimes \varphi)),$$

(the composition is written in accordance with the direction of the arrows, i.e.

$$\varphi\psi \in K[A,C] \text{ for } \varphi \in K[A,B], \;\; \psi \in K[B,C])$$

and the family of terminal morphisms does not have this property, one has only

$$\forall A, B \in |K| \; \forall \varphi \in K[A,B] \; (\varphi t_B \leq t_A 1_I = t_A),$$

where $\leq$ is the canonical partial order relation in such a category (cf. [20]), that means, especially in *Par* the natural inclusion of functions $\varphi t_B \subseteq t_A 1_I = t_A$. A total function $\varphi : A \to B$ in *Par* is characterized by $\varphi t_B = t_A 1_I = t_A$.

HOEHNKE used the notation " diagonal-halfterminal-symmetric monoidal category ", shortly $dht-$symmetric category, for such a structure and this concept includes a lot of similar approaches to partiality, for instance the considerations of DI PAOLA-HELLER [7], CALENKO-GISIN-RAIKOV [5] ASPERTI-LONGO [1], ROSOLINI [18], CURIEN-OBTULOWICZ [6], and others more.

Because of the importance of this concept, there is a proposal to use the notion Hoehnke-category instead of $dht-$symmetric category.

To each set A the total function d_A belongs to *Par*. Every such function has in *Par* the uniquely determined inverse function ∇_A, defined by $\nabla_A : A \otimes A \to A$ according to

$(a, a) \mapsto a$ and $\nabla_A(a, b)$ is not defined in case $a \neq b$. Hence, ∇_A is in general a proper partial function. For each set A one has $d_A\nabla_A = 1_A$, whereas $\nabla_A d_A \subseteq 1_{A\otimes A}$ only.

The fact, that there is at most one morphism family

$$\nabla = (\nabla_A \in K[A \otimes A, A] \mid A \in |K|)$$

in each Hoehnke-category $\underline{K}$ fulfilling the both conditions is of importance.

(D_1^*) $\quad \forall A \in |K| \; (d_A\nabla_A = 1_A)$,

(D_2^*) $\quad \forall A \in |K| \; (\nabla_A d_A d_{A\otimes A} = d_{A\otimes A}(\nabla_A d_A \otimes 1_{A\otimes A}))$.

Remark that the last condition means exactly $\nabla_A d_A \leq 1_{A\otimes A}$ in the considered Hoehnke-category $\underline{K}$.

A structure $(K^\bullet; d, t, \nabla, o)$ is called Hoehnke category with diagonal-inversions (for short $dhth\nabla s-$category, in [25,] named $dht\nabla-$symmetric category), if $(K^\bullet; d, t, o)$ is a $dhts-$category endowed with a morphism family

$$\nabla = (\nabla_A \in K[A \otimes A, A] \mid A \in |K|) \text{ fulfilling}$$

(D_1^*) $\quad \forall A \in |K| \; (d_A\nabla_A = 1_A)$,

(D_2^*) $\quad \forall A \in |K| \; (\nabla_A d_A d_{A\otimes A} = d_{A\otimes A}(\nabla_A d_A \otimes 1_{A\otimes A}))$.

The family of diagonal-inversions does not form a " natural transformation " in any Hoehnke category with diagonal-inversions, since in general only the condition

$$\forall A, B \in |K| \; \forall \varphi \in K[A, B] \; (\nabla_A \varphi \leq (\varphi \otimes \varphi)\nabla_B)$$

follows from the axioms.

The category $\underline{Par}$ is a model of this concept and only injective functions $\varphi : A \to B$ fulfil the condition $\nabla_A\varphi = (\varphi \otimes \varphi)\nabla_B$, whereas in general one has $\nabla_A\varphi \subseteq (\varphi \otimes \varphi)\nabla_B$.

The fact, that the morphism $d_A(\varphi \otimes \psi)\nabla_B$ is the infimum of two morphisms $\varphi, \psi \in K[A, B]$ with respect to the canonical order relation in any Hoehnke category with diagonal-inversions is important and this fact permits the characterization of E-equations, ECE-equations, and QE-equations (cf. [Bu.86]) in the same manner.

The categorical concept of a theory for algebras of a given type was defined by Lawvere in 1963 ([16]). Hoehnke extended this concept to partial heterogenous algebras in 1976 ([12]). A partial theory is a Hoehnke-category such that its object class is a set which forms a free algebra of type (2,0,0) with respect to the $\otimes-$operation and the distinguished objects I and O freely generated by a nonempty set J in the variety determined by the identities $Ox \approx O$ and $xO \approx O$, where O and I are the elements selected by the $0-$ary operation symbols.

If the object class of a Hoehnke-category forms even a monoid with unit element I and zero element O, then one has a strict partial theory.

Of particular interest are partial theories with diagonal-inversions since in such theories QE-equations are expressable.

Hoehnke's work in this direction culminates in the book " Partial Algebras and their Theories ", published in 2007 by SHAKER in Aachen. Hoehnke Together with his former student Schreckenberger Hoehnke presents a survey ron Hoehnke-categories, their applications to theories for partial algebras, of Malcev-clones, and the connections to Dale-monoids.

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