Scientiae Mathematicae Japonicae Online, **e-2008**, 331–340

COMMUTATION OF GEOMETRIC REALIZATION FUNCTOR AND FINITE LIMITS

S. N. HOSSEINI

Received November 28, 2007; revised March 17, 2008

ABSTRACT. The classic geometric realization functor $| - | : S^{\triangle^{op}} \longrightarrow KTop$, where S is the category of sets, $S^{\triangle^{op}}$ is the category of simplicial sets, and $KTop$ is the category of compactly generated Hausdorff topological spaces, is generalized to the functor $| - |_Y : S^{\triangle^{op}} \longrightarrow A$, where A is a category geometric over S via f and Y forms a discrete fibration over $f^*\triangle^{op}$, in $Cat(A)$, via g. It is shown that, under certain assumptions on A, f and g, this generalized functor commutes with finite limits if the collection of the inclusions of the boundary $\dot{Y}_{0n}$ of Y_{0n} into Y_{0n} is strongly initial. It is further shown, for certain geometric categories A over sets, in particular for the categories Fco, $ConsFco$, Con, Lim, PsT, $Born$, and $PreOrd$, that initiality of the inclusion of the boundary $\dot{Y}_{0n}$ of Y_{0n} into Y_{0n} guarantees commutation of the geometric realization functor and finite limits.

1. Preliminaries

Let A be a category with finite limits and coequalizers of reflexive pairs and $f : A \longrightarrow S$ be a geometric morphism. The direct and inverse images of the geometric morphism $f : A \longrightarrow S$ are denoted by $f_* : A \longrightarrow S$ and $f^* : S \longrightarrow A$, respectively, see [4] p 26. Let $g : Y \longrightarrow f^*\triangle^{op}$ be a discrete fibration in $Cat(A)$, see [4] p 50, where Y is an internal category in A, and $\triangle$ is the category of finite ordinals regarded as an internal category in S. Let $S^{\triangle^{op}}$ denote the category of simplicial sets which we regard as discrete opfibrations over $\triangle^{op}$, see [4] p50. Similarly $A^{f^*\triangle^{op}}$ denotes the category of discrete opfibrations over $f^*\triangle^{op}$, etc.

The functor $f^* : S \longrightarrow A$ induces a functor, which is still denoted by f^*, from the category $S^{\triangle^{op}}$ to the category $A^{f^*\triangle^{op}}$. The discrete fibration $g : Y \longrightarrow f * \triangle^{op}$ yields the pullback functor along g, which we denote by g^*, from the category $A^{f^*\triangle^{op}}$ to the category A^Y. Let $Colim_Y : A^Y \longrightarrow A$ be the $\underrightarrow{Lim}_Y$ defined in [4], p 51, and define:

1.1. **Definition:** The geometric realization functor, denoted by $| - |_Y$ is defined to be the composition:

$$S^{\triangle^{op}} \xrightarrow{f^*} A^{f^*\triangle^{op}} \xrightarrow{g^*} A^Y \xrightarrow{Colim_Y} A$$

In this paper we assume $| - |_Y$ preserves colimits. Conditions that guarantee $| - |_Y$ has a right adjoint, see [5], and therefore preserves colimits are given in [6] p 5, Theorem 2.4.

1.2. **Definition:**

2000 *Mathematics Subject Classification.* 18A20, 18A35, 18B99.
Key words and phrases. geometric realization functor, finite limit, filtered internal category, simplicial set, discrete fibration, topological space, initiality.
Supported partially by Mahani Research Center, Shahid Bahonar University of Kerman.

(i) A discrete fibration $\gamma : G \longrightarrow f^*C$ is said to be f-flat if f_*G is filtered and the pullback of $d_G : K_G \longrightarrow G_0 \times G_0$ along any map of the form $(\gamma_0\times\gamma_0)^* f^*(k)$, where $k \in S/(C_0\times C_0)$ is an extremal epi.
(ii) By a simplex structure in A is meant an f-flat discrete fibration over $f^*\triangle^{op}$.

1.3. **Lemma:** If for each $a \in A$, $a \times - : A \longrightarrow A$ preserves extremal epis, f_* preserves reflexive coequalizers, and reflects monos and terminals, and $g : Y \longrightarrow f^*\triangle^{op}$ is a simplex structure, then $| - |_Y$ preserves finite products and terminals.

Proof: Since f_* preserves pullbacks and reflexive coequalizers, it follows that f_* of the i-map of α is the i-map of f_* of α, for any morphism α in A, see [3] p 1. On the other hand in the category S, the i-maps are monos, see [4] p 40, and f_* reflects monos by hypothesis. Thus in A the i-maps are monos. So A is an admissible category, see [3] p 3, and therefore in A a map is an e.e. if and only if it is a coequalizer, see [3] Lemma 2.1. The proof now follows from Theorem 2.4 of [6], p 5.

□

2. The standard n-simplex

Let $n : 1 \rightarrowtail N$ be a natural number. Form the following pullbacks to get $\triangle_1(-,n)$ and $\triangle_2(-,n)$:

$$
\begin{array}{ccc}
\triangle_1(-,n) & \longrightarrow & 1 \\
{\scriptstyle i_{1n}}\downarrow & \text{pb} & \downarrow{\scriptstyle n} \\
\triangle_1 & \xrightarrow{d_1} & N
\end{array}
\qquad
\begin{array}{ccc}
\triangle_2(-,n) & \xrightarrow{\pi_{2n}} & \triangle_1(-,n) \\
{\scriptstyle i_{2n}}\downarrow & \text{pb} & \downarrow{\scriptstyle i_{1n}} \\
\triangle_2 & \xrightarrow{\pi_2} & \triangle_1 \\
{\scriptstyle \pi_1}\downarrow & \text{pb} & \downarrow{\scriptstyle d_0} \\
\triangle_1 & \xrightarrow{d_1} & N
\end{array}
$$

Diagram I

Define the internal category $\triangle[n]$ in S as $\triangle[n]_0 = \triangle_1(-,n), \triangle[n]_1 = \triangle_2(-,n)$ and let $\triangle[n]_1 \overset{d_0}{\underset{d_1}{\rightrightarrows}} \triangle[n]_0$ be the morphisms $\triangle_2(-,n) \overset{\pi_{2n}}{\underset{m_n}{\rightrightarrows}} \triangle_1(-,n)$ respectively, where $m_n = (mi_{2n}, !)$ is induced by the multiplication $m : \triangle_2 \longrightarrow \triangle_1$, and the unique morphism $! : \triangle[n]_1 \longrightarrow 1$.

A straightforward computation shows that the diagrams:

$$
\begin{array}{ccc}
\triangle[n]_1 & \overset{d_0}{\underset{d_1}{\rightrightarrows}} & \triangle[n]_0 \\
{\scriptstyle \pi_1 i_{2n}}\downarrow & & \downarrow{\scriptstyle d_0 i_{1n}} \\
\triangle_1 & \overset{d_1}{\underset{d_0}{\rightrightarrows}} & N
\end{array}
$$

commute, and the diagram with the upper maps is in fact a pullback diagram. This shows that $\triangle[n] : \longrightarrow \triangle^{op}$ is a discrete opfibration.

2.1. **Definition:** The discrete opfibration : $\triangle[n] \longrightarrow \triangle^{op}$, in $S^{\triangle^{op}}$, is called standard n-simplex. Note that this is just the standard n-simplex defined in [1], p25, regarded as a discrete opfibration over $\triangle^{op}$.

2.2. **Lemma:** $|\triangle[n]|_Y = Y_{0n}$, where Y_{0n} is the pullback of $f^*(n) : 1 \longrightarrow f^*N$ along $g_0 : Y_0 \longrightarrow f^*N$.

Proof: Since $|-|_Y = Colim_Y \circ g^* \circ f^*$, apply f^* to $\triangle[n]$ and pullback along g to get the pair $Y_{2n} \overset{\pi_{1n}}{\underset{m_n}{\rightrightarrows}} Y_{1n}$ as the following diagram shows:

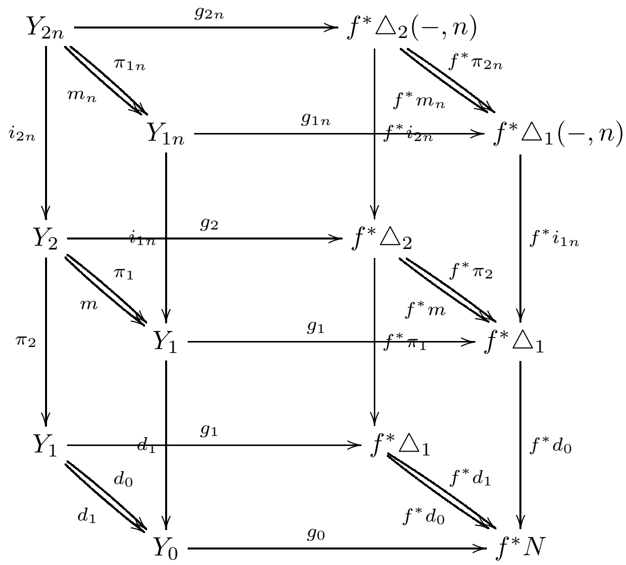

Diagram II

So $|\triangle[n]|_Y$=Coeq($Y_{2n} \overset{\pi_{1n}}{\underset{m_n}{\rightrightarrows}} Y_{1n}$).

On the other hand $Y_2 \overset{\pi_1}{\underset{m}{\rightrightarrows}} Y_1 \xrightarrow{d_0} Y_0$ is a coequalizer, since $d_0\pi_1 = d_0m$, and if a morphism h is given such that $h\pi_1 = hm$, then $h = hm(i \times 1) = h\pi_1(i \times 1) = hid_0$. Thus h factors through d_0 uniquely.

The map $d_0 : Y_1 \longrightarrow Y_0$ induces a map $d_{0n} : Y_{1n} \longrightarrow Y_{0n}$ such that the diagram:

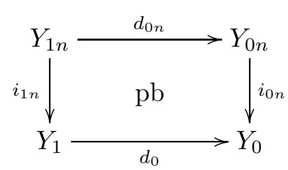

Diagram III

is a pullback diagram.

Let $i : Y_0 \longrightarrow Y_1$ be the inclusion of identities. Since $d_0 i = 1$, and Diagram III is a pullback diagram, it follows that there is a unique map $i_n : Y_{0n} \longrightarrow Y_{1n}$ such that (1) $d_{0n}i_n = 1$, and $i_{1n}i_n = ii_{0n}$. The squares $f^*i_{1n} \circ f^*m_n = f^*m \circ f^*i_{2n}$, $f^*i_{2n} \circ g_{2n} = g_2 \circ i_{2n}$, and $f^*i_{1n} \circ g_{1n} = g_1 \circ i_{1n}$ of Diagram II are pullbacks, therefore so is the square:

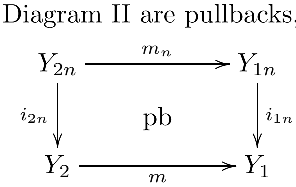

Diagram IV

Since $m(i \times 1) = 1$, Diagram IV yields a unique map $\gamma_n : Y_{1n} \longrightarrow Y_{2n}$ such that (2) $i_{2n}\gamma_n = (i \times 1)i_{1n}$, and $m_n\gamma_n = 1$. Equations (1) and (2), and Diagrams II and III, imply (3) $\pi_{1n}\gamma_n = i_n d_{0n}$. From Diagrams II and III, it follows that $d_{0n}\pi_{1n} = d_{0n}m_n$. Equations (2) and (3) imply that any map h that coequalizes π_{1n} and m_n factors uniquely through d_{0n}. Hence $Y_{2n} \underset{m_n}{\overset{\pi_{1n}}{\rightrightarrows}} Y_{1n} \xrightarrow{d_{0n}} Y_{0n}$ is a coequalizer, which proves $|\triangle[n]|_Y = Y_{0n}$.

□

2.3. **Definition:**
(i) The boundary of the standard n-simplex, $\triangle[n]$, is defined to be $\dot{\triangle}[n] = Sk^{n-1}\triangle[n]$, see [1] p 29.
(ii) The boundary of Y_{0n} is defined to be $\dot{Y}_{0n} = |\dot{\triangle}[n]|_Y$.

2.4. **Remark:** For any simplicial set X, and for each n in N, there is an inclusion $i_n : Sk^nX \rightarrowtail X$ of Sk^nX into X, see [1] p 30. It follows by Definition 2.3 that there is an inclusion $i_n : \dot{\triangle}[n] \rightarrowtail \triangle[n]$ of the boundary of $\triangle[n]$ into $\triangle[n]$.

2.5. **Lemma:** If $f_* : A \longrightarrow S$ preserves reflexive coequalizers, and f_*Y is filtered, then $f_*| - |_Y : S^{\triangle^{op}} \longrightarrow S$ preserves equalizers.

Proof: f_* preserves finite limits and reflexive coequalizers. It follows that all the squares in the following diagram commute.

$$\begin{array}{ccccccc}
S^{\triangle^{op}} & \xrightarrow{f^*} & A^{f^*\triangle^{op}} & \xrightarrow{g^*} & A^Y & \xrightarrow{Colim_Y} & A \\
 & & \downarrow{\scriptstyle f_*} & /// & \downarrow{\scriptstyle f_*} & /// & \downarrow{\scriptstyle f_*} \\
 & & S^{f_*f^*\triangle^{op}} & \xrightarrow[{[f_*(g)]^*}]{} & S^{f_*Y} & \xrightarrow[Colim_{f_*Y}]{} & S
\end{array}$$

Diagram V

Since $f_*f^* : S \longrightarrow S$ preserves equalizers, so does $f_*f^* : S^{\triangle^{op}} \longrightarrow S^{f_*f^*\triangle^{op}}$. The functor $[f_*(g)]^*$ is the pullback functor along $f_*(g)$, and so preserves equalizers, see [4] p 35. $Colim_{f_*Y} : S^{f_*Y} \longrightarrow S$ preserves equalizers, since f_*Y is filtered, see [4] p 70, Theorem 2.58. So by Diagram V, and Definition 1.1, $f_*| - |_Y$ preserves equalizers.

□

2.6. **Corollary:** If f_* preserves reflexive coequalizers, reflects monos, and f_*Y is filtered, then there is a mono $i_n : \dot{Y}_{0n} \rightarrowtail Y_{0n}$, for each n in N.

Proof: By Remark 2.4, there is a mono $i_n : \dot{\triangle}[n] \rightarrowtail \triangle[n]$. Apply the geometric realization functor to get $i_n : |\dot{\triangle}[n]|_Y \rightarrowtail |\triangle[n]|_Y$. By Definition 2.3 (ii), and Lemma 2.2, we obtain a map $i_n : \dot{Y}_{0n} \longrightarrow Y_{0n}$.

Since a mono in any topos is an equalizer, see [4] p 27, and since $S^{\triangle^{op}}$ is a topos, see [4] p 55, it follows that $i_n : \dot{\triangle}[n] \rightarrowtail \triangle[n]$ in $S^{\triangle^{op}}$ is an equalizer. Lemma 2.5 implies that $f_*(i_n) : f_*\dot{Y}_{0n} \longrightarrow f_*Y_{0n}$ is a mono. f_* reflects monos by hypothesis, thus $i_n : \dot{Y}_{0n} \longrightarrow Y_{0n}$ is a mono.

□

3. Strong Initiality

3.1. **Definition:**
(i) Given collections $\{a_n : n \in N \text{ or } n = -1\}$, and $\{b_n : n \in N\}$ of objects of a category B, and a collection $\{\alpha_n : b_n \rightarrowtail a_n\}$ of monomorphisms of B, we say the collection $\{\alpha_n\}$ is composable if:
(a) for all $n \in N$, $b_n = a_{n-1}$, and
(b) there is an object a, and monomorphisms $i_n : a_n \rightarrowtail a$ such that $i_n \alpha_n = i_{n-1}$, and if there is an object b, and monomorphisms $j_n : a_n \rightarrowtail b$ such that $j_n \alpha_n = j_{n-1}$, then there is a unique morphism $\phi : a \longrightarrow b$ such that $\phi i_n = j_n$.

If the collection $\{\alpha_n\}$ is composable, we say the composition of $\{\alpha_n\}$, denoted by $Comp\{\alpha_n\}$, is the morphism $i_{-1} : a_{-1} \rightarrowtail a$ of (b).
(ii) Let R be the image of $S^{\triangle^{op}}$ under the geometric realization functor $|-|_Y$. A collection $\{|\alpha_n|_Y : b_n \rightarrowtail a_n\}$ of monos of R is said to be $|-|_Y$-composable if the collection $\{\alpha_n\}$ is a composable collection of monos in $S^{\triangle^{op}}$.

3.2. **Lemma:** Let $\{\alpha_n : b_n \rightarrowtail a_n\}$ be a collection of monos in $S^{\triangle^{op}}$. If $\{|\alpha_n|_Y\}$ is $|-|_Y$-composable, then it is composable and $Comp\{|\alpha_n|_Y\} = |Comp\{\alpha_n\}|_Y$.

Proof: If $\{|\alpha_n|_Y\}$ is $|-|_Y$-composable, then $\{\alpha_n\}$ is composable. Since composition is a colimit, and $|-|_Y$ preserves colimits the rest follows.

□

3.3. **Notation:** Let $\alpha : a \longrightarrow b$, and $\sigma : \coprod_\Sigma a \longrightarrow c$ be morphisms in A, where $\coprod_\Sigma$ denotes the coproduct over a set Σ. If the pushout of $\coprod_\Sigma \alpha : \coprod_\Sigma a \longrightarrow \coprod_\Sigma b$ along $\sigma : \coprod_\Sigma a \longrightarrow c$ exists, we denote it by $\alpha(\Sigma, \sigma)$.

3.4. **Definition:**
(i) A morphism $\alpha : a \longrightarrow b$ of A is said to be initial with respect to f_*, if given a morphism $\beta : c \longrightarrow b$ in A, and a map $h : f_*c \longrightarrow f_*a$ in S, such that $f_*\alpha \circ h = f_*\beta$, then h can be lifted, that is, there is a map $\overline{h} : c \longrightarrow a$ such that $f_*\overline{h} = h$, and $\alpha \circ \overline{h} = \beta$.
(ii) Let R be the image of $S^{\triangle^{op}}$ under the geometric realization functor $|-|_Y$. A collection $\{\alpha_n : a_n \rightarrowtail b_n\}$ of monos of R is siad to be strongly initial if whenever sets Σ_n, and morphisms σ_n in R are given such that the collection $\{\alpha_n(\Sigma_n, \sigma_n)\}$ is a $|-|_Y$- composable collection of monos, then the composition of $\{\alpha_n(\Sigma_n, \sigma_n)\}$ is initial with respect to f_*.

3.5. **Lemma:** Let R be the image of $S^{\triangle^{op}}$ under the $|-|_Y$. Suppose $f_* : A \longrightarrow S$ preserves reflexive coequalizers, reflects monos, and f_*Y is filtered. The geometric realization functor preserves equalizers if and only if the collection $\{i_n : \dot{Y}_{0n} \rightarrowtail Y_{0n}\}$ of monos of R is strongly initial.

Proof: $\Rightarrow$: Suppose the functor $|-|_Y$ preserves equalizers. Let sets Σ_n, and morphisms $|\sigma_n|$ in R be given such that the collection $\{i_n(\Sigma_n, |\sigma_n|)\}$ is $|-|_Y$-composable. Then the collection $\{i_n(\Sigma_n, \sigma_n)\}$ is composable, and by Lemma 3.2 we have:

$$|Comp\{i_n(\Sigma_n, \sigma_n)\}|_Y = Comp\{i_n(\Sigma_n, |\sigma_n|)\}$$

The morphism $Comp\{i_n(\Sigma_n, \sigma_n)\}$ is a mono in $S^{\triangle^{op}}$ and therefore an equalizer. $|-|_Y$ preserves equalizers by hypothesis. So $Comp\{i_n(\Sigma_n, |\sigma_n|)\}$ is an equalizer. It is easy to show that equalizers in A are initial with respect to f_*. It follows that $Comp\{i_n(\Sigma_n, |\sigma_n|)\}$ is initial with respect to f_*. This proves $\{i_n : \dot{Y}_{0n} \rightarrowtail Y_{0n}\}$ is strongly initial.

$\Leftarrow$: Suppose the collection $\{i_n : \dot{Y}_{0n} \rightarrowtail Y_{0n}\}$ is strongly initial. Let $\alpha : E \rightarrowtail F$ be an equalizer in $S^{\triangle^{op}}$. There exist sets Σ_n, and morphisms σ_n in $S^{\triangle^{op}}$, see [1] p 50, such that the following diagram is a pushout in $S^{\triangle^{op}}$.

$$\begin{array}{ccc} \coprod_{\Sigma_n} \dot{\triangle}[n] & \xrightarrow{\sigma_n} & E \cup Sk^{n-1}F \\ {\scriptstyle \coprod_{\Sigma_n} i_n} \downarrow & \text{po} & \downarrow {\scriptstyle i_n(\Sigma_n, \sigma_n)} \\ \coprod_{\Sigma_n} \triangle[n] & \longrightarrow & E \cup Sk^n F \end{array}$$

Since the functor $|-|_Y$ preserves colimits, Lemma 2.2 and Definition 2.3 (ii) imply that the following diagram is a pushout in A.

$$\begin{array}{ccc} \coprod_{\Sigma_n} \dot{Y}_{on} & \xrightarrow{|\sigma_n|} & |E \cup Sk^{n-1}F| \\ {\scriptstyle \coprod_{\Sigma_n} i_n} \downarrow & \text{po} & \downarrow {\scriptstyle i_n(\Sigma_n, |\sigma_n|)} \\ \coprod_{\Sigma_n} Y_{0n} & \longrightarrow & |E \cup Sk^n F| \end{array}$$

It is easy to show $\{i_n(\Sigma_n, \sigma_n)\}$ is composable in $S^{\triangle^{op}}$ with composition $\alpha : E \rightarrowtail F$. It follows that $\{i_n(\Sigma_n, |\sigma_n|)\}$ is $|-|_Y$-composable with composition $|\alpha| : |E| \rightarrowtail |F|$. So $|\alpha|$ is initial with respect to f_*.
On the other hand by Lemma 2.5, $f_*|\alpha|$ is an equalizer. It then follows easily that $|\alpha|$ is an equalizer.

□

3.6. **Theorem:** If:
(1) for all $a \in A$, the functor $a \times - : A \longrightarrow A$ preserves e.e.'s.
(2) $f_* : A \longrightarrow S$ preserves reflexive coequalizers, reflects monos and terminals.
(3) $g : Y \longrightarrow f^*\triangle^{op}$ is a simplex structure, and,
(4) the collection $\{i_n : \dot{Y}_{0n} \rightarrowtail Y_{0n}\}$ is strongly initial,
then the geometric realization functor commutes with finite limits.

Proof: Preservation of finite products and terminals follows from Lemma 1.3. Preservation of equalizers follows from Lemma 3.5.

□

4. Applications

The categories Fco, $ConsFco$, Con, Lim, PsT, $Born$, and $PreOrd$ are topological over the category S of sets. See [7], [8], and [2]. Furthermore the forgetful functor $U : A \longrightarrow S$, where A is one of the above mentioned categories, has a left adjoint $D : S \longrightarrow A$ called the discrete functor. D preserves finite limits, that is the pair, (U, D) forms a geometric morphism. See [6] Section 5. Also U has a right adjoint, (the functor that defines the indiscrete structure on a set X), and therefore preserves colimits.

In this section we let A denote one of the categories Fco, $ConsFco$, Con, Lim, PsT, $Born$, or $PreOrd$, and we apply the previous results to the geometric morphism (U, D) with a given discrete fibration $g : Y \longrightarrow D\triangle^{op}$ in $Cat(A)$.

Let Σ be a set and for each $\sigma \in \Sigma$, let a_σ be an object of A and let $\nu_\sigma : a_\sigma \longrightarrow \coprod_\Sigma a_\sigma$ be the injection of the coproduct. If $a_\sigma = a$ for all $\sigma \in \Sigma$, we refer to the coproduct $\coprod_\Sigma a$ as the copower of a over Σ.

4.1. **Lemma:** For any set Σ, the copower of an initial mono in A, over Σ, is an initial mono.

Proof: We first show that monos are preserved under copower. Let $\alpha : X \longrightarrow Y$ be a mono in A. Since U preserves colimits, it follows that $U(\coprod_\Sigma \alpha) \equiv \coprod_\Sigma U(\alpha)$. Since U preserves monos, $U(\alpha)$ is a mono. In the category S, the functor $\coprod_\Sigma$ is easily seen to preserve monos. It follows that $U(\coprod_\Sigma \alpha) \equiv \coprod_\Sigma U(\alpha)$ is a mono. Since U reflects monos, $\coprod_\Sigma \alpha$ is a mono.

To show that initial monos are preserved we look at each category separately.

(1) $A = Fco$. Let $\alpha : (X, C_X) \rightarrowtail (Y, C_Y)$ be an initial mono. to show that the monomorphism $\beta = \coprod_\Sigma \alpha : (\coprod_\Sigma X, \overline{C}_X) \rightarrowtail (\coprod_\Sigma Y, \overline{C}_Y)$ is initial, let F be a filter on $\coprod_\Sigma X$, such that $[\beta F]$ is in $\overline{C}_Y(\alpha(x), \sigma)$. By [7], 3.2.2., we need to show F is in $\overline{C}_X(x, \sigma)$. Since $[\beta F]$ belongs to $\overline{C}_Y(\alpha(x), \sigma)$, by [7], 3.2.3, it follows that there is E in $C_Y(\alpha(x))$, such that $\nu_\sigma(E) \subseteq [\beta F]$. It is easy to show $[\alpha^{-1}E]$ is a filter, and since α is initial that it belongs to $C_X(x)$. So $[\nu_\sigma[\alpha^{-1}E]]$ is in $\overline{C}_X(x, \sigma)$. But we have $[\nu_\sigma[\alpha^{-1}E]] = [\nu_\sigma \alpha^{-1}E]$, and since the diagram

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ {\scriptstyle \nu_\sigma}\downarrow & \text{pb} & \downarrow{\scriptstyle \nu_\sigma} \\ \coprod_\Sigma X & \xrightarrow[\beta]{} & \coprod_\Sigma Y \end{array}$$

is a pullback, it follows that $[\nu_\sigma \alpha^{-1}E] = [\beta^{-1}\nu_\sigma E]$. So we have $[\nu_\sigma[\alpha^{-1}E]] = [\beta^{-1}\nu_\sigma E] = [\beta^{-1}[\nu_\sigma E]] \subseteq [\beta^{-1}\beta F] = F$. Hence F is in $\overline{C}_X(x, \sigma)$.

(2) For $A = ConsFco$, Con, and Lim, the proof is similar.

(3) For $A = PsT$, let $\alpha : X \rightarrowtail Y$ be an initial mono. To show that the monomorphism $\beta = \coprod_\Sigma \alpha : (\coprod_\Sigma X, \overline{C}_X) \rightarrowtail (\coprod_\Sigma Y, \overline{C}_Y)$ is initial, let F be in $F(X)$ such that $[\beta F]$ belongs to $\overline{C}_Y(\alpha(x), \sigma)$. We need to show F belongs to $\overline{C}_X(x, \sigma)$. Let G be an ultrafilter containing F. Then the ultrafilter $[\beta G]$ contains $[\beta F]$ and therefore belongs to $\overline{C}(\alpha(x), \sigma)$. By [7], 3.2.9, there is an ultrafilter E in $C_Y(\alpha(x))$ such that $[\nu_\sigma(E)] = [\beta(G)]$. Initiality of α implies $[\alpha^{-1}(E)]$ is in $C_X(x)$. Therefore $[\nu_\sigma[\alpha^{-1}E]] = [\nu_\sigma \alpha^{-1}E] = [\beta^{-1}\nu_\sigma E] = [\beta^{-1}\beta G] = G$ belongs to $\overline{C}_X(x, \sigma)$. Hence any ultrafilter containing F is in $\overline{C}_X(x, \sigma)$, therefore so is F.

(4) For $A = Born$, let $\alpha : (X, B) \rightarrowtail (Y, C)$ be an initial mono. To show that the monomorphism $\beta = \coprod_\Sigma \alpha : (\coprod_\Sigma X, \overline{B}) \rightarrowtail (\coprod_\Sigma Y, \overline{C})$ is initial, let D be a subset of $\coprod_\Sigma X$ such that $\beta(D)$ is in $\overline{C}$. By [7], 3.3.2, we need to show D is in $\overline{B}$. By [7], 3.3.3, there are a finite number of sets M_i in C such that $\beta(D) \subseteq \bigcup_i \nu_{\sigma_i}(M_i)$. This implies that $D = \beta^{-1}\beta(D) \subseteq \beta^{-1}(\bigcup_i \nu_{\sigma_i}(M_i)) = \bigcup_i (\beta^{-1}\nu_{\sigma_i}(M_i)) = \bigcup_i \nu_{\sigma_i}\alpha^{-1}(M_i)$. Initiality of α implies that $\alpha^{-1}(M_i)$ is in B, therefore $\nu_{\sigma_i}\alpha^{-1}(M_i)$ is in $\overline{B}$ for all i, and so is the finite union $\bigcup_i \nu_{\sigma_i}\alpha^{-1}(M_i)$. But $D \subseteq \bigcup_i \nu_{\sigma_i}\alpha^{-1}(M_i)$, therefore D is in $\overline{B}$.

(5) Finally for $A = PreOrd$, let $\alpha : (X, \leq) \rightarrowtail (Y, \leq)$ be an initial mono. To show $\beta = \coprod_\Sigma \alpha : (\coprod_\Sigma X, \leq) \rightarrowtail (\coprod_\Sigma Y, \leq)$ is initial, let (x, σ), and (x', σ') belong to $\coprod_\Sigma X$ such that $(\alpha(x), \sigma) \leq (\alpha(x'), \sigma')$. By [7], 3.1.2, we need to show $(x, \sigma) \leq (x', \sigma')$. By [7], 3.1.3, $(\alpha(x), \sigma) \leq (\alpha(x'), \sigma')$, which in the present situation implies $\alpha(x) \leq \alpha(x')$ and $\sigma = \sigma'$. Initiality of α implies $x \leq x'$. Therefore, again by applying the result of [7], 3.1.3, $(x, \sigma) \leq (x', \sigma')$ in $\coprod_\Sigma X$.

$\square$

4.2. **Lemma:** The pushout of an initial mono in A along any map is an initial mono.

Proof: We first show that monos are preserved under pushout. Let $\alpha : X \longrightarrow Y$ be a mono in A. Let $\sigma : X \longrightarrow Z$ be any morphism in A. Form the following pushout diagram:

$$\begin{array}{ccc} X & \xrightarrow{\ \sigma\ } & Z \\ {\scriptstyle\alpha}\downarrow & \text{po} & \downarrow{\scriptstyle\beta} \\ Y & \xrightarrow[\ \delta\ ]{} & T \end{array}$$

Diagram I

Since $U : A \longrightarrow S$ preserves colimits, applying U to the above diagram we get a pushout diagram in S. By [4], Lemma 1.31, it follows that $U\beta$, and thus β is a mono, since U reflects monos.

To show pushout preserves initial monos, we consider each category separately.

(1) $A = Fco$. Let $\alpha : (X, C) \rightarrowtail (Y, C)$ be an initial mono, and $\sigma : (X, C) \longrightarrow (Z, C)$ be any morphism in Fco. Suppose Diagram I is a pushout in Fco. To show β is initial, let F be a filter on Z such that $[\beta F]$ belongs to $C_T(\beta(z))$. By [7], 3.2.2, we need to show F belongs to $C_Z(z)$, Since $\{\delta, \beta\}$ is a final epi-sink, by [7], 3.2.3, it follows that either:

(i) There is a z' in Z, and E in $C(z')$ such that $\beta(z') = \beta(z)$ and $\beta(E) \subseteq [\beta F]$ or

(ii) There is a y in Y and E in $C(y)$ such that $\delta(y) = \beta(z)$ and $\delta(E) \subseteq [\beta F]$.

If (i) is the case, then $z = z'$ and it easily follows that $E \subseteq F$. Therefore $F \in \beta(z)$ since E is. If (ii) is the case, by [4], Lemma 1.28, there is a unique x in X such that $\alpha(x) = y$ and $\sigma(x) = z$. Since α is initial, it follows that $[\alpha^{-1}E]$ is in $C_X(x)$, and so $[\sigma\alpha^{-1}E]$ is in $C_Z(z)$. But $[\sigma\alpha^{-1}E] = [\beta^{-1}\delta E] \subseteq [\beta^{-1}\beta F] = F$, hence F is in $C_Z(z)$.

(2) For $A = ConsFco$ or Con the proof is similar to (1).

(3) For $A = Lim$, suppose Diagram I is a pushout in Lim. To show β is initial, let F be a filter on z such that $[\beta F]$ is in $C(\beta(z))$. By [7], 3.2.1, we need to show F is in $C(z)$. By [7], 3.2.6, there are a finite number of y_i's in Y, z_i's in Z, E_i's in $C(y_i)$, and F_i's in $C(z_i)$ such that $\delta(y_i) = \beta(z)$, $\beta(z_i) = \beta(z)$, and $\bigcap_i [\delta E_i] \cap \bigcap_i [\beta F_i] \subseteq [\beta F]$. $\delta(y_i) = \beta(z)$ implies there is x_i in X such that $\alpha(x_i) = y_i$ and $\sigma(x_i) = z$. β is a mono, therefore $z_i = z$, and so F_i is in $C(z)$ for all i. Let $F' = \bigcap_i F_i$, it follows that F' is in $C(z)$, $[\beta F'] = \bigcap_i [\beta F_i]$, and (*) $\bigcap_i [\delta E_i] \cap [\beta F'] \subseteq [\beta F]$.

On the other hand initiality of α implies that $\alpha^{-1}E_i$ is in $C(x_i)$ and so $[\sigma\alpha^{-1}E_i]$ is in $C(z)$. Hence $\bigcap_i [\sigma\alpha^{-1}E_i] \cap F'$ is in $C(z)$.

If K is in $\bigcap_i [\sigma\alpha^{-1}E_i] \cap F'$, then there are $G_i \in E_i$ such that $\sigma\alpha^{-1}(G_i) \subseteq K$, and $K \in F'$. Let $G = \bigcup_i G_i$, it follows that $\sigma\alpha^{-1}(G) \subseteq K$. Therefore $\beta^{-1}\delta(G) \subseteq K \in F'$. Since $G_i \subseteq G$, it follows that $G \in E_i$, for all i. Hence $\delta(G)$ belongs to $\bigcap_i [\delta(E_i)]$. Also $\beta(K) \in [\beta F']$. It follows that $\delta(G) \cup \beta(K)$ belongs to $\bigcap_i [\delta(E_i)] \cap [\beta(F')]$, and so by (*) it belongs to $[\beta(F)]$. But $\beta^{-1}(\delta(G) \cup \beta(K)) \subseteq K$. Therefore $K \in \beta^{-1}[\beta F] = F$. This proves $\bigcap_i [\sigma\alpha^{-1}(E_i)] \cap F' \subseteq F$. Hence F belongs to $C(z)$.

(4) $A = PsT$. Suppose Diagram I is a pushout in PsT. To show β is initial, let F be a filter on Z such that $[\beta F]$ is in $C(\beta(z))$. By [7], 3.2.2, we need to show F is in $C(z)$, which follows if we show any ultrafilter F' containing F is in $C(z)$. So let F' be an ultrafilter containing F. It easily follows that $[\beta F']$ is an ultrafilter containing $[\beta F]$. Since $[\beta F]$ is in $C(\beta(z))$, by [7], 3.2.9, it follows that either:

(i) There is z' in z and an ultrafilter E in $C(z')$ such that $\beta(z') = \beta(z)$ and $[\beta F'] = [\beta F]$ or

(ii) There is y in Y and an ultrafilter E in $C(y)$ such that $\delta(y) = \beta(z)$ and $[\delta E] = [\beta F']$.

If (i) is the case, then $z' = z$ and $F' = E$ is in $C(z)$. If (ii) is the case, then there is x in X such that $\alpha(x) = y$ and $\sigma(x) = z$. Since α is initial, it follows that $[\alpha^{-1}E]$ is in $C(x)$, and so $[\sigma\alpha^{-1}E]$ is in $C(z)$. But $F' = [\beta^{-1}\beta F'] = [\beta^{-1}\delta E] = [\sigma\alpha^{-1}E]$, and so F' belongs to $C(z)$.

(5) $A = Born$. Suppose Diagram I is a pushout in $Born$. To show β is initial, let $B \subseteq Z$ and $\beta(B) \in B_T$. By [7], 3.3.2, we need to show $B \in B_Z$. By [7], 3.3.3, there is M in B_Y and N in B_Z such that $\beta(B) \subseteq \delta(M) \cup \beta(N)$. It follows that $B \subseteq \beta^{-1}\delta(M) \cup N$. Since α is initial, it follows that $\alpha^{-1}(M)$ is in B_X, and so $\sigma\alpha^{-1}(M)$ is in B_Z. But $\sigma\alpha^{-1}(M) = \beta^{-1}\delta(M)$, and $B \subseteq \beta^{-1}\delta(M) \cup N$, hence B is in B_Z.

(6) $A = PreOrd$. Suppose Diagram I is a pushout in $PreOrd$. To show β is initial, let z and z' be in Z such that $\beta(z) \leq \beta(z')$. By [7], 3.1.2, we need to show $z \leq z'$. Note that if $\delta(y_1) = \delta(y_2)$, then because δ is pushout of σ, we have $y_1 = \alpha(x_1)$, $y_2 = \alpha(x_2)$, and $\sigma(x_1) = \sigma(x_2)$. It then follows from initiality of α and [7], 3.1.3, that β is initial.

□

4.3. **Lemma:** For each n in N, let $\alpha_n : a_n \rightarrowtail a_{n+1}$ be a mono in A. The collection $\{\alpha_n : n \in \mathsf{N}\}$ is composable, and the composition is initial if each α_n is.

Proof: Let $U(a_n) = X_n$, where U is the forgetful functor. Without loss of generality assume the monomorphism $U(\alpha_n) : X_n \rightarrowtail X_{n+1}$ is the inclusion, and let $X = \bigcup_{\mathsf{N}} X_n$, and $i_n : X_n \rightarrowtail X$ be the inclusion. To define the structure on X we consider the following cases:

(1) $A = Fco$, $ConsFco$, Con, or Lim. Let $a_n = (X_n, C_n)$, and define the structure C on X as follows:

$C(x) = \{F \in F(X) : \exists n \in \mathsf{N}, G \in C_n(x) \ni: [i_n(G)] \subseteq F\}$.

It is straightforward to check that (X, C) is in A, and that $\{\alpha_n\}$ is composable and $i_0 : (X_0, C_0) \rightarrowtail (X, C)$ is the composition of $\{\alpha_n\}$.

To show that i_0 is initial if each α_n is, note that if $[i_0 F] \in C(x)$ for some filter F on X_0, then $[i_n G] \subseteq [i_0 F]$ for some $C \in C_n(x)$. It follwos that $[i_0^{-1} i_n G] = [\alpha_0^{-1}\alpha_1^{-1}...\alpha_{n-1}^{-1}G] \subseteq F$. But the filter $[\alpha_0^{-1}\alpha_1^{-1}...\alpha_{n-1}^{-1}G]$ is in $C_0(x)$, since $G \in C_n(x)$ and α_n's are initial. Hence F is in $C_0(x)$ as desired.

(2) For $A = PsT$, define C as follows:

$C(x) = \{F \in F(x) : \forall$ ultrafilters $U \supseteq F, \exists n \in \mathsf{N}$, and an ultrafilter $G \in C_n(x) \ni: U = [i_n(G)]\}$.

(3) $A = Born$. Let $a_n = (X_n, B_n)$. Note that $B_n \subseteq B_{n+1}$, for all n. Let $B = \bigcup_{\mathsf{N}} B_n$. It easily follows that (X, B) is in $Born$, and that $\{\alpha_n\}$ is composable with composition $i_0 : (X_0, B_0) \rightarrowtail (X, B)$.

Now suppose α_n is initial for all n in N. Let $G \subseteq X_0$ such that $i_0(G) = G \in B$. Then $G \in B_n$, for some n in N, and so $G \in B_0$, since α_0, α_1, ... , α_{n-1} are all initial.

(4) $A = PreOrd$. Let $a_n = (X_n, \leq_n)$ be in $Preord$. Define the preorder $\leq$ on X by: $x \leq y$ if there is n in N such that $x \leq_n y$. It follows easily that $\{\alpha_n\}$ is composable and the composition is $i_0 : (X_0, \leq_0) \rightarrowtail (X, \leq)$.

To show i_0 is initial if each α_n is, let x, y be in X_0 such that $x \leq y$ in X. Therefore $x \leq_n y$ for some n in N. It follows that $x \leq_0 y$, since α_0, α_1, ... , α_{n-1} are all initial.

□

4.4. **Corollary:** Let $g : Y \longrightarrow D\triangle^{op}$ be a discrete fibration in A such that $U(Y)$ is filtered. The functor $|-|_Y : S^{\triangle^{op}} \longrightarrow A$ preserves equalizers if and only if for all n in N, $i_n : \dot{Y}_{0n} \rightarrowtail Y_{0n}$ is initial.

Proof: By Lemmas 4.1, 4.2, 4.3, and Definition 3.4, the collection $\{i_n : \dot{Y}_{0n} \rightarrowtail Y_{0n}\}$ is strongly initial. Since $U : A \longrightarrow S$ preserves colimits, and obviously reflects monos, the proof follows from Lemma 3.5.

□

4.5. **Corollary:** Let $g : Y \longrightarrow D\triangle^{op}$ be a simplex structure in A. The geometric realization functor $|-|_Y : S^{\triangle^{op}} \longrightarrow A$ commutes with finite limits if and only if for each n in N, $i_n : \dot{Y}_{0n} \rightarrowtail Y_{0n}$ is initial.

Proof: Since $A = Fco$, $ConsFco$, Con, Lim, PsT, $Born$ or $PreOrd$ is cartesian closed, the functor $a \times - : A \longrightarrow A$ preserves extremal epis.

The functor $U : A \longrightarrow S$ reflects terminals, and so by Theorem 3.6 and Corollary 4.4, the result follows.

□

Bibliography

[1] P. Gabriel, M. Zisman, Calculus of Fractions and Homotopy Theory, Ergebnisse der Mathematik, Vol. 35, Berlin-Heidelberg-New York, Springer 1967.

[2] H. Herrlich, Topological Functors, General Topology and its Applications 4, 1974, pp. 125-142.

[3] S. N. Hosseini, Commutation of Internal Colimit and Finite Limits, Scientiae Mathematicae Japonicae Online, e-2007, pp. 373-385.

[4] P. T. Johnstone, Topos Theory, London-New York-San Francisco, Academic Press 1977.

[5] S. MacLane, Categories for the Working Mathematician. New York-Heidelberg-Berlin, Springer 1971.

[6] M. V. Mielke, Convenient Categories for Internal Singular Algebraic Topology, Illinois Journal of Math., Vol. 27, No. 3, 1983.

[7] L. D. Nel, Initially Structured Categories and Cartesian Closedness, Can. Journal of Math., Vol. XXVII, No. 6, 1975, pp. 1361-1377.

[8] F. Schwarz, Connections Between Convergence and Nearness, Lecture Notes in Mathematics No. 719, Springer-Verlag, 1978, pp. 345357

Address:
S. N. Hosseini
Mathematics Department, Mahani Research Center
Shahid Bahonar University of Kerman
End of 22nd of Bahman Blvd, 76169-14111
Kerman, Iran
Tel: (098341)3221080
Fax: (098341)3221078 or 3221080
email: nhoseini@mail.uk.ac.ir