Scientiae Mathematicae Japonicae Online, **e-2008**, 317–322

FOUR-MEMBER COMMITTEE LOOKING FOR A SPECIALIST WITH TWO HIGH ABILITIES**

MINORU SAKAGUCHI*

Received February 4, 2008

ABSTRACT. In this paper it is shown that a four-member committee is possible to find a specialist with two different and correlated high abilities.

1 Statement and Formulation of the Problem. A 4-player (=member) committee has players I, II, III and IV. The committee wants to employ one specialist among n applicants. It interviews applicants sequentially one-by-one. Facing each applicant player I(II, III, IV) evaluates the management ability at $X_1(Y_1, Z_1, T_1)$ and computer ability at $X_2(Y_2, Z_2, T_2)$. Evaluation by the players are made independently and each player chooses, based on his evaluation, either one of R and A. The committee's choice is made by simple majority. If the players are divided in two for R and two for A, then Umpire is introduced and he votes in R and A with probabilities $\frac{1}{2}$ each. If the committee rejects the first $n-1$ applicants, then it should accept the n-th applicant. Denote

$$\xi = x_1 \wedge x_2, \eta = y_1 \wedge y_2, \zeta = z_1 \wedge z_2, \tau = t_1 \wedge t_2. \tag{1.1}$$

If the committee accepts an applicant with talents evaluated at $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t}$ by I, II, III, IV resp., then the game stops and each player is paid ξ, η, ζ, τ to I, II, III, IV, resp.. If the committee rejects an applicant, then the next applicant is interviewed and the game continues. Each player of the committee aims to maximize the expected payoff he can get.

The two different kinds of talents (management and computer abilities) for each applicant, are bivariate r.v.s, *i.i.d.* with pdf

$$h(x_1, x_2) = 1 + \gamma(1 - 2x_1)(1 - 2x_2), \ \ \forall (x_1, x_2) \in [0, 1]^2, |\gamma| \leq 1 \tag{1.2}$$

for player I. For II, III and IV, the same pdf with the same γ is used. If $X_1(X_2)$ for I is the evaluation of ability of management (foreign language), then γ will be $0 \leq \gamma \leq 1$. If X_2 is the evaluation of the computer ability, then γ may be $-1 \leq \gamma \leq 0$.

The bivariate pdf (1.2) is one of the simplest pdf that has the identical uniform marginal and correlated component variables. The correlation coefficient is equal to $\gamma/3$.

Denote the state $(j, \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t})$ where $\mathbf{x} = (x_1, x_2)$, *etc.*, to mean that @ the first $j-1$ applicants were rejected by the committee, A the j-th applicant is currently evaluated at $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t}$, by I, II, III, IV resp. and B $n-j$ applicants remain un-interviewed if the j-th is rejected by the committee. The state is illustrated by Figure 1.

2000 *Mathematics Subject Classification.* 60G40, 90C39, 90D10.

Key words and phrases. Optimal stopping game, optimal strategy, four-player two-choice game, committee.

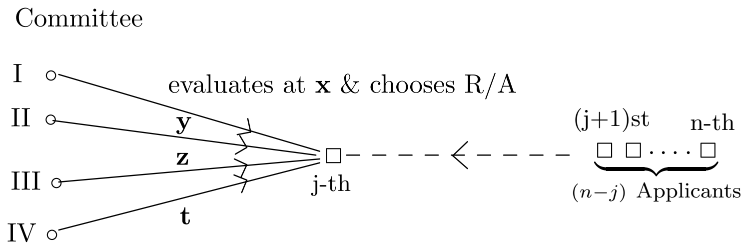

Figure 1. State $(j, \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t})$

We define u_j = Expected payoff, player I can get, if I is in state $(j, \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t})$ and all players play optimally hereafter.

Define v_j, for II, w_j, for III, and s_j for IV, similarly. Moreover we introduce a number

$$c \equiv E_{\mathbf{x}}(\xi) = 2\int_0^1 dx_1 \int_0^{x_1} x_2 h(x_1, x_2) dx_2 = \frac{1}{3} + \frac{1}{30}\gamma$$

where is in $[3/10, 11/30]$ for $\forall \gamma \in [-1, 1]$.

2 Solution to the Problem. By mathematical common sense and symmetry among the players in the problem, it is evident that :

For I in state $(j, \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t})$, R(A) dominates A(R), if $u_{j+1} > (<) \xi$. For II (III, IV), u_{j+1} and ξ are replaced by v_{j+1} and $\eta (w_{j+1}$ and ξ, s_{j+1} and $\tau)$.

Lemma 1

(2.1) $$f(u) \equiv E_{\mathbf{x}} I(\xi > u) = (\overline{u})^2 (1 + \gamma u^2), \ \ \forall u \in [0, 1].$$

This function is decreasing with values $f(0) = 1, f\left(\frac{1}{2}\right) = \frac{1}{4} + \frac{1}{16}\gamma$, *and* $f(1) = 0$. *Moreover* $f(u)$ *is convex if* $0 < \gamma \leq 1$.

(2.2) $$g(u) \equiv E_{\mathbf{x}} [\xi I(\xi > u)] = c - u^2 + \frac{2}{3}u^3 + \gamma u^3 \left(\frac{2}{3} - \frac{3}{2}u + \frac{4}{5}u^2\right)$$

is decreasing with values $g(0) = c, g\left(\frac{1}{2}\right) = \frac{1}{6} + \frac{23}{480}\gamma$, *and* $g(1) = 0$.

Proof.

$$\begin{aligned} f(u) &= \int_u^1 \int_u^1 \{1 + \gamma(1 - 2x_1)(1 - 2x_2)\} dx_1 dx_2 \\ &= (\overline{u})^2 + \gamma \left[\int_u^1 (1 - 2x_1) dx_1\right]^2 = (\overline{u})^2 \left[1 + \gamma(-u\overline{u})^2\right] \end{aligned}$$

i.e., Eq.(2.1). Moreover we obtain

$$f''(u) = 2\gamma \left[\gamma^{-1} + (1 - 6u\overline{u})\right] > 0, \ \ \forall u \in [0, 1], \ \text{ if } \ 0 < \gamma \leq 1.$$

On the other hand

$$\begin{aligned} g(u) &= \int_u^1 dx_1 \int_u^{x_1} x_2 h(x_1, x_2) dx_2 + \int_u^1 dx_2 \int_u^{x_2} x_1 h(x_1, x_2) dx_1 \\ &= 2\int_u^1 dx_1 \int_u^{x_1} x_2 \{1 + \gamma(1 - 2x_1)(1 - 2x_2)\} dx_2 \end{aligned}$$

After a bit of calculations, we have

$$2\int_u^1 (1-2x_1)dx_1 \int_u^{x_1} x_2(1-2x_2)dx_2 = \frac{1}{30} + \frac{2}{3}u^3 - \frac{3}{2}u^4 + \frac{4}{5}u^5$$

and so Eq.(2.2) follows.
Both of $f(u)$ and $g(u)$ are decreasing, because of their definitions. □

It is evident that

$$1 > f(u) > g(u) > 0, \;\; \forall u \in (0,1) \tag{2.3}$$

by the definitions of $f(u)$ and $g(u)$.

Lemma 2 *The expected payoff for I in state $(j, \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t})$ is given by $Q(u_{j+1})$, where*

$$Q(u) = u(1-3f^2+2f^3) + c\left(-\frac{1}{2}f^3 + \frac{3}{2}f^2\right) + \frac{3}{2}gf\overline{f} \tag{2.4}$$

or, equivalently

$$Q(u) = u + \frac{3}{2}gf + \left\{-3u + \frac{3}{2}(c-g)\right\} f^2 + \left(2u - \frac{1}{2}c\right) f^3 \tag{2.4$'$}$$

and f, g denote $f(u), g(u)$, resp..

We easily find that $Q(0+) = c$, and $Q(1-0) = 1$ by Lemma 1 (Note that c, f and g involve γ).

Proof. $Q(u)$ is equal to :

$$\begin{aligned}
&u\,[I(\xi<u)\,\{I(\eta<v,\zeta<w,\tau<s) + \text{ other 3 terms coming from choice triples,} \\
&\qquad\qquad \text{R-R-A, R-A-R and A-R-R}\} + I(\xi>u)I(\eta<v,\zeta<w,\tau<s)] \\
&+\xi\,[I(\xi<u)I(\eta>v,\zeta>w,\tau>s) + I(\xi>u)\,\{I(\eta>v,\zeta>w,\tau>s) \\
&\qquad\qquad \text{+other 3 terms coming from choice triples A-A-R, A-R-A, and R-A-A}\}] \\
&+\frac{1}{2}(u+\xi)\,[I(\xi>u,\eta>v,\zeta<w,\tau<s) + \text{ other 5 terms consisting of two A and two R}]
\end{aligned}$$

Committee's decision is R(A) in the first (second) brace by simple majority.

In the third brace, committee's decision depends on the outcome of the random choice. Then $E_{\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{t}}$ of the above expression is

$$\begin{aligned}
&u\left[\overline{f(u)}\left\{\overline{f(v)}\;\overline{f(w)}\;\overline{f(s)} + \overline{f(v)}\;\overline{f(w)}f(s) + \overline{f(v)}f(w)\overline{f(s)} + f(v)\overline{f(w)}\;\overline{f(s)}\right\}\right. \\
&\qquad\qquad \left. + f(u)\overline{f(v)}\;\overline{f(w)f(s)}\right] \\
&+\left[(c-g(u))f(v)f(w)f(s) + g(u)\left\{f(v)f(w)f(s) + f(v)f(w)\overline{f(s)} + f(v)\overline{f(w)}f(s)\right.\right. \\
&\qquad\qquad \left.\left. + \overline{f(v)}f(w)f(s)\right\}\right]
\end{aligned}$$

(Note that $E_{\mathbf{x}}\,[\xi I(\xi<u)] = E_{\mathbf{x}}\,[\xi(1-I(\xi>u))] = c - g(u)$)

$$+\frac{1}{2}u\left[f(u)f(v)\overline{f(w)}\;\overline{f(s)} + \text{other 5 terms consting of two } \; f \text{ and two } \overline{f}\right]$$

$$+\frac{1}{2}\left[g(u)\left\{f(v)\overline{f(w)}\ \overline{f(s)}+\overline{f(v)}f(w)\overline{f(s)}+\overline{f(v)}\ \overline{f(w)}f(s)\right\}\right.$$
$$\left.+(c-g(u))\left\{f(v)f(w)\overline{f(s)}+f(v)\overline{f(w)}f(s)+\overline{f(v)}f(w)f(s)\right\}\right]$$

Remember that, u, v, w, s, here, are $u_{j+1}, v_{j+1}, w_{j+1}, s_{j+1}$, resp.

From symmetry among the four players we can put $u = v = w = s$.

Then the above expression becomes

$$Q(u) \equiv u\left[\overline{f(u)}\left\{\overline{f(u)}^3+3(\overline{f(u)})^2 f(u)\right\}+f(u)(\overline{f(u)})^3\right]$$
$$+\left[(c-g(u))(f(u))^3+g(u)\left\{(f(u))^3+3(f(u))^2\overline{f(u)}\right\}\right]$$
$$+\frac{1}{2}\cdot 6u(f(u))^2(\overline{f(u)})^2+\frac{1}{2}\left[g(u)\cdot 3f(u)(\overline{f(u)})^2+(c-g(u))\cdot 3(f(u))^2\overline{f(u)}\right]$$

and, after several steps of computations, we finally get (2.4). □

We thus have

Theorem 1 *Optimal expected payoff to* I *satisfies the recursion*

$$u_j = Q(u_{j+1}),\ \ j = n, n-1, \cdots, 2, 1;\, u_n = E_{\mathbf{x}}(\xi) = c$$

where $Q(u)$ *is given by* $(2.4')$.

Theorem 2

$$u_1 > u_2 > \cdots > u_n = c \tag{2.5}$$

Proof. We have by Theorem 1,

$$\begin{aligned} u_j - u_{j+1} &= Q(u_{j+1}) - u_{j+1} \\ &= \left[f\left\{\frac{3}{2}g+\left(-3u+\frac{3}{2}(c-g)\right)f+\left(2u-\frac{1}{2}c\right)f^2\right\}\right]_{u=u_{j+1}} \end{aligned}$$

Let

$$m(u) \equiv \frac{3}{2}g+\left(-3u+\frac{3}{2}(c-g)\right)f+\left(2u-\frac{1}{2}c\right)f^2.$$

Clearly $m(0) = c$ and $m(1) = 0$.

We want to prove that $m(u) > 0\ \forall u \in (0, 1)$.

Suppose that $m(u_0) = 0$, for some $u_0 \in (0, 1)$. Then we get a quadratic equation

$$\left(\frac{1}{2}c-2u\right)f^2+\left(3u-\frac{3}{2}(c-g)\right)f-\frac{3}{2}g = 0,$$

so that

$$f(u) = \frac{-\left(3u-\frac{3}{2}(c-g)\right)\pm\sqrt{\left(3u-\frac{3}{2}(c-g)\right)^2+(c-4u)3g}}{c-4u} \tag{2.6}$$

for some $u = u_0 \in (0, 1)$.

The inside of the square root in the r.h.s. of (2.6) becomes negative for $u_0 = \frac{1}{2}c$, since

$$\left(\frac{3}{2}g\right)^2 - 3cg = 3g\left(\frac{3}{4}g-c\right) < 0.$$

On the other hand by (2.1) and (2.4) it must hold

$$f(u_0) = (\overline{u_0})^2(1+\gamma u_0^2) = (2.6)_{at\ u=u_0}.$$

This is impossible.

Hence $m(u) \neq 0, \forall u \in (0,1)$, and since $m(0) = c$ and $m(1) = 0$, we find that $m(u) > 0, \forall u \in (0,1)$. The theorem is proven. □

3 The Case $\gamma = 0$. Consider the special case $\gamma = 0$. Then from (2.1), (2.2) and (2.4) we obtain

$$c_0 = \frac{1}{3}, f_0(u) = \overline{u}^2, g_0(u) = \frac{1}{3} - u^2 + \frac{2}{3}u^3, \tag{3.1}$$

and

$$Q_0(u) = u(1 - 3f_0^2 + 2f_0^3) + c_0\left(-\frac{1}{2}f_0^3 + \frac{3}{2}f_0^2\right) + \frac{3}{2}g_0 f_0 \overline{f_0}. \tag{3.2}$$

For $u = \dfrac{1}{3}$, we have by (3.1) and (3.2)

$$c_0 = \frac{1}{3}, f_0\left(\frac{1}{3}\right) = \frac{4}{9}, g_0\left(\frac{1}{3}\right) = \frac{20}{81},$$

and

$$\begin{aligned} Q_0\left(\frac{1}{3}\right) &= \frac{1}{3}\left(1 - \frac{16}{27} + \frac{128}{81 \times 9}\right) + \text{ two more terms} \\ &\approx 0.19433 + 0.08413 + 0.09145 \approx 0.3699. \end{aligned}$$

Since $f_0(0.3699) \approx 0.3969, g_0(0.3699) \approx 0.2302$ we have

$$\begin{aligned} Q_0(0.3699) &\approx 0.37\left\{1 - 3(0.37)^2 + 2(0.37)^3\right\} + \text{ two more terms} \\ &\approx 0.24141 + 0.0684 + 0.08266 \approx 0.3934. \end{aligned}$$

Since $f_0(0.3934) \approx 0.3680, g_0(0.3934) \approx 0.21916$ we have

$$\begin{aligned} Q_0(0.4066) &\approx 0.3934\left\{1 - 3(0.3680)^2 + 2(0.3680)^3\right\} + \text{ two more terms} \\ &\approx 0.27278 + 0.05941 + 0.07646 \approx 0.40865. \end{aligned}$$

An example where $\gamma = 0$.

There are 4-player committee and 4 applicants. The common optimal strategy for each player is

"Choose A(R), if $x_1 \wedge x_2 > (<) u_1 = 0.4086$ in the first stage".

[For player II, III, IV, $x_1 \wedge x_2$ is replaced by $y_1 \wedge y_2, z_1 \wedge z_2, t_1 \wedge t_2, resp.$.]

If the committee rejects the first applicant, then it interviews the 2nd applicant, and

"Choose A (R), if $x_1 \wedge x_2 > (<) u_2 = 0.3934$ in the second stage"

If the committee rejects the 2nd applicant, then it interviews the 3nd applicant, and

"Choose A (R), if $x_1 \wedge x_2 > (<) u_3 = 0.3699$ in the third stage"

If the committee rejects the 3rd applicant, then it should accept the 4th applicant and each player's expected payoff is $u_4 = c_0 = 1/3$.

4 Final Remark. The present author cannot use computer by some inevitable private reasons (age, desease, *etc.*). If we can use computer, it would be interesting to make the table of

	u_n	u_{n-1}	u_{n-2}	u_{n-3}	$\cdots$	u_2	u_1
$\gamma = -1$	3/10	0.3320	······				
0	1/3	0.3699	0.3934	0.4086	······		
1	11/30	0.4073	······				

(The numbers above are obtained, from (2.1)∼(2.5) by using, a small calculator.In the real world n may be 5∼20.)

In Ref.[6] it is shown that in the three-member committee case, simple majority settles the game quickly, and the result is

	u_n	u_{n-1}	u_{n-2}	$\cdots$	u_2	u_1
$\gamma = -1$	3/10	0.3420	0.3685	$\cdots$		
0	1/3	0.3821	0.4121	$\cdots$		
1	11/30	0.4284	0.4620	$\cdots$		

In the four-member committee case each member gets less than in the three-member committee case.

References

[1] Ben Abdelaziz F., Krichen S., *An interactive method for the optimal selection problem with two decision makers,* Europ. J. Oper. Res., (to appear).

[2] T. Ferguson, *Selection by committee*, Annals of the International Society of Dynamic Games, vol. 7(2005), Advances in Dynamic Games Application to Economics, Finance, Optimization and Stochastic Control,203-209.

[3] V. V. Mazalov, M. V. Banin, *N person best choice games with voting*, Game Th. Appl., vol. IX(2003), 45-53.

[4] V. V. Mazalov, M. Sakaguchi, A. A. Falco, *Selection by committee in the best-choice problem with rank criterion*, to appear.

[5] M. Sakaguchi, *Three-member committee where odd-man's judgement is paid regard*, Sc. Math. Japon., **66**(2007) 31-36.

[6] M. Sakaguchi, *Three-member committee looking for a specialist with two high abilities*, to appear in Game Th. Appl., vol. XIV.

*3-26-4 Midorigaoka, Toyonaka, Osaka, 560-0002, Japan,
Fax: +81-6-6856-2314 E-mail: minorus@tcct.zaq.ne.jp