HENSTOCK INTEGRAL IN HARMONIC ANALYSIS

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Abstract. The feasibility of Henstock approach to integration theory is demonstrated by application of Henstock-type integrals with respect to different derivation bases to the problem of recovering the coefficients of orthogonal series by generalized Fourier formulae.

1 Introduction. The name of Henstock will always be associated first of all with a very simple definition of a generalized Riemann integral on an interval of the real line which covers the Lebesgue integral and is widely known nowadays as the gauge integral or the Henstock-Kurzweil integral. It is really amazing that such a small change introduced into classical Riemann definition yields an integral much more powerful than the integrals that can be produced by a very elaborate Lebesgue construction. No wonder that this definition has been included in many textbooks on integration theory and is used in some university courses. For example, in Math. Department at Moscow University students are taught Henstock integral within the first year undergraduate analysis course, prior to the course in the Lebesgue integration.

But having written his first papers on Riemann-type integration (see [4], [5]), Henstock realized very soon that his methods had a far greater scope than it was shown in those publications. So he introduced in [6] (see also [7] and [8]) the concept of division spaces which seems to be one of his most profound contribution to the theory of integral. Since then, it became clear that the generalized Riemann integral can be defined in a very general setting, and had such a wide generality that almost every integral known in analysis could be included in the division space theory. So very important feature of Henstock's theory is its unifying quality: it gives a unified approach to many problems which have been dealt with earlier by different methods, using different type of non-absolute integrals. Now many of them can be solved by using different types of Henstock integrals, just by choosing an appropriate basis of integration (or division space in Henstock terminology).

In this note we are going to demonstrate this by examples of application of the Henstock construction to the so-called coefficients problem in the theory of series with respect to different orthogonal systems. This problem became popular in the classical harmonic analysis in the beginning of XX century since it had been discovered that some everywhere convergent trigonometric series could fail to be Fourier-Lebesgue series of their sums. For example series

$$\sum_{n=2}^{\infty} \frac{\sin nx}{\ln n}$$

converges everywhere and hence his coefficients are defined uniquely by its sum (see [26]). But the sum is not Lebesgue integrable. So the coefficients can not be recovered by the

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classical Fourier formulae. This kind of examples can be given for many other known orthogonal systems. To integrate such series one needs nonabsolutely convergent integrals. If the sum of the series is integrable in one or another known general sense, the question is whether the coefficients can be computed by Fourier formulae in which the integral is understood in the same particular sense. The complete solution of the problem of recovering the coefficients of an orthogonal series, with respect to a certain system, is found if a general process of integration is developed so that everywhere convergence of such a series implies that it is the Fourier series of its sum in the sense of the defined integral. Convergence everywhere can be replaced here by convergence outside so-called set of uniqueness. (By the way one of the early papers by Henstock [3] is related to the sets of uniqueness for trigonometric series, but it was before he had started developing his integration theory.)

The first solution to the coefficients problem in the trigonometric case is due to Denjoy. He introduced in [2] a very complicated construction of a second order integral called the totalization T_{2s} which recaptures a function from its second Riemann symmetric derivative. Later some other authors defined Perron-type integrals to solve this problem (see [24] for details). An integral solving the coefficients problem for Haar and Walsh series was introduced in [17] in a descriptive form. Then a constructive definition of Denjoy type, based on transfinite induction, was given in [18] and later, independently, in [10]. The coefficients problem for Vilenkin system was examined in [19]. So the problem for each system was treated separately.

An advantage of application of the Henstock theory is that it provides a unifying approach to the coefficients problem for many orthogonal systems, including the multidimensional case. The point is that for many systems one can find a regular method of summation of series based on some kind of a generalized derivative. The derivative prompts the choice of an appropriate derivation basis. And in this way the coefficient problem can be reduced to the problem of recovering the primitive from a given derivative by a Henstock-type integral defined with respect to the chosen basis. We consider below several examples of applying this method.

2 Preliminaries. We remind the principal elements of the Henstock construction. We present them here, following [11] and [23], in terms of derivation basis theory instead of the one of division spaces, although Henstock prefered to use the last terminology. But in those simple cases we are dealing with here it is really a matter of language.

A *derivation basis* (or simply a *basis*) $\mathcal{B}$ in a measure space $(X, \mathcal{M}, \mu)$ is a filter base on the product space $\mathcal{I} \times X$, where $\mathcal{I}$ is a family of measurable subsets of X having positive measure μ and called *generalized intervals* or *$\mathcal{B}$-intervals*. That is, $\mathcal{B}$ is a nonempty collection of subsets of $\mathcal{I} \times X$ so that each $\beta \in \mathcal{B}$ is a set of pairs (I, x), where $I \in \mathcal{I}$, $x \in X$, and $\mathcal{B}$ has the *filter base property*: $\emptyset \notin \mathcal{B}$ and for every $\beta_1, \beta_2 \in \mathcal{B}$ there exists $\beta \in \mathcal{B}$ such that $\beta \subset \beta_1 \cap \beta_2$. So each basis is a directed set with the order given by "reversed" inclusion. We shall refer to the elements β of $\mathcal{B}$ as *basis sets*. If $x \in I$ for all the pairs (I, x) constituting each $\beta \in \mathcal{B}$ we say that the basis is a *Henstock basis*. Otherwise it is called a *McShane basis* and we do not consider such bases here. For a set $E \subset X$ and $\beta \in \mathcal{B}$ we write

$$\beta(E) := \{(I, x) \in \beta : I \subset E\} \quad \text{and} \quad \beta[E] := \{(I, x) \in \beta : x \in E\}.$$

Certain additional hypotheses guarantee some nice properties of a basis. For example, it is useful to suppose that the basis $\mathcal{B}$ *ignores no point*, i.e., $\beta[\{x\}] \neq \emptyset$ for any point $x \in X$ and for any $\beta \in \mathcal{B}$.

If X is a metric or a topological space it is usually supposed that $\mathcal{B}$ is a *Vitali basis* by which we mean that for any x and for any neighborhood $U(x)$ of x there exists $\beta_x \in \mathcal{B}$ such that $I \subset U(x)$ for each pair $(I, x) \in \beta_x$.

A *β-partition* is a finite collection π of elements of β, where the distinct elements (I', x') and (I'', x'') in π have I' and I'' disjoint (or at least non-overlapping, i.e., $\mu(I' \cap I'') = 0$). Let $L \in \mathcal{I}$. If $\pi \subset \beta(L)$ then π is called β-partition *in* L, if $\bigcup_{(I,x)\in\pi} I = L$ then π is called β-partition *of* L.

We say that a basis $\mathcal{B}$ has the *partitioning property* if the following conditions hold: (i) for each finite collection $I_0, I_1, ..., I_n$ of $\mathcal{B}$-intervals with $I_1, ...I_n \subset I_0$ the difference $I_0 \setminus \bigcup_{i=1}^n I_i$ can be expressed as a finite union of pairwise non-overlapping $\mathcal{B}$-intervals; (ii) for each $\mathcal{B}$-interval I and for any $\beta \in \mathcal{B}$ there exists a β-partition of I.

Definition 2.1. Let $\mathcal{B}$ be a basis having the partitioning property and $L \in \mathcal{I}$. A function f on L is said to be *$H_{\mathcal{B}}$-integrable* on L, with *$H_{\mathcal{B}}$-integral* A, if for every $\varepsilon > 0$, there exists $\beta \in \mathcal{B}$ such that for any β-partition π of L we have:

$$\left| \sum_{(I,x)\in\pi} f(x)\mu(I) - A \right| < \varepsilon.$$

We denote the integral value A by $(H_{\mathcal{B}}) \int_L f$.

The following extension of the previous definition is useful in many cases.

Definition 2.2. A function f defined almost everywhere on $L \in \mathcal{I}$ is *$H_{\mathcal{B}}$-integrable* on L, with *$H_{\mathcal{B}}$-integral* A, if the function

$$f_1(x) := \begin{cases} f(x) & \text{if it is defined,} \\ 0 & \text{otherwise} \end{cases}$$

is $H_{\mathcal{B}}$-integrable on L and its $H_{\mathcal{B}}$-integral is equal A.

Let F be an additive set function on $\mathcal{I}$ and E an arbitrary subset of X. For a fixed $\beta \in \mathcal{B}$, we set

$$Var(E, F, \beta) := \sup_{\pi\subset\beta[E]} \sum |F(I)|.$$

We put also

$$V_F(E) = V(E, F, \mathcal{B}) := \inf_{\beta\in\mathcal{B}} Var(E, F, \beta).$$

The extended real-valued set function $V_F(\cdot)$ is called *variational measure* generated by F, with respect to the basis $\mathcal{B}$. It is an outer measure and, in the case of a metric space X, a metric outer measure (in the last case it should be assumed that the basis is a Vitali basis).

Given a set function $F : \mathcal{I} \to \mathbb{R}$ we define the *upper* and the *lower $\mathcal{B}$-derivative* at a point x, with respect to the basis $\mathcal{B}$ and measure μ, as

$$\overline{D}_{\mathcal{B}}F(x) := \inf_{\beta\in\mathcal{B}} \sup_{(I,x)\in\beta} \frac{F(I)}{\mu(I)} \quad \text{and} \quad \underline{D}_{\mathcal{B}}F(x) := \sup_{\beta\in\mathcal{B}} \inf_{(I,x)\in\beta} \frac{F(I)}{\mu(I)}, \tag{2.1}$$

respectively. As we have assumed that $\mathcal{B}$ ignores no point then it is always true that $\overline{D}_{\mathcal{B}}F(x) \geq \underline{D}_{\mathcal{B}}F(x)$. If $\overline{D}_{\mathcal{B}}F(x) = \underline{D}_{\mathcal{B}}F(x)$ we call this common value *$\mathcal{B}$-derivative* $D_{\mathcal{B}}F(x)$. For a complex-valued set function $F = \mathrm{Re}F + i\mathrm{Im}F$ we define $\mathcal{B}$-derivative at a point x as $D_{\mathcal{B}}F(x) := D_{\mathcal{B}}\mathrm{Re}F(x) + D_{\mathcal{B}}\mathrm{Im}F(x)$.

We say that a set function F, real- or complex-valued, is *$\mathcal{B}$-continuous* at a point x, with respect to the basis $\mathcal{B}$, if $V_F(\{x\}) = 0$.

We shall need the following (see [11, Proposition 1.6.4])

Proposition 2.1. *Let an additive complex-valued function F defined on $\mathcal{I}$ be $\mathcal{B}$-differentiable on $L \in \mathcal{I}$ outside a set $E \subset L$ such that $V_F(E) = 0$. Then the function*

$$f(x) := \begin{cases} D_{\mathcal{B}}F(x) & \text{if it exists,} \\ 0 & \text{if } x \in E \end{cases}$$

is $H_{\mathcal{B}}$-integrable on L and F is its indefinite $H_{\mathcal{B}}$-integral.

The next theorem is a corollary of the above proposition.

Theorem 2.1. *Let an additive function $F : \mathcal{I} \to \mathbb{R}$ be $\mathcal{B}$-differentiable everywhere on $L \in \mathcal{I}$ outside of a set E with $\mu(E) = 0$, and $-\infty < \underline{D}_{\mathcal{B}}F(x) < \overline{D}_{\mathcal{B}}F(x) < +\infty$ everywhere on E except on a countable set $M \subset E$ where F is $\mathcal{B}$-continuous. Then the function*

$$f(x) := \begin{cases} D_{\mathcal{B}}F(x) & \text{if it exists,} \\ 0 & \text{if } x \in E \end{cases}$$

is $H_{\mathcal{B}}$-integrable on L and F is its indefinite $H_{\mathcal{B}}$-integral.

3 Dyadic basis in the theory of Walsh and Haar series. We consider here one of the simplest derivation basis, namely the dyadic basis on $X = [0, 1]$ and on $[0, 1]^m$.

In the case of $[0, 1]$ the family $\mathcal{I}$ of $\mathcal{B}$-intervals is constituted by the closures of dyadic intervals

$$J_j^{(n)} := \left[\frac{j}{2^n}, \frac{j+1}{2^n}\right), \quad 0 \le j \le 2^n - 1, \quad n = 0, 1, 2, \dots .$$

If $X = [0, 1]^m$, $\mathcal{B}$-intervals are defined as the closures of m-dimensional dyadic intervals

$$J_{\mathbf{j}}^{(\mathbf{n})} := J_{j_1}^{(n_1)} \times \dots \times J_{j_m}^{(n_m)} \tag{3.1}$$

where $\mathbf{j} = (j_1, \dots , j_m)$ and $\mathbf{n} = (n_1, \dots , n_m)$. We denote this family $\mathcal{I}_d$.

To define a dyadic basis it is enough to define basis sets β. For $X = [0, 1]$ we put

$$\beta_\delta := \{I \in \mathcal{I}_d : \quad I \subset U(x, \delta(x))\} ,$$

where δ is a so-called gauge, i.e, a positive function defined on X, and $U(x, \delta)$ denotes the neighborhood of x of radius δ. So the *dyadic basis* is defined as $\mathcal{B}_d := \{\beta_\delta : \quad \delta : X \to (0, \infty)\}$.

In the m-dimensional case we consider two dyadic basis. The first one is defined exactly as above with $\mathcal{I}_d$ being the family of all closed m-dimensional dyadic intervals. The second one is called a regular dyadic basis. To define it we use the notion of regularity. The *parameter of regularity* of a dyadic interval of the form (3.1) is defined as

$$\min_{i,l} \{|J_{j_i}^{(n_i)}|/|J_{k_l}^{(n_l)}|\}.$$

Analogously the *parameter of regularity of a vector* $\mathbf{a} = (a_1, \dots , a_m)$ is defined as

$$\min_{i,l} \{a_i/a_l\}.$$

We write $reg(J)$ (resp. $reg(\mathbf{a})$) for the parameter of regularity of a dyadic interval J (resp. of a vector $\mathbf{a}$).

Now basis sets of *ρ-regular dyadic basis* $\mathcal{B}_{d,\rho}$ we define as

$$\beta_{\delta,\rho} := \{I \in \mathcal{I}_d : \quad I \subset U(x, \delta(x)), \quad reg(I) \ge \rho\} .$$

Applying Definition 2.1 to these dyadic bases we obtain $H_{\mathcal{B}_d}$*-integral* (the *dyadic Henstock integral*) and $H_{\mathcal{B}_{d,\rho}}$*-integral* (the *ρ-regular dyadic Henstock integral*).

We show now that these integrals solve the coefficient problem for Walsh and Haar series. We remind the definitions (see [9] and [16]).

First we define the *Rademacher functions* r_n, $n = 0, 1, \dots ,$ on $[0, 1)$:

$$r_n(x) := \begin{cases} 1 & \text{if } x \in J_j^{(n+1)},\ k = 0, 2, \dots , 2^{n+1} - 2, \\ -1 & \text{if } x \in J_j^{(n+1)},\ j = 1, 3, \dots , 2^{n+1} - 1. \end{cases}$$

The *Walsh functions* are defined as products of Rademacher functions. To this end we use the dyadic representation for $n \geq 0$:

$$n = \sum_{i=0}^{\infty} n_i 2^i,$$

where $n_i = 0$ or 1 and the sum is in fact finite, and we put

$$w_n(x) := \prod_{i=0}^{\infty} (r_i(x))^{n_i}.$$

In particular $w_0 \equiv 1$.

Now we define the *Haar functions* on $[0, 1)$. Put $\chi_0(x) \equiv 1$. If $n = 2^k + i$ ($k = 0, 1, \dots ,$ $i = 0, \dots , 2^k - 1$), we put

$$\chi_n(x) := \begin{cases} 2^{k/2}, & \text{if } x \in \left[\frac{2i-2}{2^{k+1}}, \frac{2i-1}{2^{k+1}}\right), \\ -2^{k/2}, & \text{if } x \in \left[\frac{2i-1}{2^{k+1}}, \frac{2i}{2^{k+1}}\right), \\ 0, & \text{if } x \in [0, 1) \setminus \left[\frac{2i-2}{2^{k+1}}, \frac{2i}{2^{k+1}}\right). \end{cases}$$

An *m-dimensional Walsh* (resp. *Haar*) *series* is defined by

$$\sum_{\mathbf{n}=\mathbf{0}}^{\infty} b_{\mathbf{n}} w_{\mathbf{n}}(\mathbf{x}) := \sum_{n_1=0}^{\infty} \dots \sum_{n_m=0}^{\infty} b_{n_1,\dots,n_m} \prod_{i=1}^{m} w_{n_i}(x_i) \tag{3.2}$$

$$\left(\text{resp. } \sum_{\mathbf{n}=\mathbf{0}}^{\infty} a_{\mathbf{n}} \chi_{\mathbf{n}}(\mathbf{x}) := \sum_{n_1=0}^{\infty} \dots \sum_{n_m=0}^{\infty} a_{n_1,\dots,n_m} \prod_{i=1}^{m} \chi_{n_i}(x_i) \right) \tag{3.3}$$

where $a_{\mathbf{n}}$ and $b_{\mathbf{n}}$ are real numbers. If $\mathbf{N} = (N_1, \dots , N_m)$, then the $\mathbf{N}$*th rectangular partial sum* $S_{\mathbf{N}}$ of series (3.2) (resp. (3.3)) at a point $\mathbf{x} = (x_1, \dots , x_m)$ is

$$S_{\mathbf{N}}(\mathbf{x}) := \sum_{n_1=0}^{N_1-1} \dots \sum_{n_m=0}^{N_m-1} b_{\mathbf{n}} \omega_{\mathbf{n}}(\mathbf{x}) \quad \left(\text{resp. } S_{\mathbf{N}}(\mathbf{t}) := \sum_{n_1=0}^{N_1-1} \dots \sum_{n_m=0}^{N_m-1} a_{\mathbf{n}} \chi_{\mathbf{n}}(\mathbf{x}) \right).$$

The series (3.2) (or (3.3)) *rectangularly converges* to sum $S(\mathbf{x})$ at a point $\mathbf{x}$ if

$$S_{\mathbf{N}}(\mathbf{x}) \to S(\mathbf{x}) \text{ as } \min_i \{N_i\} \to \infty.$$

We consider also the regular convergence of series. Let $\rho \in (0, 1]$; then the series (3.2) (or (3.3)) *ρ-regularly converges* to a sum $S(\mathbf{x})$ at a point $\mathbf{x}$ if

$$S_{\mathbf{N}}(\mathbf{x}) \to S(\mathbf{x}) \text{ as } \min_i \{N_i\} \to \infty \text{ and } reg(\mathbf{N}) \geq \rho.$$

It is obvious that if the series (3.2) (or (3.3)) rectangularly converges to a sum $S(\mathbf{x})$ at a point $\mathbf{x}$ then for every $\rho \in (0,1]$ this series ρ-regularly converges to $S(\mathbf{x})$ at $\mathbf{x}$.

A starting point for an application of the dyadic derivative and the dyadic integral to the theory of Walsh and Haar series is an observation that due to martingale properties of the partial sums $S_{2^{\mathbf{k}}}$ of those series (here $2^{\mathbf{k}}$ stand for $(2^{k_1}, \dots, 2^{k_m})$) the integral $\int_{J_{\mathbf{j}}^{(\mathbf{k})}} S_{2^{\mathbf{k}}}$ defines an additive $\mathcal{B}_d$-interval function $\psi(I)$ on the family $\mathcal{I}_d$ of all dyadic intervals. (In dyadic analysis the function ψ is sometimes referred to as *quasi-measure* (see [16, 25]).) Since the sum $S_{2^{\mathbf{k}}}$ is constant on each $J_{\mathbf{j}}^{(\mathbf{k})}$ we get

$$S_{2^{\mathbf{k}}}(\mathbf{x}) = \frac{1}{|J_{\mathbf{j}}^{(\mathbf{k})}|} \int_{J_{\mathbf{j}}^{(\mathbf{k})}} S_{2^{\mathbf{k}}} = \frac{\psi(J_{\mathbf{j}}^{(\mathbf{k})})}{|J_{\mathbf{j}}^{(\mathbf{k})}|} \tag{3.4}$$

for any point $\mathbf{x} \in J_{\mathbf{j}}^{(\mathbf{k})}$.

Another simple observation which is essential for establishing that a given Walsh or Haar series is the Fourier series in the sense of some general integral, is the following statement (see [21, Proposition 4]).

Proposition 3.1. *Let some integration process $\mathcal{A}$ be given which produces an integral additive on $\mathcal{B}_d$. Assume a series of the form* (3.2) *or* (3.3) *is given. Let the $\mathcal{B}_d$-interval function ψ be constructed for this series by* (3.4)*. Then this series is the Fourier series of an $\mathcal{A}$-integrable function f if and only if $\psi(I) = (\mathcal{A}) \int_I f$ for any dyadic interval I.*

It is seen from formula (3.4) that, at least for points with dyadic-irrational coordinates, rectangular (respectively, ρ-regular rectangular) convergence of the series (3.2) (or (3.3)) at a point $\mathbf{x}$ to a sum $f(x)$ implies $\mathcal{B}_d$-differentiability (respectively, $\mathcal{B}_{d,\rho}$-differentiability) of the function ψ in $\mathbf{x}$ with $f(x)$ being the value of the $\mathcal{B}_d$-derivative (respectively, $\mathcal{B}_{d,\rho}$-derivative).

So in order to solve the coefficient problem it is enough to show that the function ψ is an integral of its derivative which exists at least almost everywhere. Then in view if Proposition 3.1 we get

Theorem 3.1. *If the series* (3.2) (*or* (3.3)) *is rectangular* (*respectively, ρ-regular rectangular*) *convergent to a sum f almost everywhere on $[0,1)^m$, outside a set E such that $V_\psi(E) = 0$, then the function f is $H_{\mathcal{B}_d}$-integrable* (*respectively, $H_{\mathcal{B}_{d,\rho}}$-integrable*) *and* (3.2) (*or* (3.3))*is the Fourier series of f, in the sense of the respective integral.*

To use this theorem we need some additional information related to the behavior of a series on the exceptional set which would imply that the variational measure V_ψ is equal zero on this set. Such a nice behavior of ψ on the exceptional set can be obtained either from a convergence condition or from some additional growth assumptions imposed on the series. For example, it can be easily shown, in the one dimentional case, that if the coefficients of a series 3.2 satisfy the condition $\lim_{n\to\infty} b_n = 0$ (which is a consequence of the convergence of the series at least at one dyadic-irrational point) then ψ is $\mathcal{B}_d$-continuous everywhere on $[0,1)$, and we apply Theorem 2.1 to get

Theorem 3.2. *If the series* (3.2) (*in one dimension*) *is convergent to a sum f at each dyadic irrational point, then f is $H_{\mathcal{B}_d}$-integrable and* (3.2) *is the Fourier-Walsh series of f, i.e.,*

$$b_n = (H_{\mathcal{B}_d}) \int_{[0,1)} f w_n.$$

For more details in the multidimensional case see [12], [13], [14] and [21].

4 Approximate symmetric basis and the coefficients problem for trigonometric series. A Henstock-type integral solving the coefficients problem in the case of trigonometric series was introduced by D.Preiss and B.Thomson in [15]. The idea of a gauge integral associated with a symmetric basis was mentioned in Henstock's book [6].

In [15] (see also [24]) the approximate symmetric basis was used for the construction of the integral.

Definition 4.1. *Approximate symmetric gauge is a measurable set $\mathcal{S} \subset \mathbb{R} \times (0, \infty)$ such that for any $x \in \mathbb{R}$ the set $\{t : (x, t) \in \mathcal{S}\}$ is measurable and*

$$\lim_{h \to 0^+} \frac{\mu\big(\{t \in (0, h) : (x, t) \notin \mathcal{S}\}\big)}{h} = 0. \tag{4.1}$$

Each approximate symmetric gauge $\mathcal{S}$ generate a basis set

$$\beta_{\mathcal{S}} := \big\{([x - t, x + t], x) : (x, t) \in \mathcal{S}\big\}. \tag{4.2}$$

So the approximate symmetric basis is

$$\mathcal{B}_{\mathrm{ap,\,sym}} := \big\{\beta_{\mathcal{S}} : \mathcal{S}\,\text{approximate symmetric gauge}\big\} \tag{4.3}$$

Unfortunately the approximate symmetric basis does not have the partitioning property in the form it was formulated above in Section 2. But it has this property in a weaker form (see [24]):

Theorem 4.1. *For each basis set $\beta_{\mathcal{S}}$ there is a set N of measure zero such that for every interval with endpoint $\mathbb{R} \setminus N$ there exists a $\beta_{\mathcal{S}}$-partition of this interval.*

Since in the trigonometric series theory we need to integrate only periodic functions, the following corollary of the above result is essential.

Theorem 4.2. *If $T > 0$, then for each basis set $\beta_{\mathcal{S}}$ and for some $a \in \mathbb{R}$ there exists a $\beta_{\mathcal{S}}$-partition of the interval $[a, a + T]$.*

Now with this theorem the following "periodic" definition of the approximate symmetric Henstock integral is available.

Definition 4.2. T-periodic function f is *ASH-integrable* over period T and its *ASH-integral* is A if for any $\varepsilon > 0$ there exists $\mathcal{S}$ such that

$$\left| \sum_{(I,x) \in \pi} f(x)\mu(I) - A \right| < \varepsilon$$

for all $\beta_{\mathcal{S}}$-partition π of interval $[a, a+T]$ where a is taken from theorem 4.2. We denote A by $(ASH) \int_{(T)} f$.

Now the coefficient problem for trigonometric series is solved by the following theorem.

Theorem 4.3. *If a trigonometric series*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is convergent everywhere to a function f then f and the functions $x \mapsto f(x)\cos kx$, $x \mapsto f(x)\sin kx$, $k = 1, 2, 3, \ldots$, are ASH-integrable and

$$a_0 = \frac{1}{\pi} \cdot (ASH) \int_{(2\pi)} f(x), \quad a_k = \frac{1}{\pi} \cdot (ASH) \int_{(2\pi)} f(x)\cos kx,$$
$$b_k = \frac{1}{\pi} \cdot (ASH) \int_{(2\pi)} f(x)\sin kx.$$

We note that no multidimensional analogue of ASH-integral is known up to now and the only available way to solve the coefficient problem in dimension greater then one is to use an iterated integration (see [20]).

5 Derivation basis in zero dimensional group and the respective Henstock-type integral. Now we show how a Henstock-type integral can be defined on compact subsets of a locally compact zero-dimensional abelian group and can be used to recover, by generalized Fourier formulae, the coefficients of series with respect to characters of such a group, in a compact case, and to obtain an inversion formula for multiplicative integral transforms, in a locally compact case.

Let G be a zero-dimensional locally compact abelian group G which satisfies the second countability axiom. We suppose also that the group G is periodic. It is known (see [1]) that a topology in such a group can be given by a chain of subgroups

$$\text{(5.1)} \qquad .... \supset G_{-n} \supset ... \supset G_{-2} \supset G_{-1} \supset G_0 \supset G_1 \supset G_2 ... \supset G_n \supset ...$$

with $G = \bigcup_{n=-\infty}^{+\infty} G_n$ and $\{0\} = \bigcap_{n=-\infty}^{+\infty} G_n$. The subgroups G_n are clopen sets with respect to this topology. As G is periodic, the factor group G_n/G_{n+1} is finite for each n and this implies that G_n (and so also all its cosets) is compact. Note that the factor group G_n/G_0 is also finite for any $n < 0$ and so the factor group G/G_0 is countable. We denote by K_n any coset of the subgroup G_n and by $K_n(g)$ the coset of the subgroup G_n which contains the element g, i.e., $K_n(g) := g + G_n$. For each $g \in G$ the sequence $\{K_n(g)\}$ is decreasing and $\{g\} = \bigcap_n K_n(g)$.

Let Γ denotes the dual group of G, i.e., the group of characters of the group G. It is known (see [1]) that under assumption imposed on G the group Γ is also a periodic locally compact zero-dimensional abelian group (with respect to the point-wise multiplication of characters) and we can represent it as a sum of increasing sequence of subgroups

$$\text{(5.2)} \qquad .... \supset \Gamma_{-n} \supset ... \supset \Gamma_{-2} \supset \Gamma_{-1} \supset \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset ... \supset \Gamma_n \supset ...$$

introducing a topology in Γ. Then $\Gamma = \bigcup_{i=-\infty}^{+\infty} \Gamma_i$ and $\bigcap_{i=-\infty}^{+\infty} \Gamma_i = \{\gamma^{(0)}\}$ where $(g, \gamma^{(0)}) = 1$ for all $g \in G$ (here and below (g, γ) denote the value of a character γ at a point g). For each $n \in \mathbb{Z}$ the group Γ_{-n} is the annulator of G_n, i.e.,

$$\Gamma_{-n} = G_n^{\perp} := \{\gamma \in \Gamma : (g, \gamma) = 1 \ \text{ for all } \ g \in G_n\}.$$

The factor groups $\Gamma_{-n-1}/\Gamma_{-n} = G_{n+1}^{\perp}/G_n^{\perp}$ and G_n/G_{n+1} are isomorphic and so they are of finite order for each $n \in \mathbb{Z}$. This implies that the group Γ_{-n}/Γ_0 is also finite for any $n > 0$ and Γ/Γ_0 is countable.

We denote by μ_G and μ_Γ the Haar measures on the groups G and Γ, respectively, and we normalize them so that $\mu_G(G_0) = \mu_\Gamma(\Gamma_0) = 1$.

Now we define a derivation basis $\mathcal{B}_G$ on the measure space $(G, \mathcal{M}, \mu_G)$. Take any function $\nu : G \to \mathbb{Z}$ and define a basis set by

$$\beta_\nu := \{(I, g) : g \in G, I = K_n(g), n \geq \nu(g)\}.$$

So our basis $\mathcal{B}_G$ in G is the family $\{\beta_\nu\}_\nu$ where ν runs over the set of all integer-valued functions on G. This basis has all the properties described in Section 2 for the general derivation basis, in particular it is a Vitali basis. Note that the set $\mathcal{I}_G$ of all $\mathcal{B}_G$-intervals in our case is composed by all the cosets $K_n,\ n \in \mathbb{Z}$. The partitioning property of $\mathcal{B}_G$ follows easily from compactness of any $\mathcal{B}_G$-interval and from the fact that any two $\mathcal{B}_G$-intervals K' and K'' are either disjoint or one of them is contained in the other one.

Definition 2.1 of the $H_{\mathcal{B}}$-integral can be rewritten for the basis $\mathcal{B}_G$ in the following form (see [22]):

Definition 5.1. Let $L \in \mathcal{I}_G$. A complex-valued function f on L is said to be *H_G-integrable* on L, with *H_G-integral* A, if for every $\varepsilon > 0$, there exists a function $\nu : L \to \mathbb{Z}$ such that for any β_ν-partition π of L we have:

$$\left| \sum_{(I,g)\in\pi} f(g)\mu_G(I) - A \right| < \varepsilon.$$

We denote the integral value A by $(H_G)\int_L f$.

Remark 5.1. We note that all the above definitions depend on the structure of the sequence of subgroups (5.1). So if we consider for the group Γ the definitions of the $\mathcal{B}_\Gamma$-basis and the H_Γ-integral, then we should use the sequence (5.2) in our construction.

The *upper* and the *lower $\mathcal{B}_G$-derivative* of a set function $F : \mathcal{I}_G \to \mathbb{R}$ at a point g can be rewritten, in the case of the basis $\mathcal{B}_G$ and measure μ_G, as

$$\overline{D}_G F(g) := \limsup_{n\to\infty} \frac{F(K_n(g))}{\mu_G(K_n(g))}\ , \quad \underline{D}_G F(g) := \liminf_{n\to\infty} \frac{F(K_n(g))}{\mu_G(K_n(g))}. \tag{5.3}$$

The *$\mathcal{B}_G$-derivative* at g is

$$D_G F(g) := \lim_{n\to\infty} \frac{F(K_n(g))}{\mu_G(K_n(g))}. \tag{5.4}$$

Note that in the case of our basis $\mathcal{B}_G$, given a point g, any β_ν-partition contains only one pair (I, g) with this point g. Because of this we can reformulate the definition of $\mathcal{B}$-continuity in a simpler way, saying that a set function F is *$\mathcal{B}_G$-continuous* at a point g, with respect to the basis $\mathcal{B}_G$, if $\lim_{n\to\infty} F(K_n(g)) = 0$.

As in the general case considered in Section 2, the indefinite H_G-integral on $L \in \mathcal{I}_G$ is an additive $\mathcal{B}_G$-continuous function on the set of all $\mathcal{B}_G$-subintervals of L. Moreover, it is $\mathcal{B}_G$-differentiable almost everywhere (see [22]) and $D_G F(g) = f(g)$ a.e. on L.

If the group G is compact and so the chain (5.1) is reduced to the one-side sequence $G = G_0 \supset G_1 \supset \ldots \supset G_n \supset \ldots$, then in this case the H_G-integral is defined on the whole group G. Moreover, the group Γ of characters of the group G is discrete now (see [1]) and it can be represented as a sum of increasing chain of finite subgroups $\Gamma_0 \subset \Gamma_{-1} \subset \ldots \subset \Gamma_{-n} \subset \ldots$ where $\Gamma_0 = \{\gamma^{(0)}\}$ with $(g, \gamma^{(0)}) = 1$ for all $g \in G$. So the characters γ constitute a countable orthogonal system on G with respect to normalized measure μ_G and we can consider a series

$$\sum_{\gamma\in\Gamma} a_\gamma \gamma \tag{5.5}$$

with respect to this system. We define a convergence of this series at a point g as the convergence of its partial sums

$$S_n(g) := \sum_{\gamma\in\Gamma_{-n}} a_\gamma(g,\gamma) \tag{5.6}$$

when n tends to infinity.

We associate with the series (5.5) a function F defined on each coset K_n by

$$F(K_n) := \int_{K_n} S_n(g)d\mu_G. \tag{5.7}$$

It can be proved that F is an additive function on the family of all $\mathcal{B}_G$-intervals. As the sum S_n, defined by (5.6), is constant on each K_n, then (5.7) implies

$$S_n(g) = \frac{F(K_n(g))}{\mu_G(K_n(g))}. \tag{5.8}$$

So we have a situation similar to the one we had in Section 3. An analogue of Proposition 3.1 is

Theorem 5.1. *The series* (5.5) *is the* H_G*-Fourier series of some* H_G*-integrable function* f *if and only if the function* F *associated with this series by expression* (5.7) *coincides on each* $\mathcal{B}_G$*-interval* I *with the indefinite integral* $(H_G)\int_I f$.

The following two lemmas are immediate consequences of the equality (5.8).

Lemma 5.1. *If the series* (5.5) *converges at some point* $g \in G$ *to a value* $f(g)$ *then the associated function* F (*see* (5.7)) *is* $\mathcal{B}_G$*-differentiable at* g *and* $D_GF(g) = f(g)$*. Moreover if the series* (5.5) *satisfies at a point* g *the conditions*

$$-\infty < \liminf_{n\to\infty} \mathrm{Re}S_n(g) \le \limsup_{n\to\infty} \mathrm{Re}S_n(g) < +\infty \tag{5.9}$$

$$-\infty < \liminf_{n\to\infty} \mathrm{Im}S_n(g) \le \limsup_{n\to\infty} \mathrm{Im}S_n(g) < +\infty \tag{5.10}$$

then the associated function F *satisfies the inequalities*

$$-\infty < \underline{D}_G\mathrm{Re}F(g) \le \overline{D}_G\mathrm{Re}F(g) < +\infty, \tag{5.11}$$

$$-\infty < \underline{D}_G\mathrm{Im}F(g) \le \overline{D}_G\mathrm{Im}F(g) < +\infty. \tag{5.12}$$

Lemma 5.2. *If the partial sums* (5.6) *satisfy at a point* g *the condition*

$$S_n(g) = o\left(\frac{1}{\mu_G(K_n(g))}\right) \tag{5.13}$$

then the associated function F *is* $\mathcal{B}_G$*-continuous at the point* g.

Now having in mind the above results we can use theorem 2.1 to get

Theorem 5.2. *Suppose that the partial sums* (5.6) *of the series* (5.5) *converge almost everywhere on* G *to a function* f *and satisfy the conditions* (5.9) *and* (5.10) *everywhere on* G *except on a countable set* M*, where* (5.13) *holds. Then* f *is* H_G*-integrable in the sense of Definition 2.2* (*applied to the basis* $\mathcal{B}_G$) *and* (5.5) *is the* H_G*-Fourier series of* f.

The following theorem is a particular case of Theorem 5.2.

Theorem 5.3. *Suppose that the partial sums* (5.6) *of the series* (5.5) *converge to a function* f *everywhere on* G. *Then* f *is* H_G*-integrable on* G *and the series* (5.5) *is the* H_G*-Fourier series of* f.

The following theorem is a generalization of Theorem 5.3 in a locally compact case (see Remark 5.1 for the notation).

Theorem 5.4. *If the limit*

$$\lim_{n\to\infty}(H_\Gamma)\int\limits_{\Gamma_{-n}} a(\gamma)(g,\gamma)d\mu_\Gamma$$

exists at each $g\in G$ *and its value is* $f(g)$, *where* $a(\gamma)$ *is a locally* H_Γ*-integrable function, then* f *is* H_G*-integrable on* G_{-n} *for each* n *and*

$$a(\gamma)=\lim_{n\to\infty}(H_G)\int_{G_{-n}} f(g)\overline{(g,\gamma)}d\mu_G \quad \text{a.e. on } \Gamma. \tag{5.14}$$

We mention in conclusion that the Vilenkin multiplicative system (see [1]) is a particular example of a system of characters of a zero-dimensional compact abelian group. So the previous results of this section are applicable to that system. A suitable Henstock-type integral in this case is so-called P-adic Henstock integral (see [21]).

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