THE INTEGRAL À LA HENSTOCK

Lee Peng Yee

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Abstract. Henstock provided a unified approach to many integrals in use. The author put on record what he knows about the theory and its development. Finally, he gives a personal view on the future of the theory.

1 A Unified Approach

Calculus is a gateway to advanced mathematics. The key concepts in calculus are derivative and anti-derivative. The integral is defined as an anti-derivative. It is often called the Newton integral. Hence the chain rule in differentiation becomes integration by substitution in integration, and the derivative of the product of two functions becomes integration by parts in integration. To integrate a function, we differentiate another function so that its derivative is the given function. This other function is called the primitive of the given function. Then the Newton integral of a given function is its primitive. If we integrate then differentiate or differentiate then integrate, we get back to the same function. This property is known as the fundamental theorem of calculus. The integral is regarded as a mapping of a function into another function, namely the primitive.

Another approach to integration is taking the generalized limit of Riemann sums. What we have defined is called the Riemann integral. We often define the Riemann integral on a finite interval $[a, b]$. Unfortunately, there are functions that are Newton integrable on $[a, b]$ but not Riemann integrable there. There are also functions that are Riemann integrable and not Newton integrable on $[a, b]$. When we talk about the Riemann integral, we often think of it as a mapping of a function into an integral value. The fundamental theorem of calculus does not hold for the Riemann integral without imposing further conditions. The condition imposed is usually continuity. That is, if we integrate a continuous function into its indefinite integral then when we differentiate we obtain the original continuous function.

The next step after the Riemann integral is to introduce the improper Riemann integral. We need the improper integral for applications. The improper Riemann integral includes the Riemann integral, and intersects with the Newton integral. Again the Newton integral and the improper Riemann integral do not include each other.

A student would normally learn the Newton integral first, then the Riemann integral, after which the improper Riemann integral. However when he moves on to his graduate study, he abandons everything and learns a new integral called the Lebesgue integral. The Lebesgue integral is powerful in certain ways and serves the purpose in many aspects. Still it does not include the Newton integral. In the 60s and 70s there was a movement trying to get rid of the Riemann integral in the undergraduate study. It did not succeed. The Riemann integral has its place and its use that cannot be replaced by the Lebesgue integral,

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for example, in numerical integration. The attempt of Henstock, and also that of Kurzweil, in 1957-58 was to extend the Riemann integral so that it includes the Lebesgue integral and hence to provide a unified approach to the many integrals in use.

In early days, one pleasure of Henstock was to prove that his integral includes another known integral. The Henstock integral is now known as the Kurzweil-Henstock integral, since Kurzweil defined the same integral though they went different ways in developing and applying the theory. It is common knowledge that the Kurzweil-Henstock integral is equivalent to the Denjoy integral and the Perron integral. In fact, its general form also includes the approximate Perron integral, the Haar integral, the Ito integral, and the Feynman integral. For simplicity, we shall refer to the Kurzweil-Henstock integral and its general form as the Henstock integral in this article.

It is a myth that we need the countable additivity property in order to define the Lebesgue integral or the integral in measure theory. Henstock showed that it can be done using only finite operations, namely the Riemann sums. The Henstock theory of integration is now fully developed. It has yet to be included as a standard course at the undergraduate or graduate level in the universities.

I was a student of Henstock between 1961 and 1965. I attended his first series of lectures on the integral for an honours class, that is the fourth and final year in the degree programme of a British university. His first book published in 1963 derived from this set of lecture notes. In this article, I shall put on record what I know about the theory and its development. At the end of the article, I shall also give a personal view on the future of the theory.

2 The First Idea

The first idea came from the definition of derivative. Suppose F is differentiable at a point x, and $F'(x) = f(x)$. Then for every $\epsilon > 0$ there is $\delta > 0$ such that whenever $|y - x| < \delta$ we have

$$|\frac{F(y) - F(x)}{y - x} - f(x)| < \epsilon.$$

Here δ depends on x. So we should write $\delta(x) > 0$. Alternatively, we write

$$|F(y) - F(x) - f(x)(y - x)| < \epsilon|y - x|.$$

Here y may lie on the right side of x or on the left side. So we could also write it in two inequalities:

$$|F(v) - F(x) - f(x)(v - x)| < \epsilon|v - x|$$

when $v > x$, and

$$|F(x) - F(u) - f(x)(x - u)| < \epsilon|x - u|$$

when $x > u$. Note that the first inequality above involves an interval $[x, v]$ and the left endpoint and the second inequality involves an interval $[u, x]$ and the right endpoint. Hence the first idea is not to consider a point function $f(x)$ or, as in the definition of the Riemann integral, interval functions

$$\sup_{t\in[u,v]} f(t)(v - u) \quad \text{and} \quad \inf_{t\in[u,v]} f(t)(v - u).$$

The first idea is to consider functions of interval-point pairs, namely $([u, v], x)$ where $x = u$ or v.

To facilitate the description of such functions, Henstock invented the following notation. Instead of writing

$$\sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}),$$

he writes

$$(D)\sum f(x)(v-u),$$

in which D denotes the division $x_0 < x_1 < ... < x_n$, $[u, v]$ is a typical interval in D, and $x = u$ or v as required.

Now suppose $F'(x) = f(x)$ for $x \in [a, b]$. Then for every $\epsilon > 0$ there is $\delta(x) > 0$ such that for any division D of $[a, b]$ with $([u, v], x)$ being a typical interval-point pair in D satisfying $0 \leq v - x < \delta(x)$ and $0 \leq x - u < \delta(x)$ we have

$$\begin{aligned}
&|F(b) - F(a) - (D)\sum f(x)(v-u)| \\
&= |(D)\sum\{F(v) - F(u) - f(x)(v-u)\}| \\
&\leq (D)\sum |F(v) - F(u) - f(x)(v-u)| \\
&< (D)\sum \epsilon|v-u| = \epsilon(b-a).
\end{aligned}$$

Hence we have the following definition of the Henstock integral. The above provides a proof that a derivative is integrable in the sense of Henstock. A function f is *Henstock integrable* to A on [a, b] if for every $\epsilon > 0$ there is $\delta(x) > 0$ such that for any division D of $[a, b]$ with $([u, v], x)$ being a typical interval-point pair in D satisfying $0 \leq v - x < \delta(x)$ and $0 \leq x - u < \delta(x)$ we have

$$|A - (D)\sum f(x)(v-u)| < \epsilon.$$

So Henstock succeeded in defining an integral of the Riemann type. He called it the *Riemann-complete* integral. As it happened, the integral so defined includes the Lebesgue integral and many others.

In his first attempt, Henstock considered interval-point pairs with right endpoint and left endpoint separately. More precisely, there are $\delta_1(x) > 0$ and $\delta_2(x) > 0$ such that for a typical interval-point pair $([u, v], x)$ in D we have

$$0 \leq x - u < \delta_1(x) \text{ and } 0 \leq v - x < \delta_2(x).$$

He did that in his first book [3] published in 1963. However he found out later that such division may not exist. To overcome this, he put $\delta_1(x) = \delta_2(x)$. This did not only address the short-coming and also simplified the definition. Henceforth the Henstock integral is defined as it is today. More precisely, the interval-point pairs are of the form $([u, v], x)$ in which $u \leq x \leq v$, and not restricted to $x = u$ or v with different δ functions.

3 Three Basic Concepts

The three basic concepts in the Henstock theory are: δ-fine divisions, the Saks-Henstock lemma, and the decomposability property. The theory is developed using basically these three concepts.

A division is now a collection of interval-point pairs. A division $D = ([u,v],x)$ of $[a,b]$ is said to be *δ-fine* if there is $\delta(x) > 0$ such that

$$x - \delta(x) < u \leq x \leq v < x + \delta(x).$$

When $x = a$ or b, the above inequality is taken to be one-sided. Henstock called such division *compatible with δ*. In the literature, δ-fine is used. Finally, Henstock himself used δ-fine. Henceforth it is called δ-fine thereafter.

The major issue here is the existence of δ-fine divisions. This follows from the Heine-Borel covering theorem for the case when the integral is defined on $[a,b]$. Therefore the integral is well defined. Henstock proceeded to prove that similar divisions exist for the approximate Perron integral [8], the Haar integral [5] and others. Hence many integrals now have a Riemann-type definition. For example, Xu and Lee proved it for the Ito integral (see reference in [20]) and the Kunugi integral [8], and Muldowney did it for the Feynman integral [16].

To develop the theory, the first important step is to prove the Saks-Henstock lemma. It was called Henstock's lemma at the beginning. However Henstock preferred to call it the Saks-Henstock lemma. In one sentence, the Saks-Henstock lemma says that

$$|F(b) - F(a) - (D)\sum f(x)(v-u)| < \epsilon$$

in the definition can be replaced by

$$(D)\sum |F(v) - F(u) - f(x)(v-u)| < \epsilon.$$

In other words, the absolute value sign can be moved from outside the summation sign to inside the summation sign. The lemma is used to prove the properties of the primitive of an integrable function. For example, if f is integrable $[a,b]$ then its primitive F is continuous there. If f is absolutely integrable on $[a,b]$, that is, both f and $|f|$ are integrable, then its primitive F is of bounded variation. With the Saks-Henstock lemma we are on the way to developing richer properties of the integral.

To move on further, we need to prove theorems involving the interchange of two limit operations. More precisely, suppose there is a sequence of integrable functions f_n using δ_n for $n = 1,2,3,\cdots$ We may assume that $\delta_1(x) \geq \delta_2(x) \geq ... > 0$ for $x \in [a,b]$. Suppose further $f_n(x) \longrightarrow f(x)$ as $n \longrightarrow \infty$ for every x. Henstock always says, in his theory, there is no need to consider pointwise convergence almost everywhere. It is enough to consider only pointwise convergence *everywhere*. Under certain conditions, to prove that f is also Henstock integrable on $[a,b]$, we use a standard technique called the diagonal process. In the language of Henstock, it is called *decomposability*. It says that it is possible to construct a sequence of pairwise disjoint sets $X_j, j = 1,2,3,\cdots$ with union $[a,b]$ such that f is Henstock integrable using δ with $\delta(x) = \delta_j(x)$ for $x \in X_j$. The core of the theory is now complete.

The concept of δ-fine division helps to define the integral and proves its existence. The Saks-Henstock lemma allows us to prove some algebraic and elementary properties of the integral. It is the decomposability, together with the above two concepts, that makes it possible to prove the convergence theorems involving the interchange of two limit operations.

The three basic concepts were clearly presented in an elementary paper by Henstock [4]. The paper is probably the most read paper of Henstock. It was written at the request of a referee of another of his papers. According to the referee, Henstock should write a simple paper first otherwise the reader may not be able to understand his more technical paper submitted. He obliged.

Many known integrals are now special cases of the Henstock integral. They can be given a Riemann-type definition. Then we can go through the whole process again, and at the end of it we obtain the necessary theorems for the given integral. Therefore it is only natural to ask whether there is a general theory of the integral. Indeed, there is. In the measure theory, the building blocks are measurable sets. In the Henstock theory, the building blocks are interval-point pairs. Then we state that the collection of interval-point pairs satisfies certain conditions including the existence of divisions, the conditions giving rise to the Saks-Henstock lemma, and the decomposability. So we may approach the Henstock integral axiomatically as we have done so with measure theory. For details, see [5, 8].

4 Beyond Lanzhou Lectures

Lanzhou lectures [8] was published in 1989. Though it contains some errors, it still serves as a useful platform for my research students. It contains equivalent definitions of the Henstock integral, various convergence theorems, Riemann-type definitions of other integrals, integral representation of an orthogonally additive functional, and some generalizations of the Henstock integral. It also contains substantial amount of skills necessary for doing research in the area. For a modern version of generalized absolutely continuous functions, see [9]. Some years have passed. More works have been done. For a survey written in 2003, see preprint [10]. In what follows, we give a set of random samples of further development of Henstock's ideas after the publication of Lanzhou lectures.

Henstock used the term *interval-point pair*. It came from the Riemann integral. In the definition of the Riemann integral, intervals come first. Then we choose an arbitrary point inside each interval. Hence interval-point pair. However in the case of the Henstock integral, we actually have the point x first then $\delta(x)$ afterwards and finally δ-fine divisions. Perhaps we should call point-interval pair. Zhao [13] gave an approach that we can still consider intervals first then points later as in the Riemann theory. We define the upper sum for a δ-fine division $D = ([u, v], x)$ of $[a, b]$ to be

$$s_H^u(f, D) = (D) \sum \sup f(t)(v - u)$$

in which the sup is taken over all $t \in [u, v] \subset (t - \delta(t), t + \delta(t))$. Similarly, we define the lower sum $s_H^l(f, D)$. *A function f is Henstock integrable to A on $[a, b]$ if and only if*

$$\inf_{\delta(x)} \sup_D \, s_H^u(f, D) = \sup_{\delta(x)} \inf_D \, s_H^l(f, D) = A$$

in which all divisions D of $[a, b]$ are δ-fine for given $\delta(x) > 0$.

In order to differentiate a function with respect to another function, Henstock introduced in [3] a concept called *inner variation*. Roughly speaking, variation is defined over all interval-point pairs whereas inner variation is defined over certain family of interval-point pairs. The following theorem is due to Cabral [1]: *A function is Henstock integrable on*

$[a,b]$ with primitive F if and only if for every $\epsilon > 0$ there is $\delta(x) > 0$ for $x \in [a,b]$ such that for any δ-fine partial division D in Γ_ϵ we have

$$(D)\sum |F(v) - F(u)| < \epsilon \text{ and } (D)\sum |f(x)(v-u)| < \epsilon,$$

where

$$\Gamma_\epsilon = \{([u,v],x) : |F(v) - F(u) - f(x)(v-u)| \geq \epsilon|v-u|\}.$$

The condition above is known as the *double Lusin condition*. It is an extension of inner variation. The idea of Γ_ϵ came from Lu Jitan, independently of Henstock.

I posed the following problem to Henstock: Given the space of measurable functions, recover or find all the Henstock integrable functions in the space. Henstock [6] solved it for the absolute case and Ng [17] for the nonabsolute case. Both had to introduce additional structure into the space of measurable functions. We have not seen a definition of the Henstock integral solely in the language of measure theory.

An interesting by-product of the research in the Henstcok theory is the following result [11]: *A function f is Baire one if and only if for every $\epsilon > 0$ there is $\delta(x) > 0$ for $x \in [a,b]$ such that*

$$|f(x) - f(y)| < \epsilon \text{ whenever } |x-y| < \min\{\delta(x), \delta(y)\}.$$

The original intention was to define a kind of continuity so that f is Henstock integrable if and only if f is almost everywhere continuous in the given sense. We failed. However by using the Henstock approach we manage to produce the above interesting result outside the integration theory.

The items above are those more closely connected to Henstock. We did not mention numerous good works done by Kurzweil, Mawhin, Pfeffer, Thomson, Nakanishi, Bongiorno and his group, and many others.

5 Integration without Tears

In 1964 at a conference in Leicester, England, Henstock presented a short talk titled *integration without tears*. At the end of the talk, someone came out of the room and said he finally understood what Henstock presented. It was the second time he listened to a similar talk. It shows that the concept may be simple, but it takes time for people to understand and to appreciate. In the talk, Henstock presented the definition of the integral and gave two examples of functions that are integrable. The first example was the derivative of a differentiable function and that was where the original definition came from. The second example was the Dirichlet function, and that demonstrated the power of the integral.

Henstock left Queen's University of Belfast in 1964 and took up a readership at the University of Lancaster. Readership is second to professorship in a British university. He made an attempt to teach the integral at the year one undergraduate level. It was disastrous. He had to abandon it. He never tried it again. Others have also taught the integral in real analysis at the undergraduate or graduate level. Some have written textbooks on the subject.

In [3] and later in [5] Henstock presented the proof in closed form. More precisely, he defined the variation and then presented the proof in terms of the variation. The approach was not adopted by others. Most writers use the epsilon-delta approach.

The teaching difficulty of the Henstock integral is in some way intrinsic. Though the epsilon-delta definition is supposed to be constructive, it is not so easy to construct the corresponding δ function. It is not clear how a δ-fine division is structured. There is no covering theorem using the interval-point pairs. The difficulty is similar to that of teaching elementary analysis rigorously.

There is a successful approach to teaching elementary analysis using sequences and inequalities [9]. For example, we can define continuity and uniform continuity of a function using sequences. A function f is *continuous* at a point a if and only if for every $x_n \longrightarrow a$ as $n \longrightarrow \infty$ we have $f(x_n) \longrightarrow f(a)$ as $n \longrightarrow \infty$. A function f is *uniformly continuous* on $[a, b]$ if and only if for every $x_n - y_n \longrightarrow 0$ as $n \longrightarrow \infty$ we have $f(x_n) - f(y_n) \longrightarrow 0$ as $n \longrightarrow \infty$. If we really want to prove a function f continuous at a using epsilon-delta, given $\epsilon > 0$ to find the corresponding δ we prove first the Lipschitz condition $|f(x) - f(a)| \leq M|x - a|$ for some M. Then we put $\delta = \frac{\epsilon}{M}$. We can carry on this approach all the way to results involving uniform convergence and the Riemann integrability of a function. The sequential approach is easier than using epsilon-delta. It is more easily accessible to the undergraduate students.

A sequential approach to the Henstock integral is possible. It is easy to see that f is *Henstock integrable* to A on $[a, b]$ if and only if there is a sequence of positive functions $\delta_j(x), j = 1, 2, 3, \cdots$, such that for every δ_j-fine division D_j we have

$$s(f, \delta_j, D_j) = (D_j) \sum f(x)(v - u) \longrightarrow A \text{ as } j \longrightarrow \infty.$$

This is used as a definition, for example, in [19]. When $\delta_j(x)$ is a constant function for every j, we obtain a sequential definition of the Riemann integral. It has not been tested whether such approach to the Henstock integral is easier for students.

6 Henstock as a Supervisor

He had great insight in the integral he defined and developed. When I was under his supervision, he met me once a fortnight for exactly one hour. I would present him with one sheet of paper stating the results which I thought I have proved or might be able to prove. He would start reading the results in front of me. If a result was properly stated, he could see the proof almost immediately. Then he would move on to the next statement. When in doubt, that is when he looked at the statement longer than usual, I would quickly pull out another sheet of paper with a proof of the statement. He set me hard problems. I could not solve the problems, so I wrote a paper giving a list of the problems I could not solve. Twenty years after my graduation and between 1984 and 1989, my students and I solved all those hard problems.

An instruction from Henstock to conduct my research was to convert all the summation signs in the book *Infinite matrices and sequence spaces* by R. G. Cooke (1950) into integration signs. The idea was that to do research I should find a platform and proceed from there. I copied this technique later when supervising my own students. In fact, *Lanzhou lectures* was written with this purpose in mind, that is, providing a platform for my students.

Another advice from Henstock was to look at the papers published before 1935. The reason was that during that period the Lebesgue integral had not dominated the scene, mathematicians still tried to solve problems of the nonabsolute type using nonabsolute integrals. Some of my students did exactly that and benefited from it. See, for example, [21].

One incident I could not forget was how I was sent to Henstock. I was supposed to work with S. Verblunsky on Fourier series. On the first day of my arrival, I went to see Professor Verblunsky. He said that he would take me in as his student. But, it would be better if I could work in a new area where I would have many more years to work on. So he sent me to a young mathematician down the corridor who had just invented a new integration theory. The young mathematician was Henstock. I was and still am grateful to Verblunsky. Paying forward I make sure that my students will have enough problems to work on even after they graduated.

Before I left for home, I asked what I should look out for in the next 20 years. He mentioned stability of differential equations and stochastic analysis. That was 1965. His prediction was correct. Stochastic analysis remains a popular topic for research up to these days.

7 The Last Frontier

The last frontier, as far as Henstock is concerned, is the inclusion of the Wiener integral and the Feynman integral under the general theory of Henstock. The Wiener integral, if we look at it abstractly, is nothing but an integral of Banach-valued functions. Schwabik and Ye have written a book on the Henstock integral of Banach-valued functions [18]. The other aspect of the Wiener integral involves Brownian motions. We call it the Ito integral. We have given a Riemann-type definition to the Ito integral and proved some formulae [20]. The standard proof of the uniqueness of the Ito integral is lengthy. The proof involves measure theory. If we use the Henstock approach, the proof is trivial.

The Feynman integral is defined on an infinite dimensional space. It was a long standing problem to prove the existence of divisions on an infinite dimensional space. Muldowney et al [7] finally solved the problem. Hence the last frontier has been conquered.

Perron defined his integral in order to solve differential equations. So did Kurzweil. Denjoy defined his integral in order to solve problems in trigonometric series. Recently, Lee Tuo Yeong [14, 15] was able to prove a series of theorems involving the convergence of Fourier series and using the Henstock integral. There have been attempts to apply the theory to partial differential equations and numerical integration. So far we have not seen any break through.

Henstock says: Every good mathematical theory is both simple and elegant [2]. Indeed, the Henstock theory of integration is simple and elegant. Unless and until it has applications in other fields, it will not become a common language for the mathematicians.

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Mathematics and Mathematics Education
National Institute of Education
Nanyang Technological University
1 Nanyang Walk,
Singapore 637616

E-mail : pengyee.lee@nie.edu.sg