A DIFFERENCE VERSION OF FURUTA-GIGA THEOREM ON KANTOROVICH TYPE INEQUALITY AND ITS APPLICATION

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Dedicated to Professor Masatoshi Fujii for his 60th birthday with respect and affection

Abstract. Recently, T. Furuta and M. Giga[3] gave the complementary result of Kantorovich type order preserving inequalities by Mićić-Pečarić-Seo[5]. In this note, we shall show a difference version of T. Furuta and M. Giga's result as follows: Let A and B be positive operators on a Hilbert space H such that $A \geq B \geq 0$ and $MI \geq A \geq mI > 0$ for some scalars $M > m > 0$. If $p > 1$ and $q > 1$, then the following inequality holds:

$$B^p \leq A^q + C(m, M, p, q)I,$$

where $C(m,M,p,q) = \left\{\frac{M^p - m^p}{q(M-m)}\right\}^{\frac{q}{q-1}}(q-1) + \frac{m^pM - mM^p}{M-m}$ for $m \leq \left\{\frac{M^p - m^p}{q(M-m)}\right\}^{\frac{1}{q-1}} \leq M$. In addition, we obtain Kantorovich type inequalities for the chaotic order.

1. Introduction

Throughout this note, a capital letter means a bounded linear operator on a Hilbert space H. An operator A is said to be positive(denoted by $A \geq 0$) if $(Ax, x) \geq 0$ for all $x \in H$. The Löwner-Heinz inequality asserts that $A \geq B \geq 0$ ensures $A^p \geq B^p$ for all $0 \leq p \leq 1$. However $A \geq B \geq 0$ does not ensure $A^p \geq B^p$ for $p > 1$ in general. In 1997, M. Fujii, S. Izumino, R. Nakamoto and Y. Seo[1] showed that t^2 is order preserving in the following;

$$A \geq B \geq 0 \text{ and } MI \geq A \geq mI > 0 \Rightarrow \frac{(M+m)^2}{4Mm}A^2 \geq B^2.$$

This inequality comes from the celebrated Kantorovich inequality;

$$\begin{aligned} (1.1) \qquad & MI \geq A \geq mI > 0 \text{ and } M > m > 0 \\ & \Rightarrow (A^{-1}x, x)(Ax, x) \leq \frac{(M+m)^2}{4Mm} \quad \text{for all unit vectors } x \in H. \end{aligned}$$

The constant $\frac{(M+m)^2}{4Mm}$ is called the Kantorovich constant. In [3], T. Furuta and M. Giga showed the following Theorem A which is an extension of Kantorovich type inequality:

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Theorem A. [3, Theorem 4.1] *Let A and B be positive operators on a Hilbert space H such that $A \geq B$ and $MI \geq B \geq mI > 0$ for some $M > m > 0$ and K_+ the constant defined by*

$$K_+(m,M,p,q) = \begin{cases} \frac{mM^p - Mm^p}{(q-1)(M-m)} \left(\frac{(q-1)(M^p - m^p)}{q(mM^p - Mm^p)} \right)^q & \text{if } m^{p-1}q \leq \frac{M^p - m^p}{M-m} \leq M^{p-1}q, \\ m^{p-q} & \text{if } m^{p-1}q > \frac{M^p-m^p}{M-m}, \\ M^{p-q} & \text{if } M^{p-1}q < \frac{M^p-m^p}{M-m}, \end{cases}$$

where $q \neq 1$.
If $p > 1$ and $q > 1$, then the following inequality holds:

$$K_+(m,M,p,q)A^q \geq B^p.$$

On the other hand, T. Yamazaki[6] showed the following reverse inequality. To mention it, we need the constant

$$C(m,M,p) = (p-1)\left(\frac{M^p - m^p}{p(M-m)} \right)^{\frac{p}{p-1}} + \frac{Mm^p - mM^p}{M-m}$$

for $M > m > 0$ and $p \in \mathbb{R}$.

Theorem B. *If $A \geq B \geq 0$ and $MI \geq B \geq mI > 0$ for some scalars $M > m > 0$, then*

$$A^p + C(m,M,p)I \geq B^p \text{ for all } p > 1.$$

Related to recent development of Kantorovich inequalities, we refer a textbook[4] recently published. As a result, we shall show a difference type inequalities of Theorem A. It is a difference version of Kantorovich type inequality as an extension of Theorem B. For positive invertible operators A and B, the order defined by $\log A \geq \log B$ is called the chaotic order in [2]. Since $\log t$ is an operator monotone function, the chaotic order is weaker than the usual one $A \geq B$. T. Yamazaki and M. Yanagida[7] gave some characterization of the chaotic order related with Kantorovich type inequality. In section 3, we obtain analogous inequalities to Theorem 2.1 on the chaotic order.

2. A difference version of Kantorovich type inequalities

First of all, we introduce the constant $C(m,M,f(t),q)$ which is an extension of $C(m,M,p)$: Let $f(t)$ be a real valued continuous function on an interval $[m,M]$ and $q \in \mathbb{R}$ such that $\frac{f(M)-f(m)}{q(M-m)} \geq 0$. Put

$$C(m,M,f(t),q) = \begin{cases} \frac{Mf(m)-mf(M)}{M-m} + (q-1)\left(\frac{f(M)-f(m)}{q(M-m)}\right)^{\frac{q}{q-1}} & \text{if } m \leq \left(\frac{f(M)-f(m)}{q(M-m)}\right)^{\frac{1}{q-1}} \leq M, \\ f(m) - m^q & \text{if } \left(\frac{f(M)-f(m)}{q(M-m)}\right)^{\frac{1}{q-1}} < m, \\ f(M) - M^q & \text{if } M < \left(\frac{f(M)-f(m)}{q(M-m)}\right)^{\frac{1}{q-1}}. \end{cases}$$

In particular, if $f(t) = t^p$, then the constant $C(m,M,t^p,q)$ is denoted simply by $C(m,M,p,q)$, and we note that $C(m,M,t^p,p) = C(m,M,p)$.
Moreover we prepare some notations: For $0 < m < M$ and $p > 0, q > 0$,

$$\beta_1(m,M,p,q) = max\{m^p - m^q, M^p - M^q\}$$

and

$$\beta_2(m,M,p,q)=\begin{cases} m^p-m^q & \text{if } \left(\frac{p}{q}\right)^{\frac{1}{q-p}}<m\ , \\ \left(\frac{p}{q}\right)^{\frac{p}{q-p}}-\left(\frac{p}{q}\right)^{\frac{q}{q-p}} & \text{if } m\le\left(\frac{p}{q}\right)^{\frac{1}{q-p}}\le M, \\ M^p-M^q & \text{if } M<\left(\frac{p}{q}\right)^{\frac{1}{q-p}}\ . \end{cases}$$

Under this preparation, we state our main theorem.

Theorem 2.1. *Let A and B be positive operators on a Hilbert space H such that $M_1I\ge A\ge m_1I>0$ and $M_2I\ge B\ge m_2I>0$ for some scalars $M_1>m_1>0$ and $M_2>m_2>0$. If $A\ge B\ge 0$, then the following inequalities hold:*

(a) $p>1$ *and* $q>1\Rightarrow A^q+C(m_2,M_2,p,q)I\ge B^p$.

(b) $0<p<1$ *and* $p<q\Rightarrow A^q+\beta_2(m_1,M_1,p,q)I\ge B^p$.

(c) $0<q<p<1\Rightarrow A^q+\beta_1(m_1,M_1,p,q)I\ge B^p$.

(d) $p>1$ *and* $0<q<1\Rightarrow A^q+\beta_1(m_2,M_2,p,q)I\ge B^p$.

Proof. (a) Since $f(t)=t^p$ for $p>1$ is convex on $[m,M]$, we have

$$t^p\le\frac{M^p-m^p}{M-m}(t-m)+m^p \tag{2.1}$$

for any $t\in[m,M]$. By applying the functional calculus of positive operator B to (2.1), since $M_2I\ge B\ge m_2I$, we obtain for every unit vector x,

$$(B^px,x)\le\frac{M_2^p-m_2^p}{M_2-m_2}(Bx,x)+\frac{m_2^pM_2-m_2M_2^p}{M_2-m_2}.$$

Thus,

$$(B^px,x)-(Bx,x)^q\le\frac{M_2^p-m_2^p}{M_2-m_2}(Bx,x)+\frac{m_2^pM_2-m_2M_2^p}{M_2-m_2}-(Bx,x)^q$$

for $q>1$. Let $h(t)=-t^q+\frac{M_2^p-m_2^p}{M_2-m_2}t+\frac{m_2^pM_2-m_2M_2^p}{M_2-m_2}$. Then we obtain

$$(B^px,x)-(Bx,x)^q\le\max_{m_2\le t\le M_2}h(t).$$

By a differential calculus, if $h'(t_1)=0$, then

$$t_1=\left\{\frac{M_2^p-m_2^p}{q(M_2-m_2)}\right\}^{\frac{1}{q-1}}$$

and moreover

$$h''(t_1)=-q(q-1)\left\{\frac{M_2^p-m_2^p}{q(M_2-m_2)}\right\}^{\frac{q-2}{q-1}}<0.$$

Thus, in the case of $m_2\le t_1\le M_2$, we have the upper bound $h(t_1)$ on $[m_2,M_2]$, where

$$h(t_1)=\frac{(q-1)}{q^{\frac{q}{q-1}}}\left(\frac{M_2^p-m_2^p}{M_2-m_2}\right)^{\frac{q}{q-1}}+\frac{m_2^pM_2-m_2M_2^p}{M_2-m_2}.$$

Next, if $t_1 < m_2$, then upper bound $h(m_2)$ on $[m_2, M_2]$ is given by

$$\begin{aligned} h(m_2) &= -m_2^q + \frac{M_2^p - m_2^p}{M_2 - m_2} m_2 + \frac{m_2^p M_2 - m_2 M_2^p}{M_2 - m_2} \\ &= m_2^p - m_2^q. \end{aligned}$$

Similarly, if $M_2 < t_1$, then the upper bound $h(M_2)$ on $[m_2, M_2]$ attains at $t = M_2$:

$$h(M_2) = M_2^p - M_2^q.$$

That is,

$$(B^p x, x) \leq (Bx, x)^q + C(m_2, M_2, p, q) \quad \text{for } p > 1,\ q > 1.$$

Hence we have for every unit vector x

$$\begin{aligned} (B^p x, x) &\leq (Bx, x)^q + C(m_2, M_2, p, q) \\ &\leq (Ax, x)^q + C(m_2, M_2, p, q) \quad \text{by } A \geq B > 0. \\ &\leq (A^q x, x) + C(m_2, M_2, p, q) \quad \text{by Hölder-McCarthy inequality.} \end{aligned}$$

To Prove (b)-(d), we put $h(t) = t^p - t^q$. Then we note that $h'(t) = t^{p-1}(p - qt^{q-p})$ and $h'(t_0) = 0$ for $t_0 = \left(\frac{p}{q}\right)^{\frac{1}{q-p}}$ and moreover

$$h''(t_0) = \left(\frac{p}{q}\right)^{\frac{p-2}{q-p}} p(p-q).$$

(b) Since $0 < p < 1$ and $p < q$, we have $h''(t_0) < 0$. Thus, we have

$$\max_{m_1 \leq t \leq M_1} h(t) = \beta_2(m_1, M_1, p, q)$$

and so

$$A^p \leq A^q + \beta_2(m_1, M_1, p, q) I.$$

Therefore, we have

$$B^p \leq A^p \leq A^q + \beta_2(m_1, M_1, p, q) I$$

by Löwner-Heinz inequality.

(c) Since $0 < q < p < 1$, we have $h''(t_0) > 0$. Thus, we have

$$\max_{m_1 \leq t \leq M_1} h(t) = \beta_1(m_1, M_1, p, q)$$

and so

$$A^p \leq A^q + \beta_1(m_1, M_1, p, q) I.$$

Therefore, we have

$$B^p \leq A^p \leq A^q + \beta_1(m_1, M_1, p, q) I$$

by Löwner-Heinz inequality.

(d) Since $h(t) = t^p - t^q$ for $p > 1$ and $0 < q < 1$ is convex on $[m_2, M_2]$, we have

$$\max_{m_2 \leq t \leq M_2} h(t) = \beta_1(m_2, M_2, p, q)$$

and so

$$B^p \leq B^q + \beta_1(m_2, M_2, p, q) I.$$

Therefore we have

$$B^p \leq B^q + \beta_1(m_2, M_2, p, q) I \leq A^q + \beta_1(m_2, M_2, p, q) I$$

by Löwner-Heinz inequality.

$\square$

Remark 2.2. (1) *If we put* $p = q(>1)$ *in* (a) *of Theorem 2.1, then the assumption*

$$pm_2^{p-1} \leq \frac{M_2^p - m_2^p}{M_2 - m_2} \leq pM_2^{p-1}$$

is automatically satisfied by the convexity of $f(t) = t^p$ *and so the constant* $C(m_2, M_2, p, p)$ *coincides with* $C(m_2, M_2, p)$. *Therefore we have Theorem B.*
(2) *If we put* $p = q(\in (0,1))$ *in* (b) *and* (c) *of Theorem 2.1, then it follows that* $\beta_1(m_1, M_1, p, p) = \beta_2(m_2, M_2, p, p) = 0$.

3. Application to the chaotic order

We obtain the following theorem as an application to the chaotic order.

Theorem 3.1. *Let* A *and* B *be positive invertible operators on a Hilbert space* H *such that* $\log A \geq \log B > 0$ *and* $MI \geq B \geq mI > I$ *for some scalars* $M > m > 1$. *If* $p \geq 1$ *and* $q > 1$, *then*

$$B^p \leq A^q + C(\log m, \log M, e^{pt}, q)I.$$

Proof. Since $f(t) = e^{pt}$ for $p \geq 1$ is convex on $[m, M]$, we have

$$e^{pt} \leq \frac{e^{pM} - e^{pm}}{M - m}(t - m) + e^{pm} \quad \text{for } m \leq t \leq M. \tag{3.1}$$

By applying functional calculus of positive operator $\log B$ to (3.1), since $(\log m)I \leq \log B \leq (\log M)I$, we have for every unit vector x

$$\begin{aligned}(B^p x, x) &- ((\log B)x, x)^q \\ &\leq \frac{M^p - m^p}{\log M - \log m}((\log B)x, x) + \frac{m^p \log M - M^p \log m}{\log M - \log m} - ((\log B)x, x)^q\end{aligned}$$

for $q > 1$. Let $h(t) = -t^q + \frac{M^p - m^p}{\log M - \log m}t + \frac{m^p \log M - M^p \log m}{\log M - \log m}$. Then we obtain

$$(B^p x, x) - ((\log B)x, x)^q \leq \max_{\log m \leq t \leq \log M} h(t).$$

By a differential calculus, if $h'(t_1) = 0$, then

$$t_1 = \left\{ \frac{M^p - m^p}{q(\log M - \log m)} \right\}^{\frac{1}{q-1}}$$

and moreover

$$h''(t_1) = -q(q-1) \left\{ \frac{M^p - m^p}{q(\log M - \log m)} \right\}^{\frac{q-2}{q-1}} < 0.$$

Thus, in the case of $\log m \leq t_1 \leq \log M$, we have the upper bound $h(t_1)$ on $[\log m, \log M]$,

$$h(t_1) = (q-1) \left\{ \frac{M^p - m^p}{q(\log M - \log m)} \right\}^{\frac{q}{q-1}} + \frac{m^p \log M - M^p \log m}{\log M - \log m}.$$

And also, in the case of $t_1 < \log m$, we have the upper bound $h(\log m)$ on $[\log m, \log M]$,

$$h(\log m) = m^p - (\log m)^q.$$

Moreover, in the case of $\log M < t_1$, we also have the upper bound $h(\log M)$ on $[\log m, \log M]$,

$$h(\log M) = M^p - (\log M)^q.$$

For every unit vector x, we have

$$\begin{aligned}(B^p x, x) &\leq ((\log B)x, x)^q + C \\ &\leq ((\log A)x, x)^q + C \quad \text{by } \log A \geq \log B > 0 \\ &\leq (Ax, x)^q + C \quad \text{by } A \geq \log A \\ &\leq (A^q x, x) + C \quad \text{by Hölder-McCarthy inequality for } q > 1,\end{aligned}$$

where $C := C(\log m, \log M, e^{pt}, q)$. Whence the proof is complete. □

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