CONVERGENCES ON THE HENSTOCK-KURZWEIL INTEGRAL

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I first met Professor Ralph Henstock at "Symposium in Real Analysis" held in Erice in 1995. Since then we had friendly and encouraging communications about our research, for which I am very grateful to him. I would like to express my deep respect to Prof. Henstock by introducing a part of my work that he highly valued and by explaining its relation to the Henstock-Kurzweil integral.

We concern with convergences on the Henstock-Kurzweil integral which was developed by J. Kurzweil and R. Henstock. On the real line, the Henstock-Kurweil integral is equivalent to the special Denjoy integral. We know the controlled convergence as a convergence on the Henstock-Kurzweil integral (see [5] for example). On the other hand, following K. Kunugi's idea for integration of singular functions and using the notion of ranked space introduced by Kunugi, we defined a convergence called the r-convergence for the special Denjoy integral, and showed that, under the r-convergence, the family of special Denjoy integrable functions is obtained as a completion of the family of step functions [6, 9, 11]. We discuss the controlled convergence from the viewpoint of the r-convergence [8, 10, 12].

We begin by giving the definition of ranked space introduced by K. Kunugi as a method for functional analysis in [2], 1954.

1. Ranked space ([2, 4, 7]). Let E be a non-empty set such that with each $x \in E$ there is associated a non-empty family $\mathcal{V}(x)$ consisting of subsets of E, written $V(x)$, etc., and called *preneighborhoods of* x, such that

(A) if $V(x) \in \mathcal{V}(x)$, then $x \in V(x)$.

Sometimes, we call preneighborhood of some point *preneighborhood* simply. The space E endowed with such a family of preneighborhoods is called a *ranked space* if with $n = 0, 1, 2, \cdots$, there is associated a family $\mathcal{V}_n$ of preneighorhoods so that

(a) corresponding to $x \in E, V(x) \in \mathcal{V}(x)$, and $n \in \{0, 1, 2, \cdots\}$, there are an $m \in \{0, 1, 2, \cdots\}$ and a $U(x) \in \mathcal{V}(x)$ such that $m \geq n, U(x) \subset V(x)$ and $U(x) \in \mathcal{V}_m$.

A preneighborhood $V(x) \in \mathcal{V}_n$ is called a *preneighborhood of rank n.* The ranked space E is written as $(E, \mathcal{V}(x), \{\mathcal{V}_n\}_{n=0}^{\infty})$ or simply $(E, \mathcal{V}(x), \mathcal{V}_n)$ as a preneighborhood space endowed with the *structure* $\mathcal{V}_n (n = 0, 1, 2, \cdots)$ *of ranked space.* A preneighborhood of x of rank n is denoted by $V(x, n), U(x, n)$, etc.

A sequence of preneighborhoods $\{V_i(x_i, n_i) : i = 0, 1, 2, \cdots\}$ is called a *fundamental sequence* if

(1.1) $V_0(x_0, n_0) \supset V_1(x_1, n_1) \supset \cdots$,

(1.2) $n_0 \leq n_1 \leq \cdots$, and

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(1.3) for each $n \in \{0, 1, 2, \cdots\}$, there is an i such that $i \geq n, x_i = x_{i+1}$ and $n_i < n_{i+1}$.
For a fundamental sequence $\{V_i(x_i, n_i) : i = 0, 1, 2, \cdots\}$, $\{x_i\}_{i=0}^{\infty}$ is called the *sequence of centers*. If, for every $i, x_i = x$, then the fundamental sequence is called a *fundamental sequence of center* x. A sequence of points $\{z_i\}_{i=1}^{\infty}$ is called an *r-Cauchy sequence* if there is a fundamental sequence of preneighborhoods $\{V_j\}_{j=0}^{\infty}$ such that for every V_j there is an $i_0(j)$ such that $z_i \in V_j$ for every $i \geq i_0(j)$. In this case, $\{V_j\}_{j=0}^{\infty}$ is called a *defining fundamental sequence of the r-Cauchy sequence* $\{z_i\}_{i=1}^{\infty}$. A sequence of points $\{x_i\}_{i=1}^{\infty}$ is said to be *r-convergent to* x and is written $r\text{-}\lim_{i\to\infty} x_i = x$ if there is a fundamental sequence $\{V_j\}_{j=0}^{\infty}$ of center x such that for every V_j there is an $i_0(j)$ such that $x_i \in V_j$ for every $i \geq i_0(j)$.

For sequences of monotone decreasing sets $\{A_i\}$ and $\{B_i\}$, $\{A_i\} > \{B_i\}$ means that every A_i contains some $B_{i'}$. $\{A_i\} \sim \{B_i\}$ means that $\{A_i\} > \{B_i\}$ and $\{B_i\} > \{A_i\}$. Fundamental sequences $\{U_i(x_i, m_i)\}, \{V_i(y_i, n_i)\}$, etc. are denoted by u, v, etc. For fundamental sequences $u = \{U_i\}$ and $v = \{V_i\}$, $u > v$ means $\{U_i\} > \{V_i\}$, and $u \sim v$ means $\{U_i\} \sim \{V_i\}$. Fundamental sequences u and v are said to be *in the relation* ρ if there is a fundamental sequence w in E such that $w > u$ and $w > v$. μ denotes the Lebesgue measure on one dimensional Euclidean space.

2. Kunugi's idea for integration of singular function. We show the ranked space approach to non-absolute integration by K. Kunugi published in [3], 1956, in a slightly different form from the original definition.

Let $\mathcal{E}$ be the class of real valued step functions on $[a, b]$, that is, functions f having a constant value in each of a finite number of open sub-intervals $a_{i-1} < x < a_i (i = 1, 2, \cdots, n)$ with $a_0 = a < a_1 < \cdots < a_n = b$, and as to the endpoints of these sub-intervals, we can assign values of the functions there arbitrarily. The integral of step function on a closed set, an open set, etc. is defined in the Lebesgue sense.

Associating with a closed set $F \subset [a, b]$, we consider the following two semi-norms defined on $\mathcal{E}$:

$$\|f\|_1^F = \int_F |f| dx \ ; \ \|f\|_2^F = |\int_{CF} f dx|,$$

where $CF = [a, b] - F$.

Corresponding to a closed set $F \subset [a, b]$ and $\varepsilon > 0$, we define a preneighborhood of $f \in \mathcal{E}$, written $V(f; F, \varepsilon)$, as follows:

$$V(f; F, \varepsilon) = \{g \in \mathcal{E} : \|f - g\|_1^F < \varepsilon, \|f - g\|_2^F < \varepsilon\},$$

whence we have

$$\mathcal{V}(f) = \{V(f; F, \varepsilon) : F \text{ is a closed set in } [a, b] \text{ and } \varepsilon > 0\}.$$

For $n \in \{0, 1, 2, \cdots\}$, $V(f; F, \varepsilon)$ is called a preneighborhood of rank n if and only if $\varepsilon = 1/2^n$ and $\mu([a, b] - F) < 1/2^n$, whence we have

$$\mathcal{V}_n = \{V(f; F, \varepsilon) \in \mathcal{V}(f) : \varepsilon = 1/2^n \text{ and } \mu([a, b] - F) < 1/2^n\}.$$

Kunugi showed that the space $\mathcal{E}$ endowed with $\mathcal{V}(f)(f \in \mathcal{E})$ and $\mathcal{V}_n(n = 0, 1, 2, \cdots)$ becomes a ranked space, and pointed out the following fact.

For any fundamental sequence $u = \{V_i(f_i; F_i, n_i)\}_{i=0}^{\infty} (= \{V_i(f_i; F_i, 1/2^{n_i})\}_{i=0}^{\infty})$ in the ranked space $\mathcal{E}$, the following hold.

[C, 1] $f_i(x)$ converges to a finite valued function almost everywhere x in $[a, b]$ as $i \to \infty$.

[C, 2] $\int_a^b f_i dx$ converges to a finite limit as $i \to \infty$.

Denote the limit function and the limit number by $J(u)$ and $I(u)$ respectively. Then

[C, 3] $J(u)$ is Lebesgue integrable on each $F_i (i = 0, 1, 2, \cdots)$. $\int_{F_i} J(u)dx$ converges to a finite limit as $i \to \infty$, and $\lim_{i\to\infty} \int_{F_i} J(u)dx = \lim_{i\to\infty} \int_a^b f_i dx = I(u)$.

Consider the relation ρ on the family $\mathcal{F}$ of fundamental sequences in the ranked space $\mathcal{E}$. Then, ρ is an equivalence relation on $\mathcal{F}$. Denote by $\mathcal{F}^*$ the quotient with respect to ρ of $\mathcal{F}$: $\mathcal{F}/\rho$, and elements of $\mathcal{F}^*$ by u^*, v^*, etc. Kunugi pointed out that:

[C, 4] For $u^* \in \mathcal{F}^*$; if we identify two functions which are different only on a set of measure zero, the limit function $J(u)$ is uniquely determined independently of the choice of representative u of u^*. The limit number $I(u)$ is also determined independently of the choice of representative u of u^*.

Denoting the limit function and the limit number associated with u^* by $J(u^*)$ and $I(u^*)$, respectively, Kunugi treated the limit number $I(u^*)$ as the integral of the limit function $J(u^*)$.

However, the mapping $J : u^* \to J(u^*)(u^* \in \mathcal{F}^*)$ is not always one-to-one. It may arise that $J(u^*) = J(v^*)$ even if $u^* \neq v^*(u^*, v^* \in \mathcal{F}^*)$, and therefore that $I(u^*) \neq I(v^*)$ for the same function $J(u^*) = J(v^*)$. In order to avoid this fact, K. Kunugi defined, adding a few conditions to the formation above, a new integral called *E. R. integral*, in the same paper. After that, Amemiya and Ando pointed out in [1], 1965, that the *E.R.* integrability coincides with the A-integrability and the *E.R.* integral is equal to the A-integral.

3. Completion of ranked space. Let E be a ranked space $(E, \mathcal{V}(x), \{\mathcal{V}_n\}_{n=0}^\infty)$. A ranked space E is said to be *complete* if $\cap U_i \neq \emptyset$ for every fundamental sequence $u = \{U_i\}$ in the ranked space E. A set $A \subset E$ is said to be *dense* in E if $V(x) \cap A \neq \emptyset$ for every $x \in E$ and every $V(x) \in \mathcal{V}(x)$. Let $A \subset E$ and suppose that with each $x \in A$ there is associated a family $\mathcal{V}(x; A)$ of subsets of A and with each $n = 0, 1, 2, \cdots$ there is associated a family $\mathcal{V}_n(A)$, so that

$$\mathcal{V}(x; A) = \{V(x) \cap A : V(x) \in \mathcal{V}(x)\};\ \mathcal{V}_n(A) = \{V(x, n) \cap A : V(x, n) \in \mathcal{V}_n\}.$$

Then, the set A endowed with $\mathcal{V}(x; A)$ and $\mathcal{V}_n(A)$ becomes a ranked space. The ranked space $(A, \mathcal{V}(x; A), \{\mathcal{V}_n(A)\}_{n=0}^\infty)$ is called a *ranked subspace* of the ranked space $(E, \mathcal{V}(x), \{V_n\}_{n=0}^\infty)$ if for every fundamental sequence of preneighborhoods $\{U_i(x_i, m_i)\}$ in the ranked space A, there is a fundamental sequence $\{V_i(x_i, n_i)\}$ in the ranked space E in such a way that there is an i_0 such that $m_i = n_i$ and $V_i \cap A = U_i$ for every $i \geq i_0$.

A ranked space E^* is called a *completion* of an incomplete ranked space E if it satisfies the following three conditions.

(1) The ranked space E^* is complete.

(2) The ranked space E is a ranked subspace of the ranked space E^*.

(3) The set E is dense in the ranked space E^* as a subset of E^*.

In particular, the completion E^* is called an *r-completion* if every r-Cauchy sequence in the ranked space E^* is r-convergent in the ranked space E^*.

4. The special Denjoy integral ([6, 9, 11]). Denote the class of all special Denjoy integrable functions on $[a, b]$ by $\mathcal{D}_*$. On the class $\mathcal{E}$ of all real valued step functions on $[a, b]$, corresponding to a closed set F in $[a, b]$, define two semi-norms $\|f\|_1^F$ and $\|f\|_2^F$ associated with F as follows:

$$\|f\|_1^F = \int_F |f|dx\ ;\ \|f\|_2^F = \sup_{\{I_j\}}(\sum_{j=1}^s |\int_{I_j \cap CF} f dx|),$$

where $\{I_j\}$ runs through all finite sequences $I_j (j = 1, 2, \cdots, s)$ of non-overlapping intervals in $[a, b]$ with $I_j \cap F \neq \emptyset$ for every j.

Let $f \in \mathcal{E}$. Corresponding to a non-empty closed set $F \subset [a, b]$ and an $\varepsilon > 0$ such that

[α] $\mu([a, b] - F) < \varepsilon$, and

[β] $\|f\|_2^F < \varepsilon$,

we define a preneighborhood $V(f; F, \varepsilon)$ of f by

$$V(f; F, \varepsilon) = \{g \in \mathcal{E} : \|f - g\|_1^F < \varepsilon, \|f - g\|_2^F < \varepsilon\}.$$

We say that a preneighborhood $V(f; F, \varepsilon)$ is of rank n if and only if $\varepsilon = 1/3^n$. So, we have

$$\mathcal{V}(f) = \{V(f; F, \varepsilon) \ : \ \text{non-empty closed set F} \subset [a, b] \text{ and } \varepsilon > 0 \\ \text{such that } \mu([a, b] - F) < \varepsilon \text{ and } \|f\|_2^F < \varepsilon\},$$

$$\mathcal{V}_n = \{V(f; F, \varepsilon) \in \mathcal{V}(f) : \varepsilon = 1/3^n\} \text{ for } n = 0, 1, 2, \cdots.$$

Then, they satisfy the conditions (A) and (a) to be a ranked space. In the ranked space $\mathcal{E}$ so defined, denote a preneighborhood of rank n by $V(f; F, n)$, etc. We have $\mu(F - G) = 0$ if $V(f; F, \varepsilon) \supset V(g; G, \eta)$. In order to investigate the special Denjoy integral, for a sequence of preneighborhoods $\{V_i(f_i; F_i, n_i)\}$ in $\mathcal{E}$ to be a fundamental sequence we assign, in addition to the conditions (1.1), (1.2) and (1.3) indicated in Section 1, the following condition:

[δ] $F_i \uparrow [a, b]$, that is, $F_0 \subset F_1 \subset \cdots$ and $\cup F_i = [a, b]$.

Denote the ranked space $\mathcal{E}$ treated in this way by $\mathcal{E}_{D_*}$, and the family of all fundamental sequences in $\mathcal{E}_{D_*}$ by $\mathcal{F}$.

Proposition 1. For a fundamental sequence $u = \{V_i(f_i; F_i, n_i)\}$ in $\mathcal{E}_{D_*}$, we have the properties [C, 1], [C, 2] and [C, 3] indicated in Section 2.

Now we consider the relation ρ (indicated in Section 1) on the $\mathcal{F}$. Then ρ is an equivalence relation on $\mathcal{F}$. Denote by $\mathcal{F}^*$ the quotient with respect to ρ of $\mathcal{F}$: $\mathcal{F}/\rho$. Then, as in Section 2, [C, 4] holds for the $\mathcal{F}^*$. Denote the limit function and the limit number associated with $u^* \in \mathcal{F}^*$ by $J(u^*)$ and $I(u^*)$ respectively, similarly to the case of Section 2. We have

Proposition 2. (1) $\{J(u^*) : u^* \in \mathcal{F}^*\} = \mathcal{D}_*$.

(2) The mapping $J : u^* \to J(u^*)$ is a one-to-one mapping from $\mathcal{F}^*$ onto $\mathcal{D}_*$.

(3) $I(u^*)$ is equal to the special Denjoy integral of $J(u^*)$.

Putting

$$\hat{\mathcal{E}}_{\mathcal{D}_*} = \mathcal{E} \cup \{u^* \in \mathcal{F}^* : \cap_{i=0}^{\infty} V_i = \emptyset \text{ for some } u \in u^*, u = \{V_i\}\},$$

we define the ranked space $\hat{\mathcal{E}}_{\mathcal{D}_*}$ by use of the general method of completion for the ranked space. Then

Theorem 1 ([11, Proposition 37]). The ranked space $\hat{\mathcal{E}}_{\mathcal{D}_*}$ is an r-completion of the ranked space $\mathcal{E}_{\mathcal{D}_*}$.

Now, as in the case of the ranked space $\mathcal{E}_{\mathcal{D}_*}$, associated with a closed set F in $[a, b]$, for a function $f \in \mathcal{D}_*$ which is Lebesgue integrable on F, we define two semi-norms $\|f\|_1^F$ and $\|f\|_2^F$ on $\mathcal{D}_*$, using the special Denjoy integral.

Let $f \in \mathcal{D}_*$. Corresponding to a non-empty closed set $F \subset [a, b]$ and an $\varepsilon > 0$ satisfying, in addition to the conditions $[\alpha]$ and $[\beta]$ indicated to define the ranked space $\mathcal{E}_{\mathcal{D}_*}$, the condition

$[\gamma]$ f is Lebesgue integrable on F,

we define a preneighborhood of f by

$$\hat{V}(f; F, \varepsilon) = \{g \in \mathcal{D}_* : \|f - g\|_1^F < \varepsilon, \|f - g\|_2^F < \varepsilon\}.$$

Using such preneighborhoods, we define the ranked space $\mathcal{D}_*$ similarly to the ranked space $\mathcal{E}_{\mathcal{D}_*}$.

Then, we have

Theorem 2 ([11, Theorem 9]). Define a mapping $\kappa : \hat{\mathcal{E}}_{\mathcal{D}_*} \to \mathcal{D}_*$ as follows:

$\kappa(f) = f$ on $\mathcal{E}$, and

$\kappa(u^*) = J(u^*)$ for $u^* \in \mathcal{F}^*$ such that $\cap_{i=0}^{\infty} V_i = \emptyset$ for some $u = \{V_i\} \in u^*$.

Then, κ is a one-to-one mapping, and for every fundamental sequence $\{U_i\}$ in the ranked space $\hat{\mathcal{E}}_{\mathcal{D}_*}$ there exists a fundamental sequence v in the ranked space $\mathcal{D}_*$ such that $v \sim \{\kappa(U_i)\}$, and the same statement holds for the inverse mapping κ^{-1}.

In particular, we have

(4. 1) The set $\mathcal{E}$ is dense in the ranked space $\mathcal{D}_*$ as a subset of $\mathcal{D}_*$. The ranked space $\mathcal{E}_{\mathcal{D}_*}$ is a ranked subspace of the ranked space $\mathcal{D}_*$.

(4. 2) Every r-Cauchy sequence $f_i (i = 1, 2, \cdots)$ in the ranked space $\mathcal{D}_*$ is r-convergent to a function $f \in \mathcal{D}_*$ in the ranked space $\mathcal{D}_*$.

5. Controlled convergence and r-convergence ([12, §5]). We consider the controlled convergence on $\mathcal{D}_*$ in connection with the r-convergence on $\mathcal{D}_*$ (see [5, p.39, Definition 7.4] for the definition of controlled convergence).

Proposition 3 ([10, Propositions 4, 6]). Let $f_n \in \mathcal{D}_* (n = 1, 2, \cdots)$ such that $f_n(x)$ is convergent to $f(x)$ almost everywhere x in $[a, b]$ as $n \to \infty$. Then, the following hold.

(1) If $\{f_n\}_{n=1}^{\infty}$ is r-convergent to $f \in \mathcal{D}_*$, then $\{f_n\}_{n=1}^{\infty}$ is controlled convergent to f.

(2) If $\{f_n\}_{n=1}^{\infty}$ is controlled convergent to f (as a result $f \in \mathcal{D}_*$) and f_n is Lebesgue integrable on $[a, b]$ for each $n = 1, 2, \cdots$, then $\{f_n\}_{n=1}^{\infty}$ is r-convergent to f.

We know that a sequence $\{f_n\}_{n=1}^{\infty}$ in the ranked space $\mathcal{D}_*$ is r-convergent if and only if it is r-Cauchy, by (4. 2). Hence, by Proposition 3, as far as the sequence $\{f_n\}_{n=1}^{\infty}$ such that

1) $f_n \in \mathcal{E}$ for each n, and

2) $f_n(x)$ is convergent to $f(x)$ almost everywhere x in $[a, b]$ as $n \to \infty$,

is concerned, the notions of controlled convergent sequence and r-Cauchy sequence in the ranked space $\mathcal{D}_*$ coincide. On the other hand, by (4. 1) and [11, Lemmas 47, 51], a sequence of simple functions is an r-Cauchy sequence in the ranked space $\mathcal{D}_*$ if and only if it is an r-Cauchy sequence in the ranked space $\mathcal{E}_{\mathcal{D}_*}$. Therefore, when we denote by C the family of all r-Cauchy sequences $\{f_n\}$ in the ranked space $\mathcal{E}_{\mathcal{D}_*}$ such that $f_n(x)$ is convergent almost everywhere x in $[a, b]$, we have

Proposition 4. The family C coincides with the family of all controlled convergent sequences consisting of elements of $\mathcal{E}$.

For $\{f_n\}, \{g_n\} \in C$, we say that they are in the relation ρ^* if they have a common defining fundamental sequence in $\mathcal{E}_{\mathcal{D}_*}$. Then, the relation ρ^* is an equivalence relation on C. Denote the equivalence classes of C with respect to ρ^* by $C_\xi(\xi \in \Xi)$. Then, by [11, Proposition 30] if $\{f_n\}, \{g_n\} \in C_\xi$ for the same ξ, $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} g_n(x)$ almost everywhere and $\lim_{n\to\infty} \int_a^b f_n dx = \lim_{n\to\infty} \int_a^b g_n dx$. Therefore, corresponding to each C_ξ, the limit function and the limit number are uniquely determined. Denote them by $J(C_\xi)$ and $I(C_\xi)$, respectively.

For $\{f_n\} \in C_\xi$ whose defining fundamental sequence is u, take a $u^* \in \mathcal{F}^* = \mathcal{F}/\rho$ (indicated in Section 4) such that $u \in u^*$. Then, such a u^* is uniquely determined independently of the choice of defining fundamental sequence u of $\{f_n\}$. Further, if $\{f_n\}, \{g_n\} \in C_\xi$, $\{f_n\}$ and $\{g_n\}$ have a common defining fundamental sequence. Therefore, corresponding to each C_ξ, there is one and only one $u^* \in \mathcal{F}^*$ such that for every $\{f_n\} \in C_\xi$ we can take a defining fundamental sequence u of $\{f_n\}$ with $u \in u^*$. We denote the u^* by $\tau(C_\xi)$. In this case we see that $\tau(C_{\xi_1}) \neq \tau(C_{\xi_2})$ if $\xi_1 \neq \xi_2$. Furthermore, τ is a mapping from $\{C_\xi : \xi \in \Xi\}$ onto $\mathcal{F}^*$. Because, let $u^* \in \mathcal{F}^*$. Take a $u \in u^*$ and the sequence of centers of u, say $\{f_n\}$. Then, the $\{f_n\}$ belongs to C by [C, 1] of Proposition 1 above and u is a defining fundamental sequence of $\{f_n\}$. Therefore, $\{f_n\} \in C$, so $\{f_n\} \in C_\xi$ for some ξ, and so $u^* = \tau(C_\xi)$. Thus

1) $\tau(C_\xi)$ is a one-to-one mapping from $\{C_\xi : \xi \in \Xi\}$ onto $\mathcal{F}^* = \mathcal{F}/\rho$.

Furthermore, by [11, Propsition 30] again,

2) $J(C_\xi) = J(\tau(C_\xi))$ and $I(C_\xi) = I(\tau(C_\xi))$.

As a result, by Proposition 2, (2) and (3), we obtain

Proposition 5. (1) $J(C_\xi)$ is a one-to-one mapping from $\{C_\xi : \xi \in \Xi\}$ onto $\mathcal{D}_*$.
(2) For every $\xi \in \Xi$, $I(C_\xi)$ coincides with the special Denjoy integral of $J(C_\xi)$.

These facts lead to the following results.

(5. 1) The family of all controlled convergent sequences consisting of elements of $\mathcal{E}$ is classified to be non-overlapping.

(5. 2) Under the classification of (5. 1), for each class the limit function $\lim_{n\to\infty} f_n(x)$ and the limit number $\lim_{n\to\infty} \int_a^b f_n dx$ are uniquely determined independently of the choice of the sequence $\{f_n\}$ belonging to the class. To different classes, there correspond different limit functions.

(5. 3) The class of all those limit functions obtained in (5. 2) coincides with $\mathcal{D}_*$. The limit number obtained in (5. 2) is equal to the special Denjoy integral of the corresponding limit function.

In particular, we know that:

Theorem 3. The Henstock-Kurzwei integral is obtained as a continuous extension of the integral on $\mathcal{E}$ onto $\mathcal{D}_*$ with respect to the controlled convergence.

Appendix

Letter from A. Denjoy to K. Kunugi on January 12, 1961

Cher Professeur Kunugui

Je vous remercie de votre travail sur l'intégrale. Mais cette notion si simple me paraît subir bien des efforts pour ne guère ajouter à sa limpidité. En voici les principes, selon mes jugements.

1. Un espace $\mathcal{V}$, où il n'est besoin de supposer ni l'existence d'une topologie, ni aucune hypothèse pour caractériser une distance, -- simplement la définition d'une *mesure*

sur (F) pour les *ensembles* F d'une certaine classe. Cette mesure est borérienne (si $F_1, F_2, \ldots$ sont mesurables, id. de $F_1 + F_2 + \cdots$ et $mes(\sum F_i) = \sum mes(F_i)$; etc...).

2. Une *fonction* f définie aux points de $\mathcal{V}$ (aux points où f ne serait pas définie, on fait $f = 0$). On suppose f mesurable: donc, $0 < \alpha < f < \beta$ et $-\beta < f < -\alpha < 0$ déterminent dans $\mathcal{V}$ les ensembles qui doivent être mesurables, quels que soient α et β.

Cela dit, l'intégrale $I(H, f) = \int_H f dm$ est *définie* par les conditions suivantes;

1. $I(H, f)$ est *linéaire* en f.

2. $I(H, f)$ est *complètement additive* par rapport aux ensembles $H_1, H_2, \ldots$, où f a un *signe constant*, la même pour tous les H_i.

3. si $A < f < B$ en H,

$$Ames(H) < I(H, f) < Bmes(H)$$

Et c'est fini. Les ensembles H de $\mathcal{V}$ n'ont besoin d'aucune condition topologique.

Vous avez donné le nom de "rangés" aux espaces d'une nouvelle espèce. J'ai appelé "rangés" (Enumération transfinie) les ensembles ordonnés (simplement) où chaque élément possède un *rang* propre, Leur caractère est que leurs *sections commençantes* sont toutes *dissemblables.*

En vous adressant, mon cher collèque, mes meilleurs voeux pour l'année débutante, je vous exprime mes vives félicitations pour les magnifiques activités di l'école Japonaise contemporaine, et où vos travaux brillent d'un éclat particulier.

Veuillez croire, cher collègue, à mes sentiments de très haute et très cordiale considération.

A. Denjoy

References

[1] I. Amemiya and T. Ando: *On the class of functions integrable in certain generalized sense*, J. Fac. Sci. Hokkaido Univ., **18** (1965), 128-140.

[2] K. Kunugi: *Sur les espaces complets et régulièrement complets, I, II*, Proc. Japan Acad., **30** (1954), 553-556, 912-916.

[3] K. Kunugi: *Application de la méthode des espaces rangés à la théorie de l'intégration I*, Proc. Japan Acad., **32** (1956), 215-220.

[4] K. Kunugi: *Sur la méthode des eapaces rangés, I, II*, Proc. Japan Acad., **42** (1966), 318-322, 549-554.

[5] P. Y. Lee: *Lanzhou lectures on Henstock integration*, World Scientific, Singapore (1989).

[6] S. Nakanishi: *L'intégration de Denjoy et l'intégration au moyen des espaces rangés, I-IV*, Proc. Japan Acad., **32** (1956), 678-683, **33** (1957), 13-18, 265-270, **34** (1958), 96-101.

[7] S. Nakanishi: *The method of ranked spaces proposed by Professor Kinjiro Kunugi*, Math. Japonica, **23** (1978), 291-323.

[8] S. Nakanishi: *On controlled convergence theorem for Henstock-Kurzweil integrals*, Southeast Asian Bulletin of Mathematics, **14** (1990), 135-141.

[9] S. Nakanishi: *The Denjoy integrals defined as the completion of simple functions*, Math. Japonica, **37** (1992), 89-101.

[10] S. Nakanishi: *A remark for convergences of Denjoy integrals*, Math. Japonica, **37** (1992), 473-478.

[11] S. Nakanishi: *Completions of ranked spaces and their applications to non-absolute integrals, II*, Math. Japonica, **41** (1995), 513-530.
[12] S. Nakanishi: *Approach to non-absolute integration by successive approximations*, Scientiae Mathematicae Japonicae, **54** (2001), 391-426.

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