RALPH HENSTOCK'S INFLUENCE ON THE INTEGRATION THEORY

Washek F. Pfeffer

Received Octorber 5, 2007

Professor Ralph Henstock, a distinguished analyst who devoted most of his research to integration theory, died on January 7, 2007. It was my good fortune to know Ralph for almost 30 years, and to derive much inspiration from his pioneering work. Rather than commenting briefly on a variety of Ralph's achievements, I will elaborate on that which has been the most influential: *a Riemann type definition of the Denjoy-Perron integral*.

An early recognition that the powerful Lebesgue integral does not integrate the derivatives of all differentiable functions led to the development of the Denjoy-Perron integral. Three equivalent definitions of the Denjoy-Perron integral were available at the beginning of the last century: descriptive and constructive definitions presented by Denjoy [16, Chapter 8], and a definition based on approximations by majorants and minorants due to Perron [16, Chapter 6, Section 6]. These definitions differ widely, and establishing their equivalence is not easy — for a comprehensive treatment see [4, Chapter 11]. Neither definition is simple, and attempts to generalize any of them to higher dimensions were not successful. Indeed, none of the multidimensional integrals based on these definitions integrates partial derivatives of all differentiable functions.

It was a major independent accomplishment of Henstock [5] and Kurzweil [10] to observe that a minor but ingenious change in the classical definition of the Riemann integral produces the Denjoy-Perron integral. The striking simplicity of their definition revitalized the efforts toward finding a multidimensional analog of the Denjoy-Perron integral. The initial impetus was further promoted by Henstock's diligent work on the general properties of Riemann type integrals, summarized in monographs [6, 7, 8].

After introducing the Henstock-Kurzweil result without proof, I will discuss two separate but related topics:

(1) the relationship between the Lebesgue and Denjoy-Perron integrals;

(2) a heuristic motivation for the multidimensional definition of the Denjoy-Perron integral.

1 The Henstock-Kurzweil theorem The set of all real numbers is denoted by $\mathbb{R}$, and for $E \subset \mathbb{R}$, we denote by $d(E)$ and $|E|$ the diameter and Lebesgue measure (i.e., outer Lebesgue measure) of E, respectively. We say that sets $A, B \subset \mathbb{R}$ *overlap* if $|A \cap B| > 0$. A *cell* is a nondegenerate compact subinterval of $\mathbb{R}$. Unless specified otherwise, all concepts related to measure refer to Lebesgue measure. Without additional attributes, a number is a real number, and a function is a real-valued function.

Let A be a cell. If a function f defined on a cell A is integrable, either in the sense of Lebesgue or Denjoy-Perron, we denote by $\int_B f$ the integral of f over a cell $B \subset A$. The

2000 *Mathematics Subject Classification.* Primary: 26B15, 26B20, 26B30. Secondary: 28A75, 51M25.
Key words and phrases. Riemann type integration, cells and figures, partitions, gages.

primitive of f, Lebesgue or Denjoy-Perron as the case may be, is the function

$$\int f : B \mapsto \int_B f$$

defined for each cell $B \subset A$. Since the Denjoy-Perron integral is an extension of the Lebesgue integral, our notation leads to no confusion. For emphasis, we occasionally denote the Lebesgue integral by $(L)\int$.

If F is a function defined on a cell A, we let

$$F(B) := F(d) - F(c)$$

for every cell $B := [c, d]$ contained in A. No confusion will arise from denoting by the same letter the function defined on A, as well as the associated function defined on all subcells of A. Note that F, as a function of cells, is *additive* in the usual sense.

A *weak partition* is a finite, possibly empty, collection

$$P := \big\{(A_1, x_1), \dots, (A_p, x_p)\big\}$$

where $A_1, \dots, A_p$ are nonoverlapping cells, and $\{x_1, \dots, x_p\}$ is a subset of $\mathbb{R}$. The set $[P] := \bigcup_{i=1}^p A_i$ is called the *body* of P. Given a nonnegative function δ whose domain contains $\{x_1, \dots, x_p\}$, we say that P is δ-*fine* if

$$d\big(A_i \cup \{x_i\}\big) < \delta(x_i) \text{ for } i = 1, \dots, p.$$

Note that if P is δ-fine, then $\{x_1, \dots, x_p\} \subset \{\delta > 0\}$. A weak partition P is called a *partition* whenever $x_i \in A_i$ for $i = 1, \dots, p$.

Theorem 1.1 (Henstock-Kurzweil). *A function f defined on a cell A is Denjoy-Perron integrable if and only if there is a number I having the following property: given $\varepsilon > 0$, we can find a positive function δ defined on A such that*

$$\left|\sum_{i=1}^p f(x_i)|A_i| - I\right| < \varepsilon$$

for each δ-fine partition $P := \big\{(A_1, x_1), \dots, (A_p, x_p)\big\}$ with $[P] = A$. In this case $\int_A f = I$.

Using the Perron definition of the Denjoy-Perron integral and the *variational integral* introduced by Henstock [4, Definition 11.7], the proof of Theorem 1.1 is surprisingly simple. We refer the interested reader to [4, Chapter 11].

One of the most important features of the Denjoy-Perron integral is its ability to integrate derivatives of all differentiable functions. In particular, it provides the *unrestricted fundamental theorem of calculus* (Theorem 1.2 below) — the word 'unrestricted' indicates that the integrability of derivative is not assumed but proved. This result is an immediate consequence of the Henstock-Kurzweil theorem.

Theorem 1.2. *Let F be a differentiable function defined in an open set $U \subset \mathbb{R}$. Then F' is Denjoy-Perron integrable in every cell $A \subset U$, and*

$$\int_A F' = F(A).$$

Proof. Choose an $\varepsilon > 0$, and for each $x \in U$ find a $\delta(x) > 0$ so that

$$\Big|F(B) - F'(x)|B|\Big| < \varepsilon|B|$$

for every cell $B \subset U$ with $x \in B$ and $d(B) < \delta(x)$. Now

$$\delta : x \mapsto \delta(x) : A \to \mathbb{R}$$

is a positive function. If $P := \big\{(A_1, x_1), \dots, (A_p, x_p)\big\}$ is a δ-fine partition and $[P] = A$, then

$$\left|\sum_{i=1}^{p} F'(x_i)|A_i| - F(A)\right| \leq \sum_{i=1}^{p}\Big|F'(x_i)|A_i| - F(A_i)\Big| < \varepsilon \sum_{i=1}^{p} |A_i| = \varepsilon|A|,$$

and the theorem follows from Theorem 1.1. □

The following variation of the Henstock-Kurzweil theorem will be convenient for our purposes.

Theorem 1.3. *A function f defined on a cell A is Denjoy-Perron integrable if and only if there is a continuous function F defined on A having the following property: given $\varepsilon > 0$, we can find a positive function δ defined on A such that*

$$\sum_{i=1}^{p}\Big|f(x_i)|A_i| - F(A_i)\Big| < \varepsilon$$

for each δ-fine partition $P := \big\{(A_1, x_1), \dots, (A_p, x_p)\big\}$ with $[P] \subset A$. In this case $F = \int f$ is the Denjoy-Perron primitive of f.

It is clear that Theorem 1.3 implies Theorem 1.1. The converse, often referred to as *Henstock's lemma*, is also true and easy to establish [4, Lemma 9.11].

2 The Lebesgue and generalized Riemann integrals Independently of Henstock and Kurzweil, McShane [12] obtained a Riemannian definition of the Lebesgue integral.

Theorem 2.1 (McShane). *A function f defined on a cell A is Lebesgue integrable if and only if there is a number I having the following property: given $\varepsilon > 0$, we can find a positive function δ defined on A such that*

$$\left|\sum_{i=1}^{p} f(x_i)|A_i| - I\right| < \varepsilon$$

for each δ-fine **weak** *partition $P := \big\{(A_1, x_1), \dots, (A_p, x_p)\big\}$ with $[P] = A$. In this case $\int_A f = I$.*

An easy proof of Theorem 2.1 based on the Vitali-Carathéodory theorem [16, Chapter 3, Theorem 7.6] is presented in [13, Sections 4.3 and 4.4].

While there is a striking formal similarity between Theorems 1.1 and 2.1, the actual difference is profound and not obvious. Employing the idea of Thomson [17], we illuminate the difference by finding convenient descriptive definitions of the Denjoy-Perron and Lebesgue integrals.

Theorem 2.2. *A function f defined on a cell A is Lebesgue integrable if and only if there is a continuous function F defined on A having the following property: given $\varepsilon > 0$, we can find a positive function δ defined on A such that*

$$\sum_{i=1}^{p} \Big| f(x_i)|A_i| - F(A_i) \Big| < \varepsilon$$

for each δ-fine **weak** *partition $P := \big\{(A_1, x_1), \dots, (A_p, x_p)\big\}$ with $[P] \subset A$. In this case $F = \int f$ is the Lebesgue primitive of f.*

The relationship between Theorems 2.1 and 2.2 is the same as that between Theorems 1.1 and 1.3 mentioned in Section 1. For a direct proof we refer to [13, Lemma 2.3.1].

Let F be a function defined on a cell A. For a set $E \subset A$, let

$$V_* F(E) := \inf_{\delta} \sup_{P} \sum_{i=1}^{p} \big| F(A_i) \big|$$

where δ is a positive function defined on E, and $P := \big\{(A_1, x_1), \dots, (A_p, x_p)\big\}$ is a δ-fine partition with $[P] \subset A$. A straightforward argument shows that the extended real-valued function

$$V_* F : E \mapsto V_* F(E)$$

defined for each set $E \subset A$ is a Borel measure in A. The relationship between $V_* F$ and the usual variation VF of F is based on the next observation, referred to as *Cousin's lemma.*

Lemma 2.3 (Cousin). *For each positive function δ defined on a cell A there is a δ-fine partition P with $[P] = A$.*

Cousin's lemma is proved by contradiction using the compactness of cells [13, Proposition 1.2.4].

Proposition 2.4. *If F is a function defined on a cell A, then $VF(A) = V_* F(A)$ and $VF(B) \le V_* F(B)$ for each cell $B \subset A$.*

Proof. Clearly $V_* F(A) \le VF(A)$. Proceeding toward a contradiction assume that $V_* F(B) < VF(B)$ for a cell $B \subset A$, and find a positive function δ defined on B and nonoverlapping subcells $C_1, \dots, C_k$ of B so that

$$\sum_{i=1}^{p} \big| F(A_i) \big| < \sum_{j=1}^{k} \big| F(C_j) \big|$$

for each δ-fine partition $P := \big\{(A_1, x_1), \dots, (A_p, x_p)\big\}$ with $[P] \subset B$. By Cousin's lemma, for each $j = 1, \dots, k$, there is a δ-fine partition

$$P_j := \big\{(A_{1,j}, x_{1,j}), \dots, (A_{p_j,j}, x_{p_j,j})\big\}$$

with $[P_j] = C_j$. A contradiction follows, since

$$\sum_{j=1}^{k} \big| F(C_j) \big| = \sum_{j=1}^{k} \Big| \sum_{i=1}^{p_j} F(A_{i,j}) \Big| \le \sum_{j=1}^{k} \sum_{i=1}^{p_j} \big| F(A_{i,j}) \big|$$

and $P := \bigcup_{j=1}^{k} P_j$ is a δ-fine partition with $[P] \subset B$. $\square$

Example 2.5. Let C be the Cantor ternary set in $A := [0,1]$, and let F be the associated Cantor-Vitali function (*devil's staircase*) [14, Example 3.2.3]. For each $x \in A - C$, denote by $\delta(x)$ the distance from x to C. As C is closed, $\delta : x \mapsto \delta(x)$ is a positive function defined on $A - C$. If $\{(A_1, x_1), \dots, (A_p, x_p)\}$ is a δ-fine partition, then $\sum_{i=1}^{p} |F(A_i)| = 0$. It follows that $V_* F(A - C) = 0$, and Proposition 2.4 implies

$$V_* F(C) = V_* F(A) = V F(A) = 1.$$

Proposition 2.6. *Let F be a function defined on a cell A. If the measure $V_* F$ is absolutely continuous, then F is differentiable at almost all $x \in A$.*

Proof. Denote by $\overline{F}(x)$ the upper derivative of F at $x \in A$. In view of Ward theorem [16, Chapter 4, Theorem 11.15] it suffices to show that the set

$$E := \{x \in A : \overline{F}(x) = \infty\}$$

is negligible. Seeking a contradiction, assume $|E| > 0$, and find a compact set $K \subset E$ so that $|K \cap U| > 0$ for each open set $U \subset \mathbb{R}$ with $K \cap U \neq \emptyset$. Select a negligible G_δ set $D \subset K$ dense in K, and a positive function δ defined on D. By the Baire category theorem, there is a $t > 0$ and an open set $U \subset \mathbb{R}$ such that $D \cap U \neq \emptyset$ and the set

$$D_t := \{x \in D \cap U : \delta(x) > t\}$$

is dense in $D \cap U$, and consequently in $K \cap U$. The family $\mathcal{C}$ of all cells $C \subset A$ with $F(C) > |C|/|K \cap U|$ and $d(C) < t$ is a Vitali cover of $K \cap U$. By Vitali's covering theorem, there is a disjoint countable family $\mathcal{C}_0 \subset \mathcal{C}$ that covers $K \cap U$ almost entirely. With no loss of generality, we may assume that the interior of each $C \in \mathcal{C}_0$ meets $K \cap U$ and select an $x_C \in C \cap D_t$. Since

$$\sum_{C \in \mathcal{C}_0} |F(C)| \geq \sum_{C \in \mathcal{C}_0} F(C) > \frac{1}{|K \cap U|} \sum_{C \in \mathcal{C}_0} |C_i| \geq 1,$$

there is a finite collection $\mathcal{F} \subset \mathcal{C}_0$ such that $\sum_{C \in \mathcal{F}} |F(C)| > 1$. As

$$P := \{(C, x_C) : C \in \mathcal{F}\}$$

is a δ-fine partition and $[P] \subset A$, the arbitrariness of δ implies $V_* F(D) \geq 1$, a contradiction. $\square$

Theorem 2.7. *Let F be a function defined on a cell A.*

(i) The measure $V_* F$ is absolutely continuous if and only if F is a Denjoy-Perron primitive.

(ii) The measure $V_* F$ is absolutely continuous and finite if and only if F is a Lebesgue primitive.

In either case, F is differentiable almost everywhere and $F = \int F'$.

Proof. (i) Assume $V_* F$ is absolutely continuous, and denote by N the negligible set of those $x \in A$ at which F is not differentiable (Proposition 2.6). Let

$$f(x) := \begin{cases} 0 & \text{if } x \in N, \\ F'(x) & \text{if } x \in A - N, \end{cases}$$

and choose an $\varepsilon > 0$. By our assumption, there is a positive function δ_N defined on N such that

$$\sum_{i=1}^{p} \bigl|F(A_i)\bigr| < \varepsilon$$

for each δ-fine partition $P := \bigl\{(A_1, x_1), \dots, (A_p, x_p)\bigr\}$ with $[P] \subset A$. Given $x \in A - N$, find a $\sigma(x) > 0$ so that

$$\Bigl|F(B) - f(x)|B|\Bigr| < \varepsilon |B|$$

for each cell $B \subset A$ with $x \in B$ and $d(B) < \sigma(x)$. Define a positive function δ on A by the formula

$$\delta(x) := \begin{cases} \delta_N(x) & \text{if } x \in N, \\ \sigma(x) & \text{if } x \in A - N. \end{cases}$$

Now if $P := \bigl\{(A_1, x_1), \dots, (A_p, x_p)\bigr\}$ is a δ-fine partition with $[P] \subset A$, then

$$\begin{aligned} \sum_{i=1}^{p} \Bigl| f(x_i)|A_i| - F(A_i) \Bigr| &= \sum_{x_i \in N} \bigl|F(A_i)\bigr| + \sum_{x_i \notin N} \Bigl| f(x_i)|A_i| - F(A_i) \Bigr| \\ &< \varepsilon + \varepsilon \sum_{x_i \notin N} |A_i| \le \varepsilon\bigl(1 + |A|\bigr). \end{aligned}$$

Thus F is the Denjoy-Perron primitive of f by Theorem 1.3, and since $F' = f$ almost everywhere, $F = \int F'$.

Conversely, assume that F is the Denjoy-Perron primitive of a function f defined on A, and select a negligible set $N \subset A$. With no loss of generality, we may assume $f(x) = 0$ for each $x \in N$. Given $\varepsilon > 0$, there is a positive function δ defined on A such that

$$\sum_{i=1}^{p} \Bigl| f(x_i)|A_i| - F(A_i) \Bigr| < \varepsilon$$

for each δ-fine partition $P := \bigl\{(A_1, x_1), \dots, (A_p, x_p)\bigr\}$ with $[P] \subset A$. Let δ_N be the restriction of δ to N, and let $Q := \bigl\{(B_1, y_1), \dots, (B_q, y_q)\bigr\}$ be a δ_N-fine partition with $[Q] \subset A$. Then Q is a δ-fine partition and each y_j belongs to N. It follows

$$\sum_{j=1}^{q} \bigl|F(B_j)\bigr| = \sum_{j=1}^{q} \Bigl| f(x_j)|B_j| - F(B_j) \Bigr| < \varepsilon,$$

and we infer $V_* F(N) \le \varepsilon$. The measure $V_* F$ is absolutely continuous by the arbitrariness of ε.

(ii) Assume $V_* F$ is absolutely continuous and finite, and choose an $\varepsilon > 0$. A standard argument shows that there is a $\delta > 0$ such that $V_* F(E) < \varepsilon$ for each Borel set $E \subset A$ with $|E| < \delta$. If $C_1, \dots, C_k$ is a nonoverlapping collection of cell contained in A and $\bigl|\bigcup_{i=1}^{k} C_i\bigr| = \sum_{i=1}^{k} |C_i| < \delta$, then Proposition 2.4 yields

$$\sum_{i=1}^{k} \bigl|F(C_i)\bigr| \le \sum_{i=1}^{k} VF(C_i) \le \sum_{i=1}^{k} V_* F(C_i) = V_* F\Bigl(\bigcup_{i=1}^{k} C_i\Bigr) < \varepsilon.$$

It follows that F is an absolutely continuous function, and hence the Lebesgue primitive of F'.

Conversely, assume F is a Lebesgue primitive of a function defined on A. Then F is absolutely continuous, and Proposition 2.4 yields $V_*F(A) = VF(A) < \infty$. Choose an $\varepsilon > 0$ and find a $\delta > 0$ so that $\sum_{i=1}^{k} \bigl|F(C_i)\bigr| < \varepsilon$ for each nonoverlapping collection $C_1, \ldots, C_k$ of cells contained in A for which $\sum_{i=1}^{k} |C_i| < \delta$. Given a negligible set $N \subset A$, find an open set U with $N \subset U$ and $|U| < \delta$. There is a positive function σ defined on N such that

$$\bigl[x - \sigma(x), s + \sigma(x)\bigr] \subset U$$

for each $x \in N$. Thus if $P := \bigl\{(A_1, x_1), \ldots, (A_p, x_p)\bigr\}$ is a σ-fine partition with $[P] \subset A$, then $[P] \subset U$. Hence $\sum_{i=1}^{p} |A_i| \le |U| < \delta$, which implies

$$\sum_{i=1}^{p} \bigl|F(A_i)\bigr| < \varepsilon.$$

As P and ε are arbitrary, we conclude $V_*F(N) = 0$. □

Unlike the classical characterization of Denjoy-Perron primitives by means of ACG$_*$ functions [4, Definition 7.1], the characterization presented in Theorem 2.7 does not depend on the order structure of $\mathbb{R}$. This is an important point which facilitates an extension of the Denjoy-Perron integral to higher dimensions — see Theorem 3.13 below and [14, Section 5.1].

3 Multidimensional integration In this section, we replace one-dimensional cells by cells in $\mathbb{R}^n$, i.e., by nondegenerate compact subintervals of $\mathbb{R}^n$. For a set $E \subset \mathbb{R}^n$, we denote by $d(E)$, ∂E, and $|E|$ the diameter, boundary, and n-dimensional Lebesgue measure of E, respectively. In $\mathbb{R}^n$ we use the usual Euclidean norm $|\cdot|$ induced by the inner product $x \cdot y$. The $(n-1)$-dimensional *Hausdorff measure* in $\mathbb{R}^n$ is denoted by $\mathcal{H}$. If $A \subset \mathbb{R}^n$ is a cell, then $\|A\| = \mathcal{H}(\partial A)$ is the *perimeter* of A, and we denote by ν_A the *unit exterior normal* of A, which exists $\mathcal{H}$ almost everywhere on ∂A.

Defining δ-fine partitions in $\mathbb{R}^n$ in the obvious way, Theorem 1.1 suggests a straightforward n-dimensional generalization of the Denjoy-Perron integral.

Definition 3.1. A function f defined on a cell $A \subset \mathbb{R}^n$ is called *Henstock integrable* if there is a number I having the following property: given $\varepsilon > 0$, we can find a positive function δ defined on A such that

$$\left|\sum_{i=1}^{p} f(x_i)|A_i| - I\right| < \varepsilon$$

for each δ-fine partition $P := \bigl\{(A_1, x_1), \ldots, (A_p, x_p)\bigr\}$ with $[P] = A$.

Using the n-dimensional version of Cousin's lemma [13, Section 7.3], it is easy to see that the number I of Definition 3.1 is determined uniquely by f. We call it the *Henstock integral* of f over A.

An easy argument shows that the Henstock integral has the usual properties connected with the word integral, and that it is a proper extension of the Lebesgue integral in $\mathbb{R}^n$. Notwithstanding, the Henstock integral does not integrate partial derivatives of all differentiable functions, since it satisfies Fubini's theorem [6, Chapter 6] — for the conflict between Fubini's theorem and the integrability of derivatives see [13, Section 11.1]. A substantial modification of Definition 3.1 is required to obtain an integral that provides the unrestricted fundamental theorem of calculus in $\mathbb{R}^n$.

Desired Result. *Let $U \subset \mathbb{R}^n$ be an open set, and let $v : U \to \mathbb{R}^n$ be a differentiable vector field. Then* $\operatorname{div} v$ *is integrable in every cell $A \subset U$, and*

$$\int_A \operatorname{div} v = (L)\int_{\partial A} v \cdot \nu_A \, d\mathcal{H}.$$

Although the desired result is not achieved by Henstock's integral, attempting to prove it in the same way we proved Theorem 1.2 is instructive. Hence choose an $\varepsilon > 0$, and for each $x \in U$ find a $\delta(x) > 0$ so that

$$\left| \operatorname{div} v(x)|B| - (L)\int_{\partial B} v \cdot \nu_B \, d\mathcal{H} \right| < \varepsilon d(B)\|B\|$$

for each cell $B \subset U$ with $x \in B$ and $d(B) < \delta(x)$ [14, Example 2.3.2]. Observe that $\delta : x \mapsto \delta(x)$ is a positive function defined on A, and select a δ-fine partition $P := \big\{(A_1, x_1), \ldots, (A_p, x_p)\big\}$ with $[P] = A$. Then

$$\left| \sum_{i=1}^{p} \operatorname{div} v(x_i)|A_i| - (L)\int_{\partial A} v \cdot \nu_A \, d\mathcal{H} \right| \le \sum_{i=1}^{p} \left| \operatorname{div} v(x_i)|A_i| - (L)\int_{\partial A_i} v \cdot \nu_{A_i} \, d\mathcal{H} \right| < \varepsilon \sum_{i=1}^{p} d(A_i)\|A_i\|,$$

and it is clear that without restricting the shapes of cells $A_1, \ldots, A_p$, the right side of this inequality *need not be bounded*.

The original shape restriction, due to Mawhin [11], has been refined to Vitali's type regularity condition. The *regularity* of a cell $A \subset \mathbb{R}^n$ is the number

$$r(A) := \frac{|A|}{d(A)\|A\|}.$$

A partition $\big\{(A_1, x_1), \ldots, (A_p, x_p)\big\}$ is called *η-regular* if $r(A_i) > \eta > 0$ for $i = 1, \ldots, p$. With the regularity requirement,

$$\varepsilon \sum_{i=1}^{p} d(A_i)\|A_i\| < \frac{\varepsilon}{\eta} \sum_{i=1}^{p} |A_i| = \frac{\varepsilon}{\eta}|A|.$$

Consequently, the desired result holds for the Henstock integral modified according to the following definition.

Definition 3.2. A function f defined on a cell $A \subset \mathbb{R}^n$ is called *generalized Henstock integrable* if there is a number I having the following property: given $\varepsilon > 0$, we can find a positive function δ defined on A such that

$$\left| \sum_{i=1}^{p} f(x_i)|A_i| - I \right| < \varepsilon$$

for each ε-**regular** δ-fine partition $P := \big\{(A_1, x_1), \ldots, (A_p, x_p)\big\}$ with $[P] = A$.

It is easy to see that for sufficiently small $\varepsilon > 0$, the n-dimensional analog of Cousin's lemma guarantees the existence of an ε-regular δ-fine partition P with $[P] = A$ [13, Section 7.3]. Thus the number I in Definition 3.2 is unique, and we call it the *generalized Henstock integral* of f over A.

Unfortunately, the generalized Henstock integral still has serious deficiencies. It is not rotation invariant, and not additive in the sense that integrability over nonoverlapping cells A and B whose union is a cell C need not imply integrability over C.

Since lipeomorphic images of cells can be approximated by *figures*, i.e., finite unions of cells, replacing partitions consisting of cells by partitions consisting of figures paves the way for obtaining a coordinate free integral. An *f-partition* is a finite, possibly empty, collection

$$P := \{(A_1, x_1), \dots, (A_p, x_p)\}$$

where $A_1, \dots, A_p$ are nonoverlapping figures, and $x_i \in A_i$ for $i = 1, \dots, p$. Since the perimeter $\|A\|$ and the regularity $r(A)$ are defined for each figure A, all concepts connected with partitions extend, in the obvious way, to f-partitions. Moreover, Definition 3.2 remains meaningful when partitions are replaced by f-partitions.

To achieve additivity positive functions δ must be replaced by nonnegative functions δ whose null sets $\{\delta = 0\}$ are of σ-finite measure $\mathcal{H}$. Such a function defined on a figure A is called a *gage* on A. Theorem 3.3 below, which is a simple generalization of Theorem 1.3, and the resulting corollary explain why gages yield additivity. Note that if $n = 1$, then $\mathcal{H}$ is a counting measure, and hence a gage is a nonnegative function whose null set is countable.

Theorem 3.3. *A function f defined on a cell $A \subset \mathbb{R}$ is Denjoy-Perron integrable if and only if there is a continuous function F defined on A having the following property: given $\varepsilon > 0$, we can find a* **gage** *δ on A such that*

$$\sum_{i=1}^{p} \Big| f(x_i)|A_i| - F(A_i) \Big| < \varepsilon$$

for each δ-fine partition $P := \{(A_1, x_1), \dots, (A_p, x_p)\}$ with $[P] \subset A$. In this case $F = \int f$ is the Denjoy-Perron primitive of f.

Proof. As the converse is obvious, choose an $\varepsilon > 0$ and suppose there is a gage δ on A such that

$$\sum_{i=1}^{p} \Big| f(x_i)|A_i| - F(A_i) \Big| < \varepsilon$$

for each δ-fine partition $P := \{(A_1, x_1), \dots, (A_p, x_p)\}$ with $[P] \subset A$. With no loss of generality, we may assume that $f(z) = 0$ for each z in the countable set $\{\delta = 0\}$. Order $\{\delta = 0\}$ into a sequence $\{z_k\}$ and, using the continuity of F, select $r_k > 0$ so that $|F(B)| < \varepsilon 2^{-k}$ for each cell $B \subset A$ with $z_k \in B$ and $d(B) < r_k$. The formula

$$\sigma(x) := \begin{cases} \delta(x) & \text{if } \delta(x) > 0, \\ r_k & \text{if } x = z_k, \end{cases}$$

defines a positive function σ on A. Let $P := \{(A_1, x_1), \dots, (A_p, x_p)\}$ be a σ-fine partition with $[P] \subset A$. Then $Q := \{(A_i, x_i) : \delta(x_i) > 0\}$ is a δ-fine partition and $[Q] \subset [P] \subset A$. Thus

$$\begin{aligned} \sum_{i=1}^{p} \Big| f(x_i)|A_i| - F(A_i) \Big| &= \sum_{\delta(x_i) > 0} \Big| f(x_i)|A_i| - F(A_i) \Big| + \sum_k \sum_{x_i = z_k} |F(A_i)| \\ &< \varepsilon + \varepsilon \sum_k 2^{-k} \le 2\varepsilon, \end{aligned}$$

and the theorem follows from Theorem 1.3. $\square$

Corollary 3.4. *Let a cell $A \subset \mathbb{R}$ be the union of nonoverlapping cells B and C, and let f be a function defined on A. If f is Denjoy-Perron integrable in B and C, then it is Denjoy-Perron integrable in A and*

$$\int_A f = \int_B f + \int_C f.$$

Proof. Let F_B and F_C be the Denjoy-Perron primitives of f on B and C, respectively, and let

$$F(D) := F_B(B \cap D) + F_C(C \cap D)$$

for each cell $D \subset A$. Observe that F is associated with a continuous function

$$x \mapsto F\big((-\infty, x] \cap A\big) : A \to \mathbb{R},$$

and choose an $\varepsilon > 0$. There are gages δ_B on B and δ_C on C such that

$$\sum_{i=1}^{p} \Big| f(x_i)|B_i| - F(B_i) \Big| < \varepsilon \quad \text{and} \quad \sum_{i=1}^{q} \Big| f(y_i)|C_i| - F(C_i) \Big| < \varepsilon$$

for each δ_B-fine partition $P_B := \big\{(B_1, x_1), \dots, (B_p, x_p)\big\}$ with $[P_B] \subset B$ and for each δ_C-fine partition $P_C := \big\{(C_1, y_1), \dots, (C_q, y_q)\big\}$ with $[P_C] \subset C$. Making δ_B and δ_C smaller, we may assume that $\delta_B(x)$ is smaller than or equal to the distance from $x \in B$ to ∂B, and that $\delta_C(x)$ is smaller than or equal to the distance from $x \in C$ to ∂C. Then the formula

$$\delta(x) := \begin{cases} \delta_B(x) & \text{if } x \in B, \\ \delta_C(x) & \text{if } x \in C, \end{cases}$$

defines a gage on A, which is zero on the boundaries of B and C. If $P := \big\{(A_1, x_1), \dots, (A_p, x_p)\big\}$ is a δ-fine partition, then

$$P_B := \big\{(A_i, x_i) : x_i \in B\big\} \quad \text{and} \quad P_C := \big\{(A_i, x_i) : x_i \in C\big\}$$

are δ_B-fine and δ_C-fine partitions, respectively, and $[P_B] \subset B$ and $[P_C] \subset C$; in particular $[P] \subset A$. Thus

$$\sum_{i=1}^{p} \Big| f(x_i)|A_i| - F(A_i) \Big| = \sum_{x_i \in B} \Big| f(x_i)|A_i| - F_B(A_i) \Big| + \sum_{x_i \in C} \Big| f(x_i)|A_i| - F_C(A_i) \Big| < 2\varepsilon,$$

and the corollary follows from Theorem 3.3. □

Being one-dimensional, Corollary 3.4 can be proved directly, without using gages [4, Theorem 9.8]. However, gages are indispensable in higher dimensions: they allow us to show that $P = P_B \cup P_C$, which means that if P is ε-regular, then so are P_B and P_C.

On the other hand, gages cannot be used without an additional modification of Definition 3.2. Indeed, our proof of Corollary 3.4 relies on the continuity of Denjoy-Perron primitives. Moreover, Cousin's lemma is false for gages — for a gage $\delta : x \mapsto |x|$ on a cell $A := [0, 1]^n$ in $\mathbb{R}^n$, there is no δ-fine f-partition P with $[P] = A$. Still there are δ-fine partitions P such that $[P] \subset A$ is a useful approximation of A (cf. Lemma 3.9 below). Thus a way out is in replacing Definition 3.2 by a definition modeled on Theorem 1.3. To this end, we must first introduce continuous functions defined on subfigures of $\mathbb{R}^n$, called *charges*.

As we mentioned earlier, a function F defined on a cell $A \subset \mathbb{R}$ generates a unique additive function F of all subcells of A, which in turn extends to a unique additive function of all subfigures of A, still denoted by F. If the point function F is continuous, it is uniformly continuous, which implies that the associated function of subfigures of A satisfies the following continuity condition:

If $\{B_k\}$ *is a sequence of subfigures of* A *such that* $\lim |B_k| = 0$ *and the number of connected components of each* B_k *is bounded, then* $\lim F(B_k) = 0$.

Noting that the perimeter $\|B\|$ of a figure $B \subset \mathbb{R}$ equals twice the number of the connected components of B motivates the next definition.

Definition 3.5. An additive function F of all subfigures of a figure $A \subset \mathbb{R}^n$ is called a *charge* in A if $\lim F(A_i) = 0$ for each sequence $\{A_i\}$ of subfigures of A such that $\lim |A_i| = 0$ and $\sup \|A_i\| < \infty$.

Example 3.6. Let $A \subset \mathbb{R}^n$ be a figure, and let f be a Lebesgue integrable function defined on A. It follows from the absolute continuity of the Lebesgue integral that the function

$$B \mapsto (L)\int_B f$$

defined for each figure $B \subset A$ is a charge in A, called the *Lebesgue primitive* of f and denoted by $(L)\int f$.

Example 3.7. Let $A \subset \mathbb{R}^n$ be a figure, and let $v : A \to \mathbb{R}^n$ be a continuous vector field. Approximating v uniformly by a continuously differentiable vector field an using the classical Gauss-Green theorem, it is easy to see that the function

$$B \mapsto (L)\int_{\partial B} v \cdot \nu_B \, d\mathcal{H}$$

defined for each figure $B \subset A$ is a charge in A, called the *flux* of v [14, Example 2.1.4].

Remark 3.8. Examples 3.6 and 3.7 are canonical in the following sense. Given a charge F in a figure A there are a Lebesgue integrable function $f : A \to \mathbb{R}$ and a continuous vector field $v : A \to \mathbb{R}^n$ such that

$$F(B) = (L)\int_B f + (L)\int_{\partial B} v \cdot \nu_B \, d\mathcal{H}$$

for every figure $B \subset A$. A proof of this nontrivial fact is given in [1, Section 6].

If $B \subset A \subset \mathbb{R}^n$ are figures, we denote by $A \ominus B$ the unique figure C such that $A = B \cup C$ and $|B \cap C| = 0$. The next result, due to Howard [9], is a far reaching generalization of Cousin's lemma. Its proof can be found in [9], or in [14, Corollary 2.6.5].

Lemma 3.9. *Let* F *be a charge in a figure* $A \subset \mathbb{R}^n$*, and let* δ *be a gage on* A*. For each sufficiently small* $\varepsilon > 0$*, there is an* ε*-regular* δ*-fine partition* P *such that* $[P] \subset A$ *and*

$$\Big| F\big(A \ominus [P]\big) \Big| < \varepsilon.$$

Definition 3.10. A function f defined on a figure $A \subset \mathbb{R}^n$ is called *R-integrable* if there is a charge F in A having the following property: given $\varepsilon > 0$, there is a gage δ on A such that

$$\sum_{i=1}^{p} \Big| f(x_i)|A_i| - F(A_i) \Big| < \varepsilon$$

for each ε-regular δ-fine f-partition $P := \big\{(A_1, x_1), \dots, (A_p, x_p)\big\}$ with $[P] \subset A$.

It follows from Lemma 3.9 that the charge F in Definition 3.10 is unique. We call it the *R-primitive* of f in A, denoted by $(R)\int f$. The number $(R)\int_A f := F(A)$ is called the *R-integral* of f over A. The letter 'R' emphasizes that we are still dealing with a Riemann type integral.

The R-integral extends the Lebesgue integral, but not the Henstock integral. Indeed, replacing partitions by f-partitions restricts the space of integrable functions already in dimension one [13, Example 12.3.5]. However, there are ample compensations for this restriction in higher dimensions.

Since each gage defined on a figure A is still a gage when we change its values to zero on the boundary of a figure $B \subset A$, the additivity of R-integral follows by the same argument we used in proving Corollary 3.4. As we employ figures, the R-integral is invariant with respect to lipeomorphisms. Detailed proofs of these facts are in [13, Chapter 12].

Let $A \subset \mathbb{R}^n$ be a figure. A vector field $v : A \to \mathbb{R}^n$ is called Lipschitz at an interior point x of A if

$$\limsup_{y \to x} \frac{\big|v(y) - v(x)\big|}{|y - x|} < \infty.$$

By Stepanoff's theorem [3, Theorem 3.1.9], if E is the set of interior points of A such that v is Lipschitz at each $x \in E$, then v is differentiable at almost all $x \in E$.

The following theorem, proved in [13, Theorem 12.2.5], generalizes the desired result stated above. It is a culmination of a long struggle to obtain a multidimensional version of the Denjoy-Perron integral — a memento of Henstock's seminal ideas.

Theorem 3.11. *Let $A \subset \mathbb{R}^n$ be a figure, and let $E \subset A$ be a set of σ-finite measure $\mathcal{H}$ that contains ∂A. If $v : A \to \mathbb{R}^n$ is a continuous vector field that is Lipschitz at each $x \in A - E$, then* $\operatorname{div} v$ *is R-integrable in A and*

$$(R)\int_A \operatorname{div} v = (L)\int_{\partial A} v \cdot \nu_A \, d\mathcal{H}.$$

Remark 3.12. It is noteworthy that the R-integral retain its essential properties when figures are replaced by the bounded sets of *finite perimeters* [2, Chapter 5].

In essence, Definition 3.10 is an elaboration on Henstock's lemma (Theorem 1.3). Introducing a suitable topology in the space of all figures (or all bounded sets of finite perimeter), it is possible to define the R-integral by elaborating directly on the definition of Henstock's integral (Definition 3.1). The reader interested in this approach is referred to [14, Section 5.5].

We conclude our exposition by characterizing the relationship between the R-integral and Lebesgue integral in a form analogous to Theorem 2.7.

Let F be a function defined on all subfigures of a figure A. For a set $E \subset A$, let

$$V_{\#}F(E) := \sup_{\eta>0} \inf_{\delta} \sup_{P} \sum_{i=1}^{p} \bigl|F(A_i)\bigr|$$

where δ is a gage on E and $P := \bigl\{(A_1, x_1), \dots, (A_p, x_p)\bigr\}$ is an η-regular δ-fine f-partition with $[P] \subset A$. As in the one-dimensional case, it is easy to shows that the extended real-valued function

$$V_{\#}F : E \mapsto V_{\#}F(E)$$

defined for each set $E \subset A$ is a Borel measure in A. We say that F is *derivable* at an interior point x of A if a finite limit

$$\lim \frac{F(B_k)}{|B_k|}$$

exists for each sequence $\{B_k\}$ of subfigures of A such that $x \in B_k$ for $k = 1, 2, \dots,$

$$\lim d(A_k) = 0 \quad \text{and} \quad \inf r(A_k) > 0.$$

When all these limits exist, they have the same value, denoted by $F'(x)$.

If $A \subset \mathbb{R}^n$ is a figure and F is the flux of a continuous vector field $v : A \to \mathbb{R}^n$ that is differentiable at an interior point x of A, then F is derivable at x and

$$F'(x) = \operatorname{div} v(x)$$

[14, Example 2.3.2]. In the one-dimensional case, a function F defined on a figure A is differentiable at an interior point x of A if and only if the associated function F of subfigures of A is derivable at x, in which case $F'(x)$ is the usual derivative. In particular, the notation $F'(x)$ is consistent with the standard usage.

Theorem 3.13. *Let F be a charge in a figure A.*

(i) The measure $V_{\#}F$ is absolutely continuous if and only if the charge F is an R-primitive.

(ii) The measure $V_{\#}F$ is absolutely continuous and finite if and only if the charge F is a Lebesgue primitive.

In either case, the charge F is derivable almost everywhere, and $F = (R)\int F'$.

There is no easy proof of Theorem 3.13. We refer the interested reader to [15] or [14, Chapter 3].

References

[1] T. De Pauw and W.F. Pfeffer, *Distributions for which* $\operatorname{div} v = F$ *has continuous solution*, Comm. Pure Appl. Math., to appear.

[2] L.C. Evans and R.F. Gariepy, *Measure Theory and Fine Properties of Functions*, CRC Press, Boca Raton, 1992.

[3] H. Federer, *Geometric Measure Theory*, Springer-Verlag, New York, 1969.

[4] R.A. Gordon, *The Integrals of Lebesgue, Denjoy, Perron, and Henstock*, Amer. Math. Soc., Providence, 1994.

[5] R. Henstock, *Definitions of Riemann type of the variational integrals*, Proc. London Math. Soc. **11** (1961), 79–87.

[6] ______, *Theory of integration*, Butterworth, London, 1963.

[7] ______, *Lectures on the theory of integration*, World Scientific, Singapore, 1988.

[8] ______, *The general theory of integration*, Clarendon Press, Oxford, 1991.

[9] E.J. Howard, *Analyticity of almost everywhere differentiable functions*, Proc. American Math. Soc. **110** (1990), 745–753.

[10] J. Kurzweil, *Generalized ordinary differential equations and continuous dependence on a parameter*, Czechoslovak Math. J. **82** (1957), 418–446.

[11] J. Mawhin, *Generalized Riemann integrals and the divergence theorem for differentiable vector fields*, E.B. Christoffel (Basel), Birkhäuser, 1981, pp. 704–714.

[12] E.J. McShane, *A Riemann-type Integral that Includes Lebesgue-Stieltjes, Bochner and Stochastic Integrals*, Mem. Amer. Math. Soc., 88, Providence, 1969.

[13] W.F. Pfeffer, *The Riemann Approach to Integration*, Cambridge Univ. Press, New York, 1993.

[14] ______, *Derivation and Integration*, Cambridge Univ. Press, New York, 2001.

[15] ______, *Derivatives and primitives*, Sci. Math. Japonicae **55** (2002), 399–425.

[16] S. Saks, *Theory of the Integral*, Dover, New York, 1964.

[17] B.S. Thomson, *Derivatives of Interval Functions*, Mem. Amer. Math. Soc., 452, Providence, 1991.

Department of Mathematics University of California Davis, CA 95616

E-mail address: wfpfeffer@ucdavis.edu, washek@mcn.org