DISINTEGRATION OF OPERATOR VALUED WEIGHTS

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Abstract. The present paper is devoted to giving the existence and uniqueness theorem on disintegrations of operator valued weights by applying Sutherland's theorem on disintegrations of weights on a von Neumann algebra. To do so, we study the measurability and the integration of a field of operator valued weights. The results obtained here are expected to be useful for discussing reductions of inclusion relations of von Neumann subalgebras.

1 Introduction An operator valued weight on a von Neumann algebra M is an extended notion of both a weight on M and a conditional expectation of M onto a von Neumann subalgebra N of M. U. Haagerup [Ha1],[Ha2] established this notion which has become an important tool for describing inclusions of von Neumann subalgebras. H. Kosaki[Ko] used this notion combined with Connes' spatial theory [C] and succeeded to extend Jones index theory for an arbitrary pair of factor-subfactor. C. E. Sutherland investigated the direct integral theory for weights and gave the existence and uniqueness theorem on disintegrations of weights in [Su1][Su2]. In the present paper, we give definition for the measurability of fields of operator valued weights and the extended theorem for operator valued weights. The main theorem is as follows.

Theorem. *Let $M \supset N$ be a pair of von Neumann algebras acting on a separable Hilbert space and let E be a normal semifinite faithful N-valued weight on M. Then, for a von Neumann subalgebra A of $Z(M) \cap Z(N)$, there exist a standard measure space (Γ, μ) and a measurable field of a triple $\{M_\gamma, N_\gamma, E_\gamma\}_{\gamma \in \Gamma}$ over Γ such that*

(a) *A is the diagonal operator algebra isomorphic with $\mathrm{L}^\infty(\Gamma, \mu)$,*

(b) $\{M, N, E\} \cong \int_\Gamma^\oplus \{M_\gamma, N_\gamma, E_\gamma\} d\mu(\gamma)$,

(c) $\{N', M', E^{-1}\} \cong \int_\Gamma^\oplus \{{N_\gamma}', {M_\gamma}', {E_\gamma}^{-1}\} d\mu(\gamma)$.

Moreover, the above fields are uniquely determined up to a μ-negligible set.

This theorem is a general position of tools to describe the inclusion relation of von Neumann subalgebra, especially a reduction theory on index, entropy, and indicial derivative [KY],[Ka1],[Ka2].

2 Disintegration of operator valued weights C. E. Sutherland [Su1],[Su2] developed the direct integral theory for weights on von Neumann algebras related with modular Hilbert algebras and Plancherel formulas. We shall give an operator valued weights version of his existence and uniqueness theorem on the disintegration of weights. We shall frequently use his results hereafter.

Let M be a von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$. The positive part and the extended positive part of M are written as M^+ and $\overline{M^+}$ respectively.

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Definition 2.1. *Let $\{M_\gamma, N_\gamma\}$ be a pair of measurable fields of von Neumann algebras over a standard measure space (Γ, μ) with $M_\gamma \supset N_\gamma$. A field $\gamma \mapsto E_\gamma$ of N_γ-valued weights on M_γ over (Γ, μ) is said to be measurable if a field $\gamma \mapsto \varphi_\gamma \circ E_\gamma$ is measurable for any measurable field $\gamma \mapsto \varphi_\gamma$ of weights on N_γ over (Γ, μ) in a sense of Sutherland* [Su1].

Let $\{M, N\}$ denote the direct integral of a measurable field $\{M_\gamma, N_\gamma\}$ over (Γ, μ) and $\{M, N\} \cong \int_\Gamma^\oplus \{M_\gamma, N_\gamma\} d\mu(\gamma)$. For a measurable field $\gamma \mapsto w_\gamma \in (N_\gamma)_*^+$ with the predual $w = \int_\Gamma^\oplus w_\gamma d\mu(\gamma) \in N_*^+$, the field $\gamma \mapsto w_\gamma \circ E_\gamma$ is measurable as weights on M_γ. So that a function $\gamma \mapsto w_\gamma(E_\gamma(x_\gamma)) \in [0, +\infty]$ is measurable for a measurable field $\gamma \mapsto x_\gamma \in M_\gamma^+$ of operators with $x = \int_\Gamma^\oplus x_\gamma d\mu(\gamma) \in M^+$. Hence, we get

$$E(x)(w) = \int_\Gamma w_\gamma(E_\gamma(x_\gamma)) d\mu(\gamma) \in [0, +\infty].$$

If w runs over N_*^+, $E(x)$ is defined as an extended positive operator of N, and moreover if x runs over M^+, we obtain a map E from M^+ to $\overline{N^+}$. It is easy to check that E is N-valued weights on M. Hence, the direct integral $E = \int_\Gamma^\oplus E_\gamma d\mu(\gamma)$ is constructed. Sometimes we write this direct integral system as

$$\{M, N, E\} \cong \int_\Gamma^\oplus \{M_\gamma, N_\gamma, E_\gamma\} d\mu(\gamma).$$

For a pair $M \supset N$ of von Neumann algebras, $P(M, N)$ denotes the set of all normal semifinite faithful N-valued weights on M and $P(M)$ denotes $P(M, \mathbf{C})$.

Proposition 2.1. *Let $\{M_\gamma, N_\gamma, E_\gamma\}$ be a measurable field of N_γ-valued weights E_γ on M_γ over (Γ, μ) and $\{M, N, E\} \cong \int_\Gamma^\oplus \{M_\gamma, N_\gamma, E_\gamma\} d\mu(\gamma)$. Then,*

(a) *E is in $P(M, N)$ if and only if $E_\gamma \in P(M_\gamma, N_\gamma)$ for μ-a.a.$\gamma \in \Gamma$,*

(b) *If $E(x)$ is bounded for $x = \int_\Gamma^\oplus x_\gamma d\mu(\gamma) \in M^+$, then $E(x)_\gamma = E_\gamma(x_\gamma)$ for μ-a.a.$\gamma \in \Gamma$. Namely $E_\gamma(x_\gamma)$ is bounded for μ-a.a.$\gamma \in \Gamma$.*

Proof. (a) This follows immediately from Sutherland's result on direct integrals of weights (Corollary4.16 in [Su1]) combined with the fact that N-valued weight E on M is in $P(M, N)$ if and only if $\varphi \circ E \in P(M)$ for $\varphi \in P(N)$.

(b) If we restrict the fields $\gamma \mapsto w_\gamma$ in the above definition of measurability of operator valued weights within the value in preduals $(N_\gamma)_*^+$, then it means the measurability of $E(x)$ as bounded operators and its decomposability into the direct integrands. It shows that $E(x)_\gamma = E_\gamma(x_\gamma)$ and $E_\gamma(x_\gamma)$ are bounded operators for μ-a.a.$\gamma \in \Gamma$.

□

Proposition 2.2. *Let $\{M_\gamma, N_\gamma\}$ be a measurable field of von Neumann algebras with $M_\gamma \supset N_\gamma$ and $\gamma \mapsto E_\gamma \in P(M_\gamma, N_\gamma)$ a field of operator valued weights. If a field $\gamma \mapsto \varphi_\gamma \circ E_\gamma \in P(M_\gamma)$ is measurable for some measurable field $\gamma \mapsto \varphi_\gamma \in P(N_\gamma)$ of weights, then the field $\gamma \mapsto E_\gamma$ is measurable as operator valued weights. Moreover, if another weight $\psi \in P(N)$ is represented by $\{\psi_\gamma\}$ and the direct integral of $\{E_\gamma\}$ is denoted by E, we get $(\psi \circ E)_\gamma = \psi_\gamma \circ E_\gamma$ for μ-a.a.$\gamma \in \Gamma$.*

Proof. By the assumptions, we get weights

$$\varphi = \int_\Gamma^\oplus \varphi_\gamma d\mu(\gamma) \quad \text{on} \quad N = \int_\Gamma^\oplus N_\gamma d\mu(\gamma)$$

and

$$\widetilde{\varphi} = \int_\Gamma^\oplus \varphi_\gamma \circ E_\gamma d\mu(\gamma) \quad \text{on} \quad M = \int_\Gamma^\oplus M_\gamma d\mu(\gamma).$$

Moreover, by [Su1], we see that the modular groups

$$\sigma_t^\varphi = \int_\Gamma^\oplus \sigma_t^{\varphi_\gamma} d\mu(\gamma) \;\text{ and }\; \sigma_t^{\widetilde{\varphi}} = \int_\Gamma^\oplus \sigma_t^{\varphi_\gamma \circ E_\gamma} d\mu(\gamma) \quad (t \in \mathbf{R}).$$

Since $\sigma_t^{\varphi_\gamma \circ E_\gamma} |_{N_\gamma} = \sigma_t^{\varphi_\gamma}$ for all $\gamma \in \Gamma$, we get $\sigma_t^{\widetilde{\varphi}} |_N = \sigma_t^\varphi$. Therefore, there exists $E \in P(M,N)$ such that $\widetilde{\varphi} = \varphi \circ E$ by [St]. For $\psi \in P(N)$, set $\widetilde{\psi} = \psi \circ E$ and consider their disintegrations :

$$\psi = \int_\Gamma^\oplus \psi_\gamma d\mu(\gamma) \;\text{ and }\; \widetilde{\psi} = \int_\Gamma^\oplus \widetilde{\psi_\gamma} d\mu(\gamma).$$

By the fact that $[D\widetilde{\psi} : D\widetilde{\varphi}]_t = [D\psi : D\varphi]_t$ on Connes' cocycles, it follows that, for μ-a.a.$\gamma \in \Gamma$,

$$[D\widetilde{\psi}_\gamma : D(\varphi_\gamma \circ E_\gamma)]_t = [D\psi_\gamma : D\varphi_\gamma]_t \quad (t \in \mathbf{R}).$$

On the other hand,

$$[D\psi_\gamma : D\varphi_\gamma]_t = [D(\psi_\gamma \circ E_\gamma) : D(\varphi_\gamma \circ E_\gamma)]_t \quad (t \in \mathbf{R}).$$

Hence, we get

$$[D\widetilde{\psi}_\gamma : D(\varphi_\gamma \circ E_\gamma)]_t = [D(\psi_\gamma \circ E_\gamma) : D(\varphi_\gamma \circ E_\gamma)]_t \quad (t \in \mathbf{R}).$$

This implies that $\widetilde{\psi_\gamma} = \psi_\gamma \circ E_\gamma$, namely, the field $\gamma \mapsto \psi_\gamma \circ E_\gamma$ is measurable and $(\psi \circ E)_\gamma = \psi_\gamma \circ E_\gamma$. This is the desired conclusion. □

Theorem 2.3. *Let $M \supset N$ be a pair of von Neumann algebras acting on a separable Hilbert space and $E \in P(M,N)$. To each von Neumann subalgebra A in $Z(M) \cap Z(N)$, there corresponds a measurable field $\{M_\gamma, N_\gamma, E_\gamma\}$ of von Neumann algebras equipped with operator valued weights over a standard measure space (Γ, μ) such that*

(a) *A is the diagonal operator algebra isomorphic with $\mathrm{L}^\infty(\Gamma, \mu)$,*

(b) $\{M, N, E\} = \int_\Gamma^\oplus \{M_\gamma, N_\gamma, E_\gamma\} d\mu(\gamma)$,

(c) $\sigma_t^E = \int_\Gamma^\oplus \sigma_t^{E_\gamma} d\mu(\gamma)$.

Moreover, the above field is uniquely determined up to a μ-negligible set.

Proof. It is well-known that (a) and $\{M, N\} \cong \int_\Gamma^\oplus \{M_\gamma, N_\gamma\} d\mu(\gamma)$. Take $\varphi \in P(N)$ and set $\widetilde{\varphi} = \varphi \circ E \in P(M)$. Applying Sutherland's results on disintegrations of weights (Theorem4.18 in [Su1]), we get

$$\varphi = \int_\Gamma^\oplus \varphi_\gamma d\mu(\gamma) \quad \text{and} \quad \widetilde{\varphi} = \int_\Gamma^\oplus \widetilde{\varphi}_\gamma d\mu(\gamma).$$

Moreover, we see that

$$\sigma_t^\varphi = \int_\Gamma^\oplus \sigma_t^{\varphi_\gamma} d\mu(\gamma) \quad \text{and} \quad \sigma_t^{\widetilde{\varphi}} = \int_\Gamma^\oplus \sigma_t^{\widetilde{\varphi}_\gamma} d\mu(\gamma).$$

Since $\sigma_t^{\tilde{\varphi}} |_N = \sigma_t^{\varphi}$ for all $t \in \mathbf{R}$, it is easy to see that for μ-a.a.$\gamma \in \Gamma$, $\sigma_t^{\widetilde{\varphi_\gamma}} |_{N_\gamma} = \sigma_t^{\varphi_\gamma}$ for all $t \in \mathbf{R}$. Therefore, there exists $E_\gamma \in P(M_\gamma, N_\gamma)$ such that $\widetilde{\varphi_\gamma} = \varphi_\gamma \circ E_\gamma$ for μ-a.a.$\gamma \in \Gamma$. Applying Proposition 2.2, we see that the field $\gamma \mapsto E_\gamma$ is measurable as operator valued weights. Uniqueness of the above field up to a μ-negligible set and statement (c) is easy to check in a similar way to the above. □

The following is concerning to Connes' cocycle and the commutativity of operator valued weights discussed in [IK].

Proposition 2.4. *Let $M \supset N$ be a pair of von Neumann algebras disintegrated as $\{M, N\} \cong \int_\Gamma^\oplus \{M_\gamma, N_\gamma\} d\mu(\gamma)$ and let $E, F \in P(M, N)$ be also disintegrated as*

$$E = \int_\Gamma^\oplus E_\gamma d\mu(\gamma) \;\; \textit{and} \;\;\; F = \int_\Gamma^\oplus F_\gamma d\mu(\gamma).$$

Then, we have the followings:

(a) $[DE : DF]_t = \int_\Gamma^\oplus [DE_\gamma : DF_\gamma]_t d\mu(\gamma) \quad (t \in \mathbf{R})$,

(b) *E commutes with F if and only if E_γ commutes with F_γ for μ-a.a.$\gamma \in \Gamma$.*

Proof. (a) The statement (a) follows immediately from

$$[D(\varphi \circ E) : D(\varphi \circ F)]_t = \int_\Gamma^\oplus [D(\varphi_\gamma \circ E_\gamma) : D(\varphi_\gamma \circ F_\gamma)]_t \, d\mu(\gamma) \quad (t \in \mathbf{R})$$

for $\varphi = \int_\Gamma^\oplus \varphi_\gamma d\mu(\gamma) \in P(N)$.
(b) Assume that E commutes with F. Then, applying Theorem 4 in [IK], Connes' cocycle $[DE : DF]_t$ is a strongly continuous one parameter group of unitaries in $M \cap N'$. Therefore, we see that $[DE_\gamma : DF_\gamma]_t$ must be also one parameter groups for μ-a.a.$\gamma \in \Gamma$ by (a), which implies that E_γ commutes with F_γ by Theorem 4 in [IK]. It is easy to check the converse. □

3 Disintegration of spacial derivatives Let M be a von Neumann algebras and $A \cong \mathrm{L}^\infty(\Gamma, \mu)$ a subalgebra of $Z(M)$ acting on a separable Hilbert space $\mathcal{H}$. Then M and M' are disintegrated as

$$\{M, M'\} = \int_\Gamma^\oplus \{M_\gamma, M'_\gamma\} \, d\mu(\gamma) \;\text{ on }\; \mathcal{H} = \int_\Gamma^\oplus \mathcal{H}_\gamma \, d\mu(\gamma).$$

In this chapter we use the same notation in §7 [St]. Take $\psi = \int_\Gamma^\oplus \psi_\gamma d\mu(\gamma) \in P(M')$. We denote that

$$\mathcal{N}(M', \psi) = \{x \in M' : \psi(x^* x) < +\infty\}$$

and

$$D(\mathcal{H}, M', \psi) = \{\eta \in \mathcal{H} : \exists \lambda > 0 \quad \text{s.t.} \parallel x\eta \parallel^2 \leq \lambda^2 \psi(x^* x) \quad \text{for all } \; x \in \mathcal{N}(M', \psi)\}.$$

It is well-known that $\mathcal{N}(M', \psi)$ is a left ideal of M' and $D(\mathcal{H}, M', \psi)$ is a dense subspace of $\mathcal{H}$. For $\psi \in P(M')$ and $\eta \in D(\mathcal{H}, M', \psi)$, the bounded operator R^ψ_η from $\mathcal{H}_\psi$, GNS construction of ψ, onto $\mathcal{H}$ is defined by $R^\psi_\eta(x_\psi) = x\eta$ for $x \in \mathcal{N}(M', \psi)$.

Lemma 3.1. *Under the above situation, if $\eta \in D(\mathcal{H}, M', \psi)$ is represented as $\{\eta_\gamma\}_{\gamma \in \Gamma}$, then $\eta_\gamma \in D(\mathcal{H}_\gamma, M'_\gamma, \psi_\gamma)$ for μ-a.a.$\gamma \in \Gamma$.*

Proof. Consider weights $\Psi = \psi + w_\eta$ and $\Psi_\gamma = \psi_\gamma + w_{\eta_\gamma}$, where $w_\xi(z) = (z\xi, \xi)$. Then $\mathcal{N}(M', \psi) = \mathcal{N}(M', \Psi)$ and $\mathcal{N}(M'_\gamma, \psi_\gamma) = \mathcal{N}(M'_\gamma, \Psi_\gamma)$. If we choose a sequence $\{x^j\}$ such that $\{x^j_\Psi\}$ is dense in $\mathcal{H}_\Psi$, then $\{x^j_{\gamma,\Psi_\gamma}\}$ is also dense in $\mathcal{H}_{\Psi_\gamma}$ for μ-a.a.γ.

Let B be a Borel subset of Γ and χ_B a characteristic function on B. Since $\chi_B \in Z(M)$, we have $\chi_B x^j \in \mathcal{N}(M', \psi)$. It is obvious that for $x^j = \int_\Gamma^\oplus x^j_\gamma d\mu(\gamma)$,

$$\int_B \| x^j_\gamma \eta_\gamma \|^2 d\mu(\gamma) \le \lambda^2 \int_B \psi_\gamma((x^j_\gamma)^* x^j_\gamma) d\mu(\gamma).$$

For μ-a.a.γ, it holds that $\| x^j_\gamma \eta_\gamma \|^2 \le \lambda^2 \psi_\gamma((x^j_\gamma)^* x^j_\gamma)$ for all j. For an arbitrary $x_\gamma \in \mathcal{N}(M'_\gamma, \psi_\gamma)$, there exists a sub-sequence $\{x^k_\gamma\}$ such that $x^k_{\gamma,\Psi_\gamma} \to x_{\gamma,\Psi_\gamma} \in \mathcal{H}_{\Psi_\gamma}$. This means that $x^k_\gamma \eta_\gamma \to x_\gamma \eta_\gamma$ in $\mathcal{H}_\gamma$ and $x^k_{\gamma,\psi_\gamma} \to x_{\gamma,\psi_\gamma}$ in $\mathcal{H}_{\psi_\gamma}$. Therefore, $\| x_\gamma \eta_\gamma \|^2 \le \lambda^2 \psi_\gamma(x^*_\gamma x_\gamma)$. By Definition 4.1 in [Su1], this shows that $\eta_\gamma \in D(\mathcal{H}_\gamma, M'_\gamma, \psi_\gamma)$ for μ-a.a.γ. □

Lemma 3.2. *Under the above situation, if R^ψ_η is represented as $\{(R^\psi_\eta)_\gamma\}$, then $(R^\psi_\eta)_\gamma = R^{\psi_\gamma}_{\eta_\gamma}$ for μ-a.a.$\gamma \in \Gamma$.*

Proof. After taking $\eta \in D(\mathcal{H}, M', \psi)$ represented as $\{\eta_\gamma\}$ and a fundamental sequence $x^j \in \mathcal{N}(M', \psi)$ in $\mathcal{H}_\psi$ for fields $\gamma \mapsto x^j_{\gamma,\psi_\gamma} \in \mathcal{H}_{\psi_\gamma}$. By Lemma 3.1, we have

$$R^\psi_\eta(x^j_\psi) = x^j \eta = \int_\Gamma^\oplus x^j_\gamma \eta_\gamma d\mu(\gamma) = \int_\Gamma^\oplus R^{\psi_\gamma}_{\eta_\gamma}(x^j_{\gamma,\psi_\gamma}) d\mu(\gamma).$$

This implies that $(R^\psi_\eta)_\gamma(x^j_\gamma) = R^{\psi_\gamma}_{\eta_\gamma}(x^j_\gamma)$ for μ-a.a.γ. Thus we have the conclusion. □

For $\varphi = \int_\Gamma^\oplus \varphi_\gamma d\mu(\gamma) \in P(M)$, one can define the densely defined quadratic forms $q(\eta) = \varphi(R^\psi_\eta (R^\psi_\eta)^*)$ on $\mathcal{H}$ with the domain :

$$D(q) = \{\eta \in D(\mathcal{H}, M', \psi) : q(\eta) < +\infty\}$$

and $q_\gamma(\eta_\gamma) = \varphi_\gamma(R^{\psi_\gamma}_{\eta_\gamma} (R^{\psi_\gamma}_{\eta_\gamma})^*)$ on $\mathcal{H}_\gamma$ with the domain :

$$D(q_\gamma) = \{\eta_\gamma \in D(\mathcal{H}_\gamma, M'_\gamma, \psi_\gamma) : q_\gamma(\eta_\gamma) < +\infty\}.$$

For such quadratic forms q and q_γ, it is known that they are closable and there exist canonical selfadjoint positive operators T and $T(\gamma)$ respectively such that

$$q(\eta) = (T\eta, \eta), \quad D(\overline{q}) = D(T^{1/2}),$$
$$q_\gamma(\eta_\gamma) = (T(\gamma)\eta_\gamma, \eta_\gamma), \quad D(\overline{q_\gamma}) = D(T(\gamma)^{1/2}).$$

Lemma 3.3. *Under the above situation, the field $\gamma \mapsto T(\gamma)$ is measurable as closed operators, namely $\gamma \mapsto (1 + T(\gamma))^{-1}$ is measurable as bounded operators, and*

$$T = \int_\Gamma^\oplus T(\gamma) d\mu(\gamma).$$

Proof. Let a unitary $u \in A$ be fixed, then $u\eta \in D(\mathcal{H}, M', \psi)$ and

$$q(u\eta) = \varphi(R^\psi_{u\eta}(R^\psi_{u\eta})^*) = \varphi(uR^\psi_\eta(R^\psi_\eta)^* u^*) = \varphi(R^\psi_\eta(R^\psi_\eta)^*) = q(\eta).$$

For the positive sesquilinear form $q(\xi, \eta)$ corresponding to $q(\eta)$, we denote the scalar product $(\xi, \eta)_q = (\xi, \eta) + q(\xi, \eta)$ for $\xi, \eta \in D(\mathcal{H}, M', \psi)$, then $(u\xi, u\eta)_q = (\xi, \eta)_q$. If we use a bounded positive operator $S \leq 1$ such that $(\xi, \eta) = (S\xi, \eta)_q$ and $T = S^{-1} - 1$, then $(\xi, \eta) = (u\xi, u\eta) = (Su\xi, u\eta)_q = (u^*Su\xi, \eta)_q$. The uniqueness of S shows that $S = u^*Su$. Therefore, S commutes with the algebra A. Hence, there exists a disintegration $S = \int_\Gamma^\oplus S_\gamma d\mu(\gamma)$. Put $T_\gamma = (S_\gamma)^{-1} - 1$, then we obtain $T = \int_\Gamma^\oplus T_\gamma d\mu(\gamma)$.

Taking a fundamental sequence $\{\eta^i \in D(q)\}$ of $\mathcal{H}$ with $\overline{T^{1/2}|_{\{\eta^i\}}} = T^{1/2}$ and fields $\gamma \mapsto \eta^i_\gamma \in \mathcal{H}_\gamma$ with $\overline{T_\gamma^{1/2}|_{\{\eta^i_\gamma\}}} = T_\gamma^{1/2}$, it follows from Lemma 3.2 that

$$(T_\gamma \eta^i_\gamma, \eta^i_\gamma) = (T(\gamma)\eta^i_\gamma, \eta^i_\gamma) \qquad \text{a.a.}\gamma \in \Gamma.$$

This implies that $T_\gamma \subset T(\gamma)$ for μ-a.a.$\gamma \in \Gamma$. Since T_γ and $T(\gamma)$ are selfadjoint, we get $T_\gamma = T(\gamma)$ for μ-a.a.$\gamma \in \Gamma$. Hence, the desired conclusion is obtained. □

The selfadjoint positive operator T discussed above is called the spatial derivative [C] for $\varphi \in P(M)$ and $\psi \in P(M')$ and T is denoted by $\Delta(\varphi/\psi)$.

Let N be a von Neumann subalgebra of M. Then, there exists an order reversing isomorphism :

$$P(M, N) \ni E \mapsto E^{-1} \in P(N', M')$$

uniquely determined with the property

$$\Delta(\varphi/\psi\circ E^{-1}) = \Delta(\varphi\circ E/\psi) \quad \text{ for } \varphi \in P(N) \text{ and } \psi \in P(M').$$

Lemma 3.4. *Let M be a von Neumann algebra on a separable Hilbert space $\mathcal{H}$. Then to each von Neumann subalgebra $A \cong \mathrm{L}^\infty(\Gamma, \mu)$ in the center $Z(M)$ of M, for $\varphi \in P(M)$ and $\psi \in P(M')$, the multiple $\{M, M', \varphi, \psi\}$ is disintegrated as*

$$\{M, M', \varphi, \psi\} \cong \int_\Gamma^\oplus \{M_\gamma, M'_\gamma, \varphi_\gamma, \psi_\gamma\} d\mu(\gamma)$$

where $\phi_\gamma \in P(M_\gamma)$, $\psi_\gamma \in P(M'_\gamma)$ and

$$\Delta(\varphi/\psi) = \int_\Gamma^\oplus \Delta(\varphi_\gamma/\psi_\gamma) d\mu(\gamma).$$

Proof. The conclusion is just a translation of Lemma 3.3. □

Theorem 3.5. *Let $M \supset N$ be a pair of von Neumann algebras acting on a separable Hilbert space and let E be a normal semifinite faithful N-valued weight on M. Then, for a von Neumann subalgebra A of $Z(M) \cap Z(N)$, there exist a standard measure space (Γ, μ) and a measurable field of a triple $\{M_\gamma, N_\gamma, E_\gamma\}_{\gamma\in\Gamma}$ over Γ such that*

(a) *A is the diagonal operator algebra isomorphic with $\mathrm{L}^\infty(\Gamma, \mu)$,*

(b) $\{M, N, E\} \cong \int_\Gamma^\oplus \{M_\gamma, N_\gamma, E_\gamma\} d\mu(\gamma)$,

(c) $\{N', M', E^{-1}\} \cong \int_\Gamma^\oplus \{{N_\gamma}', {M_\gamma}', {E_\gamma}^{-1}\} d\mu(\gamma)$.

Moreover, the above fields are uniquely determined up to a μ-negligible set.

Proof. The former part of the result of Theorem 3.5 is already shown in Theorem 2.3. By Theorem 2.3, E^{-1} is also disintegrated as

$$E^{-1} = \int_\Gamma^\oplus (E^{-1})_\gamma d\mu(\gamma).$$

By Lemma 3.4 and Proposition 2.2, we get

$$\Delta(\varphi\circ E/\psi) = \int_\Gamma^\oplus \Delta(\varphi_\gamma\circ E_\gamma/\psi_\gamma)d\mu(\gamma),$$
$$\Delta(\varphi/\psi\circ E^{-1}) = \int_\Gamma^\oplus \Delta(\varphi_\gamma/\psi_\gamma\circ(E^{-1})_\gamma)d\mu(\gamma).$$

By the uniqueness of the disintegration of closed operators, we see that

$$\Delta(\varphi_\gamma\circ E_\gamma/\psi_\gamma) = \Delta(\varphi_\gamma/\psi_\gamma\circ(E^{-1})_\gamma) \text{ for } \mu\text{-a.a.}\gamma\in\Gamma.$$

It implies that

$$(E_\gamma)^{-1} = (E^{-1})_\gamma \qquad \mu\text{-a.a.}\gamma\in\Gamma.$$

This completes the proof. □

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