A CHARACTERIZATION OF THE HOMOGENEOUS REAL HYPERSURFACE OF TYPE (B) WITH TWO DISTINCT CONSTANT PRINCIPAL CURVATURES IN A COMPLEX HYPERBOLIC SPACE

SADAHIRO MAEDA

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Dedicated to emeritus Professor Masanori Kôzaki of Saga University

Abstract. A tube of radius r $(0 < r < \infty)$ around totally real totally geodesic $\mathbb{R}H^n(c/4)$ is called a homogeneous real hypersurface of type (B) in $\mathbb{C}H^n(c)$. It is known that every type (B) hypersurface with radius $r \neq (1/\sqrt{|c|})\log_e(2+\sqrt{3})$ has three distinct constant principal curvatures and in the case of $r = (1/\sqrt{|c|})\log_e(2+\sqrt{3})$ the real hypersurface M^{2n-1} of type (B) has two distinct constant principal curvatures. The main purpose of this paper is to characterize this real hypersurface M.

1. Introduction

For a non-zero constant c, $M_n(c)$ denotes a complex n-dimensional complete and simply connected Kähler manifold of constant holomorphic sectional curvature c (with complex structure J). That is, $M_n(c)$ is holomorphically isometric to a complex projective space $\mathbb{C}P^n(c)$ when $c > 0$, and it is holomorphically isometric to a complex hyperbolic space $\mathbb{C}H^n(c)$ when $c < 0$.

We consider a real hypersurface M^{2n-1} (with unit normal local vector field $\mathcal{N}$) in the ambient space $\widetilde{M} = M_n(c)$ $(n \geqq 2,\ c \neq 0)$. M is said to be a *Hopf hypersurface* if the characteristic vector $\xi(:= -J\mathcal{N})$ of M is a principal curvature vector of M in $\widetilde{M}$ at its each point. It is known that tubes of sufficiently small constant radius around Kähler submanifolds in $\widetilde{M} = M_n(c)$ are Hopf hypersurfaces. This means that Hopf hypersurfaces are natural examples of real hypersurfaces and that they make an abundant class in the theory of real hypersurfaces in $\widetilde{M}$.

In this paper, first of all we give a characterization of Hopf hypersurfaces M in $\widetilde{M} = M_n(c)$ in terms of integral curves of the characteristic vector field ξ of M (Proposition 1).

We next recall the following related to the fact that there exist no totally umbilic real hypersurfaces in $\widetilde{M}$ (see [6, 7]). In $\mathbb{C}P^n(c)$ $(n \geqq 3)$, a connected real hypersurface M has at most two distinct principal curvatures at each point of M if and only if M is locally congruent to a geodesic sphere $G(r)$ of radius r $(0 < r < \pi/\sqrt{c})$. In $\mathbb{C}H^n(c)$ $(n \geqq 3)$, a connected real hypersurface M has at most two distinct principal curvatures at each point of M if and only if M is locally congruent to either a geodesic sphere $G(r)$ $(0 < r < \infty)$ in $\mathbb{C}H^n(c)$, a tube of radius r $(0 < r < \infty)$ over a complex hyperplane $\mathbb{C}H^{n-1}(c)$, a horosphere

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or a tube of radius $r = (1/\sqrt{|c|})\log_e(2+\sqrt{3})$ over an n-dimensional totally real totally geodesic real hyperbolic space $\mathbb{R}H^n(c/4)$ of constant sectional curvature $c/4$ in $\mathbb{C}H^n(c)$. Note that these real hypersurfaces in $\widetilde{M}$ have two distinct constant principal curvatures. Moreover, each tube of radius $r \neq (1/\sqrt{|c|})\log_e(2+\sqrt{3})$ over $\mathbb{R}H^n(c/4)$ has three distinct constant principal curvatures. In this context, it is interesting to characterize just the real hypersurface, which is a tube over $\mathbb{R}H^n(c/4)$ with radius $r = (1/\sqrt{|c|})\log_e(2+\sqrt{3})$, in the ambient space $\mathbb{C}H^n(c)$ (see Theorem).

2. Hopf hypersurfaces in $\widetilde{M} = M_n(c)$, $c \neq 0$

Let M^{2n-1} be a real hypersurface of $\widetilde{M} = M_n(c)$ ($n \geqq 2$, $c \neq 0$) and $\mathcal{N}$ a unit local normal vector field on M. The Riemannian connections $\widetilde{\nabla}$ of $\widetilde{M}$ and ∇ of M are related by the following formulas of Gauss ans Weingarten:

$$\widetilde{\nabla}_X Y = \nabla_X Y + \langle AX, Y\rangle \mathcal{N}, \tag{2.1}$$

$$\widetilde{\nabla}_X \mathcal{N} = -AX \tag{2.2}$$

for any vector fields X and Y on M, where $\langle\ ,\ \rangle$ is the Riemannian metric of M induced from the standard metric of the ambient space $\widetilde{M}$ and A is the shape operator of M in $\widetilde{M}$. An eigenvector X of the shape operator A is called a *principal curvature vector* and an eigenvalue λ of A is called a *principal curvature*.

It is known that M has an almost contact metric structure induced from the complex structure of the ambient space $\widetilde{M}$, namely we have a quartet $(\phi, \xi, \eta, \langle\ ,\ \rangle)$ defined by

$$\langle \phi X, Y\rangle = \langle JX, Y\rangle, \quad \xi = -J\mathcal{N} \quad \text{and} \quad \eta(X) = \langle \xi, X\rangle = \langle JX, \mathcal{N}\rangle$$

which satisfy

$$\phi^2 X = -X + \eta(X)\xi, \quad \langle \xi, \xi\rangle = 1, \quad \phi\xi = 0. \tag{2.3}$$

It follows from (2.1), (2.2) and $\widetilde{\nabla} J = 0$ that

$$(\nabla_X \phi)Y = \eta(Y)AX - \langle AX, Y\rangle\xi, \tag{2.4}$$

$$\nabla_X \xi = \phi AX. \tag{2.5}$$

Let R denote the curvature tensor of M. We have the equations of Gauss and Codazzi given by

$$\begin{aligned}\langle R(X,Y)Z, W\rangle =& (c/4)\{\langle Y,Z\rangle\langle X,W\rangle - \langle X,Z\rangle\langle Y,W\rangle + \langle \phi Y,Z\rangle\langle \phi X,W\rangle \\ & - \langle \phi X,Z\rangle\langle \phi Y,W\rangle - 2\langle \phi X,Y\rangle\langle \phi Z,W\rangle\} \\ & + \langle AY,Z\rangle\langle AX,W\rangle - \langle AX,Z\rangle\langle AY,W\rangle,\end{aligned} \tag{2.6}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = (c/4)\{\eta(X)\phi Y - \eta(Y)\phi X - 2\langle \phi X, Y\rangle\xi\}. \tag{2.7}$$

The following lemma clarifies a fundamental property which is a useful tool in the theory of real hypersurfaces in $\widetilde{M} = M_n(c)$ ($n \geqq 2$, $c \neq 0$).

Lemma A. *For a Hopf hypersurface M^{2n-1} ($n \geqq 2$) with principal curvature α corresponding to the characteristic vector field ξ in the ambient space $\widetilde{M} = M_n(c)$, $c \neq 0$, we have the following*:

1. *α is locally constant on M;*
2. *If X is a tangent vector of M perpendicular to ξ with $AX = \lambda X$, then $(2\lambda - \alpha)A\phi X = (\alpha\lambda + (c/2))\phi X$. In particular, we get $A\phi X = \frac{\alpha\lambda + (c/2)}{2\lambda - \alpha}\phi X$ in the case of $c > 0$.*

Remark 1. When $c<0$, in Lemma A(2) there exists a case that both of equations $2\lambda-\alpha=0$ and $\alpha\lambda+(c/2)=0$ hold. In fact, for example we take a horoshere in $\mathbb{C}H^n(c)$. It is known that this real hypersurface has two distinct constant principal curvatures $\lambda=\sqrt{|c|}/2,\ \alpha=\sqrt{|c|}$ or $\lambda=-\sqrt{|c|}/2,\ \alpha=-\sqrt{|c|}$. Hence, when $c<0$, we must consider two cases of $2\lambda-\alpha=0$ and $2\lambda-\alpha\neq 0$.

The following gives a geometric meaning of Hopf hypersurfaces in $\widetilde{M}$.

Proposition 1. *Let M be a real hypersurface (with unit normal local vector field $\mathcal{N}$) in $\widetilde{M}=M_n(c)$ $(n\geqq 2,\ c\neq 0)$. Then the following two conditions are equivalent.*

(1) *M is a Hopf hypersurface in $\widetilde{M}$.*
(2) *At each point $p\in M$ there exists such a totally geodesic complex curve $M_1(c)$ in $\widetilde{M}$ through p with $T_pM_1(c)=\{\xi_p,\mathcal{N}_p(=J\xi_p)\}_{\mathbb{R}}$ that the normal section $N_p=M\cap M_1(c)$ given by $M_1(c)$ is the integral curve through the point p of the characteristic vector field ξ of M.*

Proof. It follows from (2.1) and (2.5) that $\widetilde{\nabla}_\xi\xi=\phi A\xi+\langle A\xi,\xi\rangle\mathcal{N}$. This equation implies that the condition (1) in our proposition is equivalent to saying that

$$\widetilde{\nabla}_\xi\xi=\langle A\xi,\xi\rangle\mathcal{N}=\langle A\xi,\xi\rangle J\xi,$$

which is nothing but the condition (2). □

We next recall the following property of the holomorphic distribution $T^0M=\{X\in TM|\ X\perp\xi\}$ of a Hopf hypersurface M in $\widetilde{M}$.

Proposition 2. *The holomorphic distribution $T^0M=\{X\in TM|\ X\perp\xi\}$ of each Hopf hypersurace M in $\widetilde{M}=M_n(c)$ $(n\geqq 2,\ c\neq 0)$ is not integrable.*

Proof. Suppose that T^0M is integrable for some Hopf hypersurface M in $\widetilde{M}$. Then we have

$$\langle\nabla_XY-\nabla_YX,\xi\rangle=0\quad\text{for}\ \forall\ X,Y\in T^0M.$$

This, together with (2.5), implies

$$\langle(\phi A+A\phi)X,Y\rangle=0\quad\text{for}\ \forall\ X,Y\in T^0M.\tag{2.8}$$

Hence, from (2.8) and the assumption that ξ is principal we see that $\phi A+A\phi$ vanishes identically on M, which is a contradiction (see page 252 in [8]). □

For a real hypersurface M^{2n-1} in $\mathbb{C}H^n(c)$ $(n\geqq 2)$ we usually set $V_\lambda=\{X\in TM|\ AX=\lambda X\}$, which is so-called the principal foliation on M^{2n-1} with respect to a principal curvature λ. Also, $V^0_\lambda(:=\{X\in T^0M|\ AX=\lambda X\})$ is said to be a *restricted principal foliation* associated with a principal curvature λ of M. In the following, we consider Hopf hypersurfaces in $\mathbb{C}H^n(c)$ $(n\geqq 2)$. In order to prove our Theorem, we recall the following classification theorem of Hopf hypersurfaces with constant principal curvatures in $\mathbb{C}H^n(c)$, which is due to Berndt ([2]).

Theorem A. *Let M be a Hopf hypersurface all of whose principal curvatures are constant in $\mathbb{C}H^n(c)$ $(n\geqq 2)$. Then M is locally congruent to one of the following:*

(A_0) *a horosphere in $\mathbb{C}H^n(c)$,*
($A_{1,0}$) *a geodesic sphere of radius r $(0<r<\infty)$,*
($A_{1,1}$) *a tube of radius r around totally geodesic $\mathbb{C}H^{n-1}(c)$, where $0<r<\infty$,*
(A_2) *a tube of radius r around totally geodesic $\mathbb{C}H^k$ $(1\leqq k\leqq n-2)$, where $0<r<\infty$,*
(*B*) *a tube of radius r around totally real totally geodesic $\mathbb{R}H^n(c/4)$, where $0<r<\infty$.*

These real hypersurfaces are said to be of type (A_0), (A_1), (A_2) and (B). Here, type (A_1) means either type ($A_{1,0}$) or type ($A_{1,1}$). Summing up real hypersurfaces of type (A_0), (A_1) and (A_2), we call them real hypersurfaces of type (A). A real hypersurface of type (B) with radius $r = (1/\sqrt{|c|})\log_e(2+\sqrt{3})$ has two distinct constant principal curvatures. Except this, the numbers of distinct principal curvatures of these real hypersurfaces are $2, 2, 2, 3, 3$, respectively. The principal curvatures of these real hypersurfaces are as follows.

	λ_1	λ_2	α
(A_0)	$\frac{\sqrt{\lvert c\rvert}}{2}$	—	$\sqrt{\lvert c\rvert}$
($A_{1,0}$)	$\frac{\sqrt{\lvert c\rvert}}{2}\coth\frac{\sqrt{\lvert c\rvert}r}{2}$	—	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}r)$
($A_{1,1}$)	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh\frac{\sqrt{\lvert c\rvert}r}{2}$	—	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}r)$
(A_2)	$\frac{\sqrt{\lvert c\rvert}}{2}\coth\frac{\sqrt{\lvert c\rvert}r}{2}$	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh\frac{\sqrt{\lvert c\rvert}r}{2}$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}r)$
(B)	$\frac{\sqrt{\lvert c\rvert}}{2}\coth\frac{\sqrt{\lvert c\rvert}r}{2}$	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh\frac{\sqrt{\lvert c\rvert}r}{2}$	$\sqrt{\lvert c\rvert}\tanh(\sqrt{\lvert c\rvert}r)$

The restricted principal foliation $V^0_{\lambda_i}$ associated with λ_i, satisfies the following (see Lemma A).

1. $V^0_{\lambda_i}$ is invariant under the action of ϕ for a hypersurface of type (A).
2. $\phi(V^0_{\lambda_1}) = V^0_{\lambda_2}$, $\phi(V^0_{\lambda_2}) = V^0_{\lambda_1}$ for a hypersurface of type (B).

All examples of Theorem A are homogeneous real hypersurfaces in $\mathbb{C}H^n(c)$, namely they are orbits under some subgroups of the full isometry group $\mathrm{I}(\mathbb{C}H^n(c))$ of the ambient space $\mathbb{C}H^n(c)$. However, in general a homogeneous real hypersurface in $\mathbb{C}H^n(c)$ is *not* necessarily a Hopf hypersurface. There exist many homogeneous non-Hopf hypersurfaces as well as many homogeneous Hopf hypersurfaces (for details, see [3]).

At the end of this section we review the notion of circles in Riemannian geometry. Let $\gamma = \gamma(s)$ be a regular smooth curve parametrized by its arclength s in a Riemannian manifold M (with Riemannian connection ∇). The curve γ is a *circle* of curvature k on M if γ satisfies the following ordinary differential equations: $\nabla_{\dot\gamma}\dot\gamma = kY_s$, $\nabla_{\dot\gamma}Y_s = -k\dot\gamma$, where $k(\geqq 0)$ is a constant and Y_s is the unit normal principal vector along the curve γ. A circle of null curvature is nothing but a geodesic.

3. Main Theorem

In view of Proposition 2 we establish the following:

Theorem. *A connected real hypersurface M in $\mathbb{C}H^n(c)$, $n \geqq 2$ is a type (B) hypersurface with radius $r = (1/\sqrt{|c|})\log_e(2+\sqrt{3})$ if and only if M satisfies the following two conditions.*

(1) *The holomorphic distribution $T^0M = \{X \in TM \mid X \perp \xi\}$ of M is decomposed as the direct sum of restricted principal foliations $V^0_{\lambda_i} = \{X \in T^0M \mid AX = \lambda_i X\}$. Moreover, every $V^0_{\lambda_i}$ is integrable and its each leaf is a totally geodesic submanifold of the real hypersurface M.*

(2) *At some point $p \in M$, for some positive constant k there exist two geodesics $\gamma_i = \gamma_i(s)$ on M through the point $p = \gamma_i(0)$ $(i = 1, 2)$ satisfying the following:*
   (a) $\langle \dot\gamma_i(0), \xi_p\rangle = 0$ $(i = 1, 2)$,
   (b) *the curves γ_i $(i = 1, 2)$, considered as curves in the ambient space $\mathbb{C}H^n(c)$, are circles of positive curvature k and $3k$, respectively.*

Proof. We first show that M is a type (B) hypersurface if and only if M satisfies the condition (1). The following discussion is an improvement of that in [5]. Note that there exists a gap in [5] (see line -5 in page 141).

Suppose that M is of type (B). Then T^0M is decomposed as the following direct sum of principal foliations: $T^0M = V^0_{\lambda_1} \oplus V^0_{\lambda_2}$, where $\lambda_1 = (\sqrt{|c|}/2)\coth(\sqrt{|c|}\,r/2)$, $\lambda_2 = (\sqrt{|c|}/2)\tanh(\sqrt{|c|}\,r/2)$, $\phi V^0_{\lambda_1} = V^0_{\lambda_2}$ and $A\xi = \sqrt{|c|}\tanh(\sqrt{|c|}\,r)\xi$. For each $X, Y \in V^0_{\lambda_i}$ $(i = 1, 2)$, we have

$$A\nabla_X Y = \nabla_X(AY) - (\nabla_X A)Y = \lambda_i \nabla_X Y - (\nabla_X A)Y.$$

Since $\langle \phi X, Y\rangle = 0$ and A is symmetric, Codazzi equation (2.7) implies, for any $Z \in TM$, that

$$\begin{aligned}\langle(\nabla_X A)Y, Z\rangle &= \langle(\nabla_X A)Z, Y\rangle = \langle(\nabla_Z A)X, Y\rangle\\ &= \langle \nabla_Z(AX) - A\nabla_Z X, Y\rangle = \langle(\lambda_i I - A)\nabla_Z X, Y\rangle\\ &= \langle \nabla_Z X, (\lambda_i I - A)Y\rangle = 0.\end{aligned}$$

These two equations show that $A(\nabla_X Y) = \lambda_i \nabla_X Y$ for any $X, Y \in V^0_{\lambda_i}$, so that $V^0_{\lambda_i}$ is integrable and each leaf T_{λ_i} is a totally geodesic submanifold of the real hypersurface M of type (B). Then we can obtain the condition (1).

Next, suppose the condition (1). Without loss of generality we assume that $c = -4$. First of all note that our real hypersurface M is a Hopf hypersurface. In fact, for each $X = \sum_i X^i v_i \in T^0M$, where v_i is a unit vector in the restricted principal foliation $V^0_{\lambda_i}$ of the condition (1), we find that $\langle A\xi, X\rangle = \langle \xi, AX\rangle = \sum_i \langle \xi, X^i\lambda_i v_i\rangle = 0$. In the following, our discussion is divided into two cases (I) $\dim T^0M = 2$ and (II) $\dim T^0M \geqq 4$.

Case (I). In this case the holomorphic distribution T^0M is decomposed as $T^0M = V^0_{\lambda_1} \oplus V^0_{\lambda_2}$ with $\lambda_1 \neq \lambda_2$ and $\dim V^0_{\lambda_1} = \dim V^0_{\lambda_2} = 1$ (see Proposition 2). Furthermore, all integral curves of $V^0_{\lambda_1}$ and $V^0_{\lambda_2}$ are geodesics on M. We take a local field of orthonormal frames $\{e_1, e_2, \xi\}$ on M in such a way that $e_i \in V^0_{\lambda_i}$ $(i = 1, 2)$ and $e_2 = \phi e_1$. As M is a Hopf hypersurface, Equation (2.5) tells us that $\nabla_\xi \xi = 0$. By hypothesis we also have $\nabla_{e_1} e_1 = \nabla_{e_2} e_2 = 0$. Equation (2.5) implies

$$\nabla_{e_1}\xi = \lambda_1 e_2,\ \nabla_{e_2}\xi = -\lambda_2 e_1,\ \nabla_{e_1} e_2 = -\lambda_1 \xi,\ \nabla_{e_2} e_1 = \lambda_2 \xi. \tag{3.1}$$

Codazzi equation (2.7) shows

$$(\nabla_{e_1} A)e_2 - (\nabla_{e_2} A)e_1 = 2\xi.$$

On the other hand, Equation (3.1) yields

$$\begin{aligned}(\nabla_{e_1} A)e_2 - (\nabla_{e_2} A)e_1 &= \nabla_{e_1}(Ae_2) - A\nabla_{e_1} e_2 - \nabla_{e_2}(Ae_1) + A\nabla_{e_2} e_1\\ &= (e_1\lambda_2)e_2 + (\lambda_2 I - A)\nabla_{e_1} e_2 - (e_2\lambda_1)e_1 - (\lambda_1 I - A)\nabla_{e_2} e_1\\ &= -(e_2\lambda_1)e_1 + (e_1\lambda_2)e_2 + \{\alpha(\lambda_1 + \lambda_2) - 2\lambda_1\lambda_2\}\xi.\end{aligned}$$

It follows from these two equations that

$$2 = \alpha(\lambda_1 + \lambda_2) - 2\lambda_1\lambda_2, \tag{3.2}$$

$$e_2\lambda_1 = 0, \tag{3.3}$$

$$e_1\lambda_2 = 0. \tag{3.4}$$

We here show that $\xi\lambda_1 = \xi\lambda_2 = 0$. It follows from (2.7) that $(\nabla_{e_1} A)\xi - (\nabla_\xi A)e_1 = e_2$. On the other hand, we also have

$$\begin{aligned}(\nabla_{e_1} A)\xi - (\nabla_\xi A)e_1 &= \nabla_{e_1}(A\xi) - A(\nabla_{e_1}\xi) - \nabla_\xi(Ae_1) + A(\nabla_\xi e_1)\\ &= \alpha\lambda_1 e_2 - \lambda_1\lambda_2 e_2 - (\xi\lambda_1)e_1 - (\lambda_1 I - A)(\nabla_\xi e_1).\end{aligned}$$

Since $(\lambda_1 I - A)(\nabla_\xi e_1)$ is orthogonal to e_1, these two equations show that $\xi\lambda_1 = 0$. Similarly, we also have $\xi\lambda_2 = 0$.

Next, since α is locally constant, (3.2) and (3.3) give

$$(\alpha - 2\lambda_1)(e_2\lambda_2) = 0. \tag{3.5}$$

Simlarly, from (3.2) and (3.4) we see that

$$(\alpha - 2\lambda_2)(e_1\lambda_1) = 0. \tag{3.6}$$

It suffices to consider the following three cases in (I).

Case (I_a): $\alpha \equiv 2\lambda_2$ locally and $\alpha \neq 2\lambda_1$ at some point $p \in M$. In this case, Equation (3.2) shows $(\lambda_2)^2 = 1$. Without loss of generality, we may assume $\lambda_2 = 1$ and hence $\alpha = 2$. For simplicity, putting $\lambda_1 = \lambda$, we find

$$\begin{cases} \nabla_{e_1}e_1 = \nabla_{e_2}e_2 = \nabla_\xi\xi = 0, \\ \nabla_{e_1}e_2 = -\lambda\xi, \ \nabla_{e_2}e_1 = \xi, \\ \nabla_{e_1}\xi = \lambda e_2, \ \nabla_{e_2}\xi = -e_1. \end{cases}$$

It follows from the continuity of λ that $\lambda \neq 1$ on some neighborhood $\mathcal{U}$ of the point p. Putting $\nabla_\xi e_1 = \mu e_2$, we see from Codazzi equation (2.7) that

$$(\nabla_{e_1}A)\xi - (\nabla_\xi A)e_1 = e_2.$$

On the other hand, we obtain

$$\begin{aligned}(\nabla_{e_1}A)\xi - (\nabla_\xi A)e_1 &= 2(\nabla_{e_1}\xi) - A(\nabla_{e_1}\xi) - \nabla_\xi(\lambda e_1) + A(\nabla_\xi e_1) \\ &= -(\xi\lambda)e_1 + (\lambda+\mu)e_2 - \lambda\mu e_2.\end{aligned}$$

These two equations yield $\lambda + \mu - \lambda\mu = 1$, so that $\mu = 1$ on $\mathcal{U}$. Hence we have $\nabla_\xi e_1 = e_2$ and $\nabla_\xi e_2 = -e_1$. Let R denote the curvature tensor of the real hypersurface M. Then, by the definition of R we see that

$$\begin{aligned}\langle R(e_1, e_2)e_2, e_1\rangle &= \lambda\langle\nabla_{e_2}\xi, e_1\rangle + \lambda\langle\nabla_\xi e_2, e_1\rangle + \langle\nabla_\xi e_2, e_1\rangle \\ &= -2\lambda - 1.\end{aligned}$$

On the other hand, Gauss equation (2.6) yields $\langle R(e_1, e_2)e_2, e_1\rangle = -4 + \lambda$. Thus $\lambda = 1$ on $\mathcal{U}$, which is a contradiction. Hence, Case (I_a) cannot occur.

Case (I_b): $\alpha \equiv 2\lambda_1$ locally and $\alpha \neq 2\lambda_2$ at some point $p \in M$. This case cannot occur by the same discussion as in Case (I_a).

Case (I_c): $\alpha \neq 2\lambda_1$ and $\alpha \neq 2\lambda_2$ at some point $p \in M$. In this case, equations (3.3), (3.4), (3.5) and (3.6) yield

$$e_1\lambda_1 = e_1\lambda_2 = e_2\lambda_1 = e_2\lambda_2 = 0$$

on some neighborhood $\mathcal{U}$ of the point p. Then we can see that all principal curvatures α, λ_1, λ_2 are constant on our connected real hypersurface M. Hence, Theorem A tells us that M is of either type (A) or type (B). However, each type (A) hypersurface M does not satisfy $\phi V^0_{\lambda_1} = V^0_{\lambda_2}$. Therefore we find that M is of type (B).

Case (II). By assumption for any $X, Y \in V^0_{\lambda_i}$ we have $A\nabla_X Y = \lambda_i \nabla_X Y$, so that $(\nabla_X A)Y = (X\lambda_i)Y$. We divide our discussion into two cases.

Case (II_a): $\dim V^0_{\lambda_i} \geqq 2$. In this case we see that

$$(\nabla_X A)Y - (\nabla_Y A)X = (X\lambda_i)Y - (Y\lambda_i)X \quad \text{for } \forall X, Y \in V^0_{\lambda_i}.$$

On the other hand, Codazzi equation (2.7) yields

$$(\nabla_X A)Y - (\nabla_Y A)X = 2\langle\phi X, Y\rangle\xi \quad \text{for } \forall X, Y \in V^0_{\lambda_i}.$$

Choosing X, Y as arbitrary two independent vectors in $V^0_{\lambda_i}$, we know from these two equations that $X\lambda_i = Y\lambda_i = \langle\phi X, Y\rangle = (\nabla_X A)Y = 0$. This means that

$$(\nabla_X A)Y = \langle\phi X, Y\rangle = 0 \quad \text{for } \forall X, Y \in V^0_{\lambda_i}. \tag{3.7}$$

Therefore, for each unit vector $X \in V^0_{\lambda_i}$ and each $Z \in TM$, from (2.7), (3.7) and the symmetry of the shape operator A we obtain

$$\begin{aligned} 0 &= \langle(\nabla_X A)X, Z\rangle = \langle(\nabla_X A)Z, X\rangle \\ &= \langle(\nabla_Z A)X, X\rangle = \langle\nabla_Z(AX) - A\nabla_Z X, X\rangle \\ &= \langle(Z\lambda_i)X + (\lambda_i I - A)\nabla_Z X, X\rangle = Z\lambda_i, \end{aligned} \tag{3.8}$$

so that λ_i is constant.

Case (II$_b$): $\dim V^0_{\lambda_i} = 1$. As α is constant by Lemma A, we only need to consider the case that $2\lambda_i - \alpha \neq 0$ on some neighborhood of an arbitrary fixed point p.

Let e be a unit vector in $V^0_{\lambda_i}$ so that $Ae = \lambda_i e$. Then Lemma A implies $A\phi e = \dfrac{\alpha\lambda_i - 2}{2\lambda_i - \alpha}\phi e$. Hence, $\phi e \in V^0_{\lambda_j}$ for some j with $\lambda_j = \dfrac{\alpha\lambda_i - 2}{2\lambda_i - \alpha}$ $(\neq \lambda_i)$. This equation is equivalent to

$$(2\lambda_j - \alpha)\lambda_i = \alpha\lambda_j - 2. \tag{3.9}$$

When $2\lambda_j - \alpha \neq 0$, we have $\lambda_i = \dfrac{\alpha\lambda_j - 2}{2\lambda_j - \alpha}$. So, when $\dim V^0_{\lambda_j} \geqq 2$, we see that λ_i is constant (see the discussion in Case (II$_a$)).

Next, we consider the case of $2\lambda_j - \alpha = 0$. Hence Equation (3.9) yields $\alpha\lambda_j - 2 = 0$. Solving these equations, we get $\lambda_j = 1,\ \alpha = 2$ or $\lambda_j = -1,\ \alpha = -2$. In the following, it suffices to study the case of $\lambda_j = 1$ and $\alpha = 2$. For simplicity, we set $\lambda = \lambda_i$. So we see that $Ae = \lambda e$ $(\lambda \neq 1)$, $A\phi e = \phi e$, $A\xi = 2\xi$ and $\nabla_e e = 0$. We shall verify some equalities in order to show that the case $2\lambda_j - \alpha = 0$ does not occur. Codazzi equation (2.7) gives

$$(\nabla_\xi A)\phi e - (\nabla_{\phi e} A)\xi = e.$$

On the other hand, from (2.5) we find that

$$\begin{aligned} (\nabla_\xi A)\phi e - (\nabla_{\phi e} A)\xi &= \nabla_\xi(A\phi e) - A\nabla_\xi(\phi e) - \nabla_{\phi e}(A\xi) + A\nabla_{\phi e}\xi \\ &= \nabla_\xi(\phi e) - A\nabla_\xi(\phi e) - 2\nabla_{\phi e}\xi + A\nabla_{\phi e}\xi \\ &= (I - A)\nabla_\xi(\phi e) - 2\phi A\phi e + A\phi A\phi e \\ &= (I - A)\nabla_\xi(\phi e) + (2 - \lambda)e. \end{aligned}$$

Taking the inner products of these equations with the vector e, we obtain

$$1 = \langle\nabla_\xi(\phi e), (1 - \lambda)e\rangle + 2 - \lambda.$$

Since $\lambda \neq 1$, this means that

$$\langle\nabla_\xi(\phi e), e\rangle = -1. \tag{3.10}$$

Again, by using Codazzi equation (2.7), we see that

$$(\nabla_e A)\phi e - (\nabla_{\phi e} A)e = 2\xi.$$

On the other hand, from (2.4) we get

$$\begin{aligned} (\nabla_e A)\phi e - (\nabla_{\phi e} A)e &= \nabla_e(A\phi e) - A\nabla_e(\phi e) - \nabla_{\phi e}(Ae) + A\nabla_{\phi e}e \\ &= \nabla_e(\phi e) - A\nabla_e(\phi e) - (\phi e\lambda)e - \lambda\nabla_{\phi e}e + A\nabla_{\phi e}e \\ &= (\nabla_e\phi)e + \phi\nabla_e e - A\{(\nabla_e\phi)e + \phi\nabla_e e\} \\ &\quad - (\phi e\lambda)e + (A - \lambda I)\nabla_{\phi e}e \\ &= -\lambda\xi + \lambda A\xi - (\phi e\lambda)e + (A - \lambda I)\nabla_{\phi e}e \\ &= \lambda\xi - (\phi e\lambda)e + (A - \lambda I)\nabla_{\phi e}e. \end{aligned}$$

These equations, combined with the fact that $\lambda \neq 1$ and $\langle \nabla_{\phi e} e, e\rangle = 0$, show that $\nabla_{\phi e} e = \langle \nabla_{\phi e} e, \xi\rangle \xi$, so that

$$\begin{aligned}\nabla_{\phi e} e &= -\langle e, \nabla_{\phi e}\xi\rangle \xi \\ &= -\langle e, \phi A \phi e\rangle \xi \quad \text{(from (2.5))} \\ &= -\langle e, \phi^2 e\rangle \xi = \xi.\end{aligned}$$

Then we have

$$\nabla_{\phi e} e = \xi. \tag{3.11}$$

It follows from (2.3), (2.4) and (3.11) that

$$\nabla_{\phi e}(\phi e) = 0. \tag{3.12}$$

We here recall the following

$$\nabla_e(\phi e) = -\lambda \xi. \tag{3.13}$$

Indeed,

$$\begin{aligned}\nabla_e(\phi e) &= (\nabla_e \phi)e + \phi \nabla_e e \\ &= \eta(e)Ae - \langle Ae, e\rangle \xi \\ &= -\lambda \xi.\end{aligned}$$

Using these equalities (2.5), (3.10), (3.11), (3.12) and (3.13), we compute $\langle R(e, \phi e)\phi e, e\rangle$, where R is the curvature tensor of the real hypersurface M of $\mathbb{C}H^n(-4)$.

$$\begin{aligned}R(e, \phi e)\phi e &= \nabla_e \nabla_{\phi e}(\phi e) - \nabla_{\phi e}\nabla_e(\phi e) - \nabla_{[e,\phi e]}(\phi e) \\ &= \nabla_{\phi e}(\lambda \xi) - \nabla_{-\lambda\xi - \xi}(\phi e) \\ &= (\phi e \lambda)\xi + \lambda \phi A \phi e + (\lambda + 1)\nabla_\xi(\phi e) \\ &= (\phi e \lambda)\xi - \lambda e + (\lambda + 1)\nabla_\xi(\phi e).\end{aligned}$$

Hence,

$$\begin{aligned}\langle R(e, \phi e)\phi e, e\rangle &= -\lambda + (\lambda + 1)\langle \nabla_\xi(\phi e), e\rangle \\ &= -2\lambda - 1.\end{aligned}$$

On the other hand, Gauss equation (2.6) yields

$$\begin{aligned}\langle R(e, \phi e)\phi e, e\rangle &= -4 + \langle A\phi e, \phi e\rangle \langle Ae, e\rangle \\ &= -4 + \lambda.\end{aligned}$$

Thus we have $\lambda = 1$, which is a contradiction. Therefore our case (II_b) reduces to the case of $\dim V_{\lambda_i} = \dim V_{\lambda_j} = 1$.

We set $\mathfrak{T} = \{\xi, e, \phi e\}_{\mathbb{R}}$ with $Ae = \lambda e$ and $A\phi e = \dfrac{\alpha\lambda - 2}{2\lambda - \alpha}\phi e$. For simplicity we put $\mu = \dfrac{\alpha\lambda - 2}{2\lambda - \alpha}$. Note that $\lambda \neq \mu$. We shall now prove that $\mathfrak{T}$ is integrable and its each leaf is a totally geodesic submanifold of the real hypersurface M in $\mathbb{C}H^n(-4)$. We first remark that $\nabla_e e = \nabla_{\phi e}(\phi e) = 0$, since both $\{e\}_{\mathbb{R}}$ and $\{\phi e\}_{\mathbb{R}}$ satisfy the condition (1) in our Theorem. Also, we see easily that $\nabla_\xi \xi$, $\nabla_e \xi$, $\nabla_{\phi e}\xi \in \mathfrak{T}$ and $\nabla_e(\phi e) \in \mathfrak{T}$. Next, we prove $\nabla_\xi e \in \mathfrak{T}$. For this purpose, we observe that

$$\begin{aligned}(\nabla_\xi A)e - (\nabla_e A)\xi &= \nabla_\xi(Ae) - A\nabla_\xi e - \nabla_e(A\xi) + A\nabla_e \xi \\ &= (\xi\lambda)e + (\lambda I - A)\nabla_\xi e - \alpha\lambda\phi e + \lambda\mu\phi e.\end{aligned}$$

On the other hand, Codazzi equation (2.7) gives

$$(\nabla_\xi A)e - (\nabla_e A)\xi = -\phi e.$$

Hence we find that

$$(\lambda I - A)\nabla_\xi e = -\{\lambda(\mu - \alpha) + 1\}\phi e.$$

This, together with the fact that $\nabla_\xi e$ is perpendicular to V_λ, yields that $\nabla_\xi e \in \{\phi e\}_{\mathbb{R}} \subset \mathfrak{T}$. Similarly, we have $\nabla_\xi(\phi e) \in \mathfrak{T}$. We next verify $\nabla_{\phi e} e \in \mathfrak{T}$. It follows from $Ae = \lambda e$ and $A\phi e = \mu\phi e$ that

$$(\nabla_e A)\phi e - (\nabla_{\phi e} A)e = (e\mu)\phi e + (\mu I - A)\nabla_e(\phi e) - (\phi e\lambda)e - (\lambda I - A)\nabla_{\phi e} e.$$

Here, from (2.4) and $\nabla_e e = 0$ we have

$$(\mu I - A)\nabla_e(\phi e) = -\lambda(\mu - \alpha)\xi.$$

Moreover, Codazzi equation (2.7) shows

$$(\nabla_e A)\phi e - (\nabla_{\phi e} A)e = 2\xi.$$

These three equations tell us that

$$2\xi = (e\mu)\phi e - \lambda(\mu - \alpha)\xi - (\phi e\lambda)e - (\lambda I - A)\nabla_{\phi e} e,$$

which implies $\nabla_{\phi e} e \in \{\xi, \phi e\}_{\mathbb{R}} \subset \mathfrak{T}$. Consequently, $\mathfrak{T}$ is an integrable distribution and its each leaf (, say) T is a totally geodesic submanifold of M.

We here set $\mathfrak{L} = \{e, \phi e, \xi, \mathcal{N}\}_{\mathbb{R}}$. Then by similar computation we find that $\widetilde{\nabla}_X Y \in \mathfrak{L}$ for all $X, Y \in \mathfrak{L}$, which implies that the distribution $\mathfrak{L}$ is integrable and its each leaf (, say) L is a complex 2-dimensional totally geodesic Kähler submanifold of the ambient space $\mathbb{C}H^n(c)$. Note that each leaf L is nothing but totally geodesic $\mathbb{C}H^2(-4)$ in $\mathbb{C}H^n(-4)$. Then we can see that every leaf T of the distribution $\mathfrak{T}$ is a real hypersurface of totally geodesic $\mathbb{C}H^2(-4)$ in $\mathbb{C}H^n(-4)$. So the discussion in Case (I) implies that λ is constant locally on T, so that $(\nabla_e A)e = 0$ along T. This, combined with the computation in (3.8), yields that $Z\lambda = 0$ on M for any $Z \in TM$.

Hence we can see that every real hypersurface M (of $\mathbb{C}H^n(-4)$) satisfying the condition (1) is of type (A) or type (B). However, there exists no type (A) hypersurface (say,) M satisfying the condition (1). In fac, this real hypersurface M satisfies neither $\dim V^0_{\lambda_i} = 1$ nor $\dim V^0_{\lambda_i} \geqq 2$. Note that $\phi(V^0_{\lambda_i}) \perp V^0_{\lambda_i}$ if $\dim V^0_{\lambda_i} \geqq 2$. Therefore we conclude that M is of type (B).

Next, we suppose that M is of type (B) with principal curvatures $\lambda_1 = (\sqrt{|c|}/2)\cdot \coth(\sqrt{|c|}\,r/2)$, $\lambda_2 = (\sqrt{|c|}/2)\tanh(\sqrt{|c|}\,r/2)$ and $\alpha = \sqrt{|c|}\tanh(\sqrt{|c|}\,r)$ in $\mathbb{C}H^n(c)$. Then by the above argument we know that every leaf T_{λ_1} (resp. T_{λ_2}) of the restricted principal foliation $V^0_{\lambda_1}$ (resp. $V^0_{\lambda_2}$) is totally geodesic in the real hypersurface M. Moreover, this leaf is a non-totally geodesic but totally umbilic hypersurface of constant sectional curvature k_1 (resp. k_2) with $\sqrt{k_1 - (c/4)} = \lambda_1$ (resp. $\sqrt{k_2 - (c/4)} = \lambda_2$) in a real n-dimensional totally real totally geodesic submanifold $\mathbb{R}H^n(c/4)$ in the ambient space $\mathbb{C}H^n(c)$. Hence we find that every geodesic $\gamma = \gamma(s)$ on the real hypersurface M with $\dot\gamma(0) \in V^0_{\lambda_1}$ (resp. $\dot\gamma(0) \in V^0_{\lambda_2}$) is a circle of positive curvature λ_1 (resp. λ_2) in $\mathbb{C}H^n(c)$.

On the other hand, needless to say $\lambda_1 > \lambda_2 (> 0)$ holds. Then by solving the equation $\lambda_1 = 3\lambda_2$, we see that $r = (1/\sqrt{|c|})\log_e(2 + \sqrt{3})$. In this case, $\lambda_1 = \alpha = \sqrt{3|c|}/2$ and $\lambda_2 = \sqrt{|c|}/(2\sqrt{3})$.

At the end of the proof we recall the following fact. We take a geodesic $\gamma = \gamma(s)$ on M^n which is a hypersurface isometrically immersed into a Riemannian manifold $\widetilde{M}^{n+1}$. Suppose that the curve γ is a circle of positive curvature (, say) k in the ambient manifold $\widetilde{M}^{n+1}$. Then, we find easily that the shape operator A of M^n in $\widetilde{M}^{n+1}$ satisfies $A\dot\gamma(s) = k\dot\gamma(s)$ for each s or $A\dot\gamma(s) = -k\dot\gamma(s)$ for each s (see (2.1) and (2.2)).

Hence by the above three facts we can conclude that $r = (1/\sqrt{|c|}\,)\log_e(2+\sqrt{3}\,)$ if and only if a type (B) hypersurface M satisfies the condition (2). Thus we obtain the desirable conclusion. □

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Department of Mathematics, Saga University, 1 Honzyo, Saga, 840-8502, Japan

E-mail address: smaeda@ms.saga-u.ac.jp