ON HYPER R-SUBGROUPS OF HYPERNEAR-RINGS

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Abstract. The study of hypernear-rings is extremely challenging, offering curiously beautiful results to one who is willing to look for structure where symmetry is not so abundant. The purpose of this paper is to present certain results arising from a study of hypernear-rings. In particular, we define the notion of hyper R-subgroup of a hypernear-ring and we investigate some properties of hypernear-rings with respect to the hyper R-subgroups.

1 Introduction The theory of hypergroups has been introduced by Marty [7] in 1934 during the 8^{th} Congress of the Scandinavian Mathematicians. Since the hypergroup is a very general hyperstructure, several researchers endowed it with more stronger or less strong axiom. As a result we are really dealing now with a big number of hypergroups. As an example, we can mention the canonical hypergroup introduced by Mittas [8]. Polygroups [3] or quasi canonical hypergroups [1] are a generalization of canonical hypergroups. In the context of canonical hypergroup some mathematicians studied multi-valued systems whose additive structure is just a quasi canonical hypergroup. For instance, in [4], Dasic has introduced the notion of hypernear-ring in a particular case. Gontineac [6] called this zero symmetric hypernear-ring and studied the concept of hypernear-rings in a general case. In [5], Davvaz introduced the notion of an H_v-near ring generalizing the notion of a hypernear-ring. The study of hypernear-rings is extremely challenging, offering curiously beautiful results to one who is willing to look for structure where symmetry is not so abundant. In this paper, we consider the definition of hypernear-rings according to the definition of Dasic [4] and Gontineac [6]. We define the notion of hyper R-subgroup of a hypernear-ring and we investigate some properties of hypernear-rings with respect to the hyper R-subgroups.

2 Preliminaries First we shall present the fundamental definitions.

Definition 2.1. Let H be a non-empty set. A *hyperoperation* $*$ on H is a mapping of $H \times H$ into the family of non-empty subsets of H.

The concept of hyperstructures constitute a generalization of the well-known algebraic structures (groups, rings, modules, near-rings and so on). A comprehensive review of the theory of hyperstructures appears in [2] and [9]. This paper deals with hyperstructures mainly with hypernear-rings. We recall the following definition from [4] and [6].

Definition 2.2. A *hypernear-ring* is an algebraic structure $(R, +, \cdot)$ which satisfies the following axioms:

1) $(R, +)$ is a *quasi canonical hypergroup* (not necessarily commutative), i.e., in $(R, +)$ the following hold:

a) $x + (y + z) = (x + y) + z$ for all $x, y, z \in R$;

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b) There is $0 \in R$ such that $x + 0 = 0 + x = x$ for all $x \in R$;

c) For every $x \in R$ there exists one and only one $x' \in R$ such that $0 \in x + x'$, (we shall write $-x$ for x' and we call it the opposite of x);

d) $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$.

If $x \in R$ and A, B are subsets of R, then by $A + B$, $A + x$ and $x + B$ we mean

$$A + B = \bigcup_{\substack{a \in A \\ b \in B}} a + b, A + x = A + \{x\}\ ,\ x + B = \{x\} + B.$$

2) With respect to the multiplication, $(R, \cdot)$ is a semigroup having absorbing element 0, i.e., $x \cdot 0 = 0$ for all $x \in R$. But, in general, $0x \neq 0$ for some $x \in R$.

3) The multiplication is distributive with respect to the hyperoperation $+$ on the left side, i.e., $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in R$.

A hypernear-ring R is called *zero symmetric* if $0x = x0 = 0$ for all $x \in R$.

Note that for all $x, y \in R$, we have $-(-x) = x, 0 = -0, -(x + y) = -y - x$ and $x(-y) = -xy$.

Example 2.3. [4]. Let $(H, +)$ be a hypergroup and let $M_0(H)$ be a set of all mappings $f : H \longrightarrow H$ such that $f(0) = 0$. For all $f, g \in M_0(H)$ we define the hyperoperation $f \oplus g$ of mappings as follows:

$$(f \oplus g)(x) = \{h \in M_0(H) \mid \forall x \in H,\ h(x) \in f(x) + g(x)\}$$

where $+$ on the right side is the hyperoperation in $(H, +)$. If $h \in f \oplus g$, $h(0) \in f(0) + g(0) = 0$, so $h(0) = 0$. Furtheremore $(f \oplus g)(x) = f(x) + g(x)$. A multiplication is a substitution of mappings. Then $(M_0(H), \oplus, \cdot)$ is a zero symmetric hypernear-ring

Definition 2.4. Let R be a hypernear-ring. A non-empty subset S of R is called a *subhypernear-ring* if $(S, +)$ is a subhypergroup of $(R, +)$ and $(S, \cdot)$ is a subsemigroup of $(R, \cdot)$.

3 Hyper R-subgroups of hypergroups We first consider the notion of hyper R-subgroup of a hypernear-ring R.

Definition 3.1. A *two-sided hyper R-subgroup* of a hypernear-ring R is a subset H of R such that

1) $(H, +)$ is a subhypergroup of $(R, +)$, i.e.,
   1i) $a, b \in H$ implies $a + b \subseteq H$,
   1ii) $a \in H$ implies $-a \in H$,

2) $RH \subseteq H$,

3) $HR \subseteq H$.

If H satisfies (1) and (2), then it is called a *left hyper R-subgroup* of R. If H satisfies (1) and (3), then it is called a *right hyper R-subgroup* of R.

Definition 3.2. Let $(R, +, \cdot)$ be a hypernear-ring.

(i) The subset $R_0 = \{x \in R \mid 0x = 0\}$ of R is called a *zero-symmetric part* of R.

(ii) The subset $R_c = \{x \in R \mid xy = y,\ \forall y \in R\}$ is called *constant part* of R.

(iii) If $R = R_0$ (resp. $R = R_c$), we say that R is a *zero-symmetric* (resp. *constant*) *hypernear-ring,* respectively.

Example 3.3. Consider hypernear-ring $R = \{0, a, b, c\}$ with addition and multiplication tables below:

$+$	0	a	b	c
0	$\{0\}$	$\{a\}$	$\{b\}$	$\{c\}$
a	$\{a\}$	$\{0, a\}$	$\{b\}$	$\{c\}$
b	$\{b\}$	$\{b\}$	$\{0, a, c\}$	$\{b, c\}$
c	$\{c\}$	$\{c\}$	$\{b, c\}$	$\{0, a, b\}$

$\cdot$	0	a	b	c
0	0	a	b	c
a	0	a	b	c
b	0	a	b	c
c	0	a	b	c

Since $R = R_c$, so R is a constant hypernear-ring.

Lemma 3.4. If $0y = y$ for each y in a hypernear-ring R, then R is a constant hypernear-ring.

Proof. Take $x \in R$. Then $xy = x(0y) = (x0)y = 0y = y$. □

Lemma 3.5. Let $(R, +, \cdot)$ be a constant hypernear-ring. Then R is the only right hyper R-subgroup of R.

Proof. Let $(H, +)$ be a subhypergroup of $(R, +)$. If $HR \subseteq H$, then we have $0R \subseteq H$, and so $R \subseteq H$ since $0R = R$. Thus R is the only right hyper R-subgroup of R. □

Lemma 3.6. Let $(R, +, \cdot)$ be a constant hypernear-ring. If $(H, +)$ is a subhypergroup of $(R, +)$ then H is a left hyper R-subgroup of R.

Proof. Let $(H, +)$ be a subhypergroup of $(R, +)$. Since $xy = y$ for all $x, y \in R$, we have $RH \subseteq H$, and so H is a left hyper R-subgroup of R, for all subhypergroups □

Proposition 3.7. Let R be a hypernear-ring. Then R_0 is a zero symmetric subhypernear-ring of R.

Proof. Let $x, y \in R_0$ and $z \in x + y$, arbitrary. Then we have $0x = 0$ and $0y = 0$, and so $0z \in 0(x+y) = 0x+0y = 0+0 = 0$. Hence we obtain $z \in R_0$. This implies $x+y \subseteq R_0$. From $x(-y) = -xy$, we have $0(-x) = -(x0) = -0 = 0$, for all $x \in R_0$. This implies that $-x \in R_0$. Also, we have $0(xy) = (0x)y = 0y = 0$, that is., $R_0R_0 \subset R_0$. This implies $R_0R \subset R_0$. This completes the proof. □

Now let H be a subhypernear-ring of hypernear-ring R. Define the sets by

$$B_t := \{0x \mid x \in H\} \text{ and } B := \{0r \mid r \in R\}.$$

Theorem 3.8. Let H be a subhypernear-ring of hypernear-ring R. Then we have

(i) B_t is a two-sided hyper H-subgroup of H,

(ii) B_t is a left hyper B-subgroup of B.

Proof. (i) Let $x, y \in B_t$ and $z \in x + y$. Then there exists $s_1, s_2 \in H$ such that $x = 0s_1, y = 0s_2$. Hence we have $z \in x + y = 0s_1 + 0s_2 = 0(s_1 + s_2) \subseteq B_t$ since $s_1 + s_2 \subseteq H$. This implies that $x + y \subset B_t$. Next, let $x \in B_t$. Then there exists $s \in H$ such that $x = 0s$. Hence we have $-x = -(0s) = 0(-s)$. This implies that $-x \in B_t$ since $-s \in H$. Let $a \in H$ and $b \in B_t$. Then there exists $x \in H$ such that $b = 0x$ and $ab = a(0x) = (a0)x = 0x \in H$. Therefore $HB_t \subset B_t$. Similarly, Let $a_1 \in H$ and $b_1 \in B_t$. Then there exists $x_1 \in H$ such that $b_1 = 0x_1$. Hence $b_1a_1 = (0x_1)a_1 = 0(x_1a_1)$. Since H is a subhypernear-ring, we obtain $x_1a_1 \in H$ for a_1 and $x_1 \in H$. Therefore $b_1a_1 \in B_t$. This implies that $B_tH \subset H$. Hence B_t is a two-sided hyper H-subgroup of H.

(ii) Let $0r, 0s \in B$ and $z \in 0r + 0s$. Then $z \in 0r + 0s = 0(r + s) \subseteq B$. Hence we have $z \in B$. Also, we have $0r \cdot 0s = 0s$ since $(0r) \cdot (0s) = ((0r)0)s = 0s$. Therefore B is a subhypernear-ring and from (i), $(B_t, +)$ is a subhypergroup of $(B, +)$. It remains to show that $BB_t \subset B_t$. Let $a \in B$ and $b \in B_t$. Then there exists $x \in H$ and $r \in R$ such that $a = 0r, b = 0x$. Thus $ab = (0r) \cdot (0x) = 0(r0)x = 0(0)x = 0x \in B_t$. Therefore we have $BB_t \subset B_t$. This completes the proof. □

Definition 3.9. A subhypergroup $A \subseteq R$ is called *normal* if for all $x \in R$, we have $x + A - x \subseteq A$.

Proposition 3.10. Let A is a normal subhypergroup of R. Then

(i) $A + x = x + A$ for all $a \in R$,

(ii) $(A + x) + (A + y) = A + x + y$ for all $x, y \in R$.

Proof. (i) Suppose that $y \in A + x$. Then there exists $a \in A$ such that $y \in a + x$. Hence $y \in A + x = 0 + a + x \subseteq (x - x) + a + x = x + (-x + a + x) \subseteq x + A$, and so $A + x \subseteq x + A$. Similarly $x + A \subseteq A + x$.

(ii) We have $(A + x) + (A + y) = A + (x + A) + y = A + (A + x) + y = A + x + y$. □

Theorem 3.11. Let R be a hypernear-ring and H a subhypernear-ring. Then $H = (R_0 \cap H) + B_t$ and $(R_0 \cap H) \cap B_t = \{0\}$.

Proof. We have $(R_0 \cap H) \cap B_t \subset R_0 \cap B$ and $R_0 \cap B = \{0\}$. Thus $(R_0 \cap H) \cap B_t = \{0\}$. Finally, if $a \in H$, we have $0 \in 0a - 0a = 0a - (00)a = 0a - 0(0a) = 0(a - 0a)$. So there exists $y \in a - 0a$ such that $0 = 0y$. Since $a \in H$, we have $a - 0a \subseteq H$, and so $y \in H$. From $y \in a - 0a$, using condition (d) in Definition 2.2, we get $a \in y + 0a$. Since $a \in H$, then $0a \in B_t$. Since $0 = 0y$, then $y \in R_0$. Therefore we have $a \in (R_0 \cap H) + B_t$. □

Definition 3.12. For an element x of a hypernear-ring R, the *(right) annihilator* of x is $Ann(x) = \{r \in R \mid xr = 0\}$. For a non-empty subset B of a hypernear-ring R, the annihilator of B is $Ann(B) = \cap\{Ann(x) \mid x \in B\}$.

Proposition 3.13. For any element x of a zero symmetric hypernear-ring R, $Ann(x)$ is a right R-subgroup of R.

Proof. Certainly $0 \in Ann(x)$. If $a, b \in Ann(x)$, then $x(a + b) = xa + xb = 0$, so for every $c \in a + b$, we have $xc = 0$ which implies that $a + b \subseteq Ann(x)$. Also, we have $x(-a) = -xa = -0 = 0$ and so $-a \in Ann(x)$. On the other hand, for $r \in R$ and $a \in Ann(x)$, we have $x(ar) = (xa)r = 0r = 0$ so $ar \in Ann(x)$ which yields $Ann(x)R \subseteq Ann(x)$. □

Proposition 3.14. If e is any element of a hypernear-ring R, then $eR = \{er \mid r \in R\}$ is a right hyper R-subgroup of R.

Proof. It is straightforward. □

Definition 3.15. An element e of a hypernear-ring R is an *idempotent* if $e^2 = e$.

Lemma 3.16. For a hypernear-ring R, if $e \in R_c$ then $e^2 = e$, so e is an idempotent.

Proof. Since each element of a constant hypernear-ring is a left identity, it is also an idempotent. □

Theorem 3.17. Let e be an idempotent element of a zero symmetric hypernear-ring R. Then

(i) $Ann(e) \cap eR = \{0\}$.

(ii) For all $r \in R$, there exists a unique element $a \in Ann(e)$ and there exists a unique element $b \in eR$ such that $r \in A + b$.

Proof. (i) Let $x \in Ann(e) \cap eR$. Then $x = er$ for some $r \in R$. So

$$0 = ex = e(er) = (ee)r = er = x,$$

hence $Ann(e) \cap eR = \{0\}$.

(ii) For $r \in R$, we have

$$0 \in er - er = er - e^2r = er - e(er) = e(r - er).$$

So there exists $y \in r - er$ such that $0 = ey$. ¿From $y \in r - er$, using condition (d) in Definition 2.2, we get $r \in y + er$. Since $ey = 0$, then $y \in Ann(e)$. We put $a = y$ and $b = er$. Then $x \in a + b$. If we take another $a' \in Ann(e)$ and $b' \in eR$ with $x \in a' + b'$, then $x \in (a + b) \cap (a' + b')$. ¿From $x \in a' + b'$, we get $b' \in -a' + x$, and so $b' \in -a' + (a + b) = (-a' + a) + b$. Hence there exists $y \in -a' + a$ such that $b' \in y + b$, and so $y \in b' - b$. Therefore $(-a' + a) \cap (b' - b) \neq \emptyset$. Since $-a' + a \subseteq Ann(e)$ and $b' - b \subseteq eR$ and $eR \cap Ann(e) = \emptyset$, we obtain $-a' + a = b' - b = \{0\}$. Therefore $a = a'$ and $b = b'$. □

With hypernear-rings, as with other mathematical structure, we shall be interested in mappings that preserve some or all of the properties of the hypernear-rings. One could summarize a lot of research effort by saying that it is an investigation of properties that are preserved relevant to the structure.

Definition 3.18. Let R and R' be two hypernear-rings. Then the map $f : R \to R'$ is called a *homomorphism* if for all $x, y \in R_1$,

(i) $f(x + y) = f(x) + f(y)$,

(ii) $f(x \cdot y) = f(x) \cdot f(y)$,

(iii) $f(0) = 0$.

If f is surjective, that is, one to one and onto, then f is an isomorphism.

Definition 3.19. If f is a homomorphism from R into R', then the *kernel* of f is the set $kerf = \{x \in R \mid f(x) = 0\}$.

It is easy to see that $kerf$ is a left hyper R-subgroup of R, but in general is not normal in R.

Proposition 3.20. Let $f : R \to R'$ be a homomorphism of hypernear-rings. Then the following statements are true.

(1) If f is onto and M is a hyper R-subgroup of R, then $f(M)$ is a hyper R'-subgroup of R'.

(2) If N is a hyper R'-subgroup of R', then $f^{-1}(N)$ is a hyper R-subgroup of R.

(3) $f(R_0) \subseteq R'_0$.

(4) $f(R_c) \subseteq R'_c$.

(5) If f is an isomorphism, then so is f^{-1}.

Proof. The proof is so easy that it will be omitted. □

Definition 3.21. A normal subhypergroup A of the hypergroup $(R, +)$ is

(i) a *left hyperidel* of R if $x \cdot a \in A$ for all $x \in R$ and $a \in A$.

(ii) a *right ideal* of R if $(x + A) \cdot y - x \cdot y \subseteq A$ for all x and $y \in R$.

(iii) a *hyperideal* of R if $(x + A) \cdot y - x \cdot y \cup z \cdot A \subseteq A$ for all x, y and $z \in R$.

Theorem 3.22. Let $(R, +, \cdot)$ be a hypernear-ring.

(i) If K is a left hyperideal of R and L is a left hyper R-subgroup of R, then $L + K$ is a left hyper R-subgroup of R.

(ii) If K is a right hyperideal of R and L is a right hyper R-subgroup of R, then $L + K$ is a right hyper R-subgroup of R.

Proof. In each case, $L + K = K + L$ is a subhypergroup which is normal if L is normal.
(i) If $RL \subseteq L$ and $RK \subseteq K$, then $r(l + k) = rl + rk \subseteq L + K$ for all $r \in R, l \in L$ and $k \in K$. This completes the proof of (i).
(ii) Now assume that K is a right hyperideal and K is a right hyper R-subgroup. Let $r \in R, l \in L, k \in K$. Then $(l + k)r - lr \subseteq K$ as K is a right hyperideal of R. So for some $k_1 \in K$, we have

$$(l + k)r = k_1 + lr \subseteq K + LR \subseteq K + L.$$

Hence $L + K = K + L$ is a right hyper R-subgroup. □

Lemma 3.23. Let R be a hypernear-ring, S a subhypernear-ring of R and H a left (resp. *right, two-sided*) hyper R-subgroup of R. Then $H \cap S$ is a left (resp. *right, two-sided*) hyper R-subgroup of R.

Proof. The proof is so easy that it will be omitted. □

Let H be a normal hyper R-subgroup of hypernear-ring R. If we define a relation

$$x \sim y \ (\bmod H) \text{ if and only if } x - y \cap H \neq \emptyset$$

for all $x, y \in H$, then this relation is a congruence on H.

Let $\rho(x)$ be the equivalence class of the element $x \in H$ and define R/H as follows;

$$R/H = \{\rho(x) \mid x \in H\}.$$

Define the hyperoperation $\oplus$ and multiplication $\odot$ on R/H by

$$\rho(a) \oplus \rho(b) = \{\rho(c) \mid c \in \rho(a) \oplus \rho(b)\} \text{ and } \rho(a) \odot \rho(b) = \rho(a \cdot b).$$

Theorem 3.24. $(R/H, \oplus, \odot)$ is a hypernear-ring, *factor hypernear-ring.*

Lemma 3.25. Let H be a normal hyper R-subgroup of R. Then $\rho(x) = H + x$.

Proof. Suppose that $y \in H + x$, then there exists $a \in H$ such that $y \in a + x$, which implies that $a \in y - x$ and so $(y - x) \cap H \neq \emptyset$ or $y \in \rho(x)$. Thus $H + x \subseteq \rho(x)$. Similarly, we have $\rho(x) \subseteq H + x$. □

Theorem 3.26. (First isomorphism theorem). Let f be a homomorphism from R into R' with kernel K such that K is a normal hyper R-subgroup of R, then $R/H \cong Imf$.

Proof. The proof is straightforward. □

Theorem 3.27. Let R be a hypernear-ring and K a normal hyper R-subgroup of R. Then the following statement are equivalent:

(i) K is the kernel of a hypernear-ring homomorphism.

(ii) $(a + x)y - xy \subseteq K$ for all $x, y \in R$ and all $a \in K$.

(iii) $-xy + (a + x)y \subseteq K$ for all $x, y \in R$ and all $a \in K$.

Proof. (i)$\longrightarrow$(ii): Suppose that K is the kernel of a hypernear-ring homomorphism f. Then

$$f((a + x)y - xy) = (f(a) + f(x))f(y) - f(x)f(y) = 0$$

if $x, y \in R$ and $a \in K$. Hence $(a + x)y - xy \subseteq K$.

(ii)$\longrightarrow$(i): For the normal hyper R-subgroup K of R, there is the quotient R/K. We consider the natural map $\pi : R \longrightarrow R/K$ where $\pi(x) = x + K$. Clearly by Proposition 3.10, we have $\pi(x + y) = \pi(x) + \pi(y)$ and $\pi(0) = 0$. We show that $\pi(xy) = \pi(x)\pi(y)$, that is, $K + xy = (K + x)(K + y)$ and to do this, we only need to show $(K + x)(K + y) = K + xy$ is well defined binary operation. We take $K + x' = K + x$ and $K + y' = K + y$. So there are $a, b \in K$ such that $x' \in a + x$ and $y' \in b + y$. Hence $x'y' \in (a + x)(b + y) = (a + x)b + (a + x)y$. Now $x'y' - xy \subseteq (a + x)b + [(a + x)y - xy] \subseteq K + K \subseteq K$. This means $K + x'y' = K + xy$, which in turn means $(K + x)(K + y) = K + xy$ is well defined.

(ii)$\longrightarrow$(iii): For any $a \in K$ and $x, y \in R$, we have

$$-xy + (a + x)y = -[-(a + x)y + xy] = -[(a + y)(-y) - x(-y)] \subseteq K$$

since by (ii) $(a + y)(-y) - x(-y) \subseteq K$.

(iii)$\longrightarrow$(ii): The proof is similar to (ii)$\longrightarrow$(iii). □

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