STRONG LAWS OF LARGE NUMBERS FOR WEIGHTED SUMS OF SET-VALUED RANDOM VARIABLES IN RADEMACHER TYPE P BANACH SPACE

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ABSTRACT. In this paper, we shall prove some strong laws of large numbers(SLLN's) for weighted sums of set-valued random variables in the sense of Hausdorff metric d_H for which the basic space being Rademacher type p $(1 \le p \le 2)$ Banach space. We partially follow the results of classical SLLN's for $\mathfrak{X}$-valued random variables in [2], extending it to more general set-valued case.

1. Introduction As it is well known, the strong law of large numbers is essential theory in probability, statistics and the related fields. In 1975, Artstein and Vitale used an embedding theorem to prove a strong law of large numbers for independent and identically distributed set-valued random variables whose basic space is a d-dimensional Euclidean space $\mathbb{R}^d$ in [3], and Hiai extended it to the case that basic space is a separable Banach space $\mathfrak{X}$ in [8]. Taylor and Inoue proved SLLN's for only independent case in Banach space in [19]. Many other authors such as Giné, Hahn and Zinn [6], Hess [7], Puri and Ralescu [17] discussed SLLN's under different settings for set-valued random variables where the underlying space is a separable Banach space.

On the other hand, SLLN's for weighted sums of random variables are important in probability theory and often used in practice. Taylor [18] discussed different types of LLN's for weighted sums of random elements in normed linear spaces. In 1985, Taylor and Inoue proved the SLLN for weighted sums of independent set-valued random variables in [19], where the weights is a triangular array of constants. Since the growth behaviors of weights will affect the convergence of sums, and the weights are not always a triangular array of constants, it is necessary to discuss SLLN's for more general weights. In general, additional restrictions on the distributions or the Banach spaces are needed to obtain some results for which the identical distribution is not assumed. In [2], Adler etc. proved this type of SLLN for $\mathfrak{X}$-valued random variables.

In this paper, what we are concerned is strong laws of large numbers for weighted sums of set-valued random variables in Racemacher type p Banach space. First we prove some properties for the Rademacher type p of $\mathbf{K}_k(\mathfrak{X})$, the space of all compact subsets of $\mathfrak{X}$. Then we obtain SLLN's for weighted sums of set-valued random variables in the sense of d_H.

The organization of this paper is as follows. In section 2, we shall briefly introduce some definitions of set-valued random variables and two Lemmas. In section 3, we shall prove SLLN's for weighted sums of set-valued random variables in the sense of d_H.

2. Preliminaries on Set-Valued Random Variables Throughout this paper, we assume that $(\Omega, \mathcal{A}, P)$ is a complete probability space, $(\mathfrak{X}, \|\cdot\|)$ is a real separable Banach space, $\mathbf{K}(\mathfrak{X})$ is the family of all nonempty closed subsets of $\mathfrak{X}$, $\mathbf{K_k}(\mathfrak{X})$ is the family of all

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nonempty compact subsets of $\mathfrak{X}$, and $\mathbf{K_{kc}}(\mathfrak{X})$ is the family of all nonempty compact convex subsets of $\mathfrak{X}$. If E is some metric space, let $\mathcal{B}(E)$ denote the Borel field of E.

Let A and B be two nonempty subsets of $\mathfrak{X}$ and let $\lambda \in \mathbb{R}$, the set of all real numbers. We define addition and scalar multiplication by

$$A + B = \{a + b : a \in A, b \in B\},$$

$$\lambda A = \{\lambda a : a \in A\}.$$

The Hausdorff metric on $\mathbf{K}_k(\mathfrak{X})$ is defined by

$$d_H(A, B) = \max\{\sup_{a\in A} \inf_{b\in B} \|a - b\|,\ \sup_{b\in B} \inf_{a\in A} \|a - b\|\}$$

for $A,\ B \in \mathbf{K}_k(\mathfrak{X})$. For an A in $\mathbf{K}_k(\mathfrak{X})$, let $\|A\|_{\mathbf{K}} = d_H(\{0\}, A)$.

The metric space $(\mathbf{K}_k(\mathfrak{X}), d_H)$ is complete and separable, and $\mathbf{K}_{kc}(\mathfrak{X})$ is a closed subset of $(\mathbf{K}_k(\mathfrak{X}), d_H)$ (cf. [14], Theorems 1.1.2 and 1.1.3).

A set-valued mapping $F : \Omega \to \mathbf{K}_k(\mathfrak{X})$ is called set-valued random variable (or measurable) if, for each open subset O of $\mathfrak{X}$, $F^{-1}(O) = \{\omega \in \Omega : F(\omega) \cap O \neq \emptyset\} \in \mathcal{A}$. Notice that we usually define a set-valued random variable as $F : \Omega \to \mathbf{K}(\mathfrak{X})$, but we focus on $\mathbf{K}_k(\mathfrak{X})$-valued case in this paper.

For each set-valued random variable F, the expectation of F, denoted by $E[F]$, is defined as

$$E[F] = \Big\{ \int_\Omega f dP : f \in S_F \Big\},$$

where $\int_\Omega f dP$ is the usual Bochner integral in $L^1[\Omega, \mathfrak{X}]$, the family of integrable $\mathfrak{X}$-valued random variables, and $S_F = \{f \in L^1[\Omega, \mathfrak{X}] : f(\omega) \in F(\omega), a.e.\}$. This kind of integral is also called the Aumann integral, as it was introduced by Aumann in [4].

Let $A, B \in \mathbf{K}(\mathfrak{X})$. If there exists a $W \in \mathbf{K}(\mathfrak{X})$ such that $A = B + W$ then W is called the Hukuhara difference of A and B, denoted by $A \ominus B$. We have $A \ominus B + B = A$.

Let A, B be nonempty subsets of $\mathfrak{X}$. If there exists $x \in \mathfrak{X}$ such that $x + B \subset A$, then $A \ominus B$ exists, and if $A, B \in \mathbf{K}_{kc}(\mathfrak{X})$, we have

$$A \ominus B = \{x \in \mathfrak{X} : x + B \subset A\}.$$

We call $\{V_n : n \geq 1\}$ is stochastically dominated by a set-valued random element V, if for some constant $D < \infty$, such that for each $n \geq 1$,

$$P\{\|V_n\|_{\mathbf{K}} > t\} \leq DP\{\|DV\|_{\mathbf{K}} > t\},\ t \geq 0. \tag{2.1}$$

Of course, (2.1) is automatic with $V = V_1$ and $D = 1$ if the $\{V_n : n \geq 1\}$ are identically distributed set-valued random variables.

Now we will introduce an important definition and two Lemmas, which will be used later.

Let $\{\varepsilon_i : i \geq 1\}$ are independent and identically distributed random variables with $P\{\varepsilon_1 = 1\} = P\{\varepsilon_1 = -1\} = 1/2$, we usually call $\{\varepsilon_i\}$ a Bernoulli sequence or a Rademacher series.

Definition 1 Let $\{\varepsilon_i\}$ be a Rademacher series, $1 \leq p \leq 2$, $\mathfrak{X}$ is called Rademacher type p Banach space if there is a constant C such that for all finite sequences $x_i \in \mathfrak{X}$,

$$E\Big[\|\sum_{i=1}^n \varepsilon_i x_i\|^p\Big] \leq C \sum_{i=1}^n E[\|x_i\|^p].$$

Hoffmann-Jorgensen and Pisier [10] proved that for $1 \leq p \leq 2$, a real separable Banach space is of Rademacher type p Banach space iff there exists a constant $0 < C < \infty$ such that

$$E\Big[\|\sum_{i=1}^{n} x_i\|^p\Big] \leq C\sum_{i=1}^{n} E[\|x_i\|^p]$$

for every finite collection $\{x_1, \cdots, x_n\}$ of independent random elements with $E[x_i] = 0, E[\|x_i\|^p] < \infty, 1 \leq i \leq n$.

We can easily extend Lemmas 2 and 3 in [2] to the case of set-valued random variables.

Lemma 1 Let $\{V_n : n \geq 1\}$ and V be set-valued random variables such that $\{V_n : n \geq 1\}$ is stochastically dominated by V in the sense that (2.1) holds. Let $\{c_n : n \geq 1\}$ be positive constants such that

$$(\max_{1\leq j\leq n} c_j^q)\sum_{j=n}^{\infty}\frac{1}{c_j^q} = O(n) \ for \ some \ q > 0$$

and

$$\sum_{n=1}^{\infty} P\{\|V\|_{\mathbf{K}} > Dc_n\} < \infty.$$

Then for all $0 < M < \infty$, we have

$$\sum_{j=1}^{\infty}\frac{1}{c_j^q}E\Big[\|V_j\|_{\mathbf{K}}^q I_{\{\|V_j\|_{\mathbf{K}}\leq Mc_j\}}\Big] < \infty.$$

Lemma 2 Let V_0 and V be set-valued random variables such that V_0 is stochsatically dominated by V in the sense that (2.1) holds. Then for $x \geq 0$,

$$E[\|V_0\|_{\mathbf{K}} I_{\{\|V_0\|_{\mathbf{K}}>x\}}] = \int_x^{\infty} P\{\|V_0\|_{\mathbf{K}} > t\}dt + xP\{\|V_0\|_{\mathbf{K}} > x\},$$

and

$$E[\|V_0\|_{\mathbf{K}} I_{\{\|V_0\|_{\mathbf{K}}>x\}}] \leq D^2 E[\|V\|_{\mathbf{K}} I_{\{\|DV\|_{\mathbf{K}}>x\}}].$$

3. Main Results In order to get strong laws of large numbers for weighted sums of set-valued random variables in Rademacher type p Banach space, we need to prove several Lemmas. We note that, since we often deal with the constants in our proofs, we may use the same symbol $C (0 < C < \infty)$ to denote different constants in order to simplify the notation.

Lemma 3 If $\{F_i : i \geq 1\}$ is a sequence of independent compact set-valued random variables with $E[F_i] = \{0\}$, then

(i) for any $x_i \in S_{F_i}$, $E[x_i] = 0$;

(ii) for any finite n, if $x_i \in S_{F_i}, i = 1, \cdots, n$, then $x_1, \cdots, x_n$ are independent.

Proof. (i) is obviously. (ii) It only needs to prove the case $n = 2$. Since $\mathfrak{X}$ is separable and $x_i, i = 1, 2$ are $\mathfrak{X}$-valued random variable, there exist measurable function $\varphi_i : (\mathbf{K}_k(\mathfrak{X}), \mathcal{B}(\mathbf{K}_k(\mathfrak{X}))) \to (\mathfrak{X}, \mathcal{B}(\mathfrak{X}))$ such that for each $i = 1, 2$, $x_i(\omega) = \varphi_i(F_i(\omega))$ for every $\omega \in \Omega$. Now we prove that x_1, x_2 are independent. Indeed, for any Borel set $A, B \in \mathcal{B}(\mathfrak{X})$,

$$\begin{aligned} P\{\varphi_1(F_1(\omega)) \in A, \varphi_2(F_2(\omega)) \in B\} &= P\{F_1(\omega) \in \varphi_1^{-1}(A), F_2(\omega) \in \varphi_2^{-1}(B)\} \\ &= P\{F_1(\omega) \in \varphi_1^{-1}(A)\}P\{F_2(\omega) \in \varphi_2^{-1}(B)\} \\ &= P\{\varphi_1(F_1(\omega)) \in A\}P\{\varphi_2(F_2(\omega)) \in B\}. \end{aligned}$$

Hence, x_1, x_2 are independent. □

Then we can have the following Lemma.

Lemma 4 If $\mathfrak{X}$ is a Rademacher type $p(1 < p \leq 2)$ Banach space, $\{F_i : i \geq 1\}$ are independent compact set-valued random variables with $E[F_i] = \{0\}$, $E[\|F_i\|_{\mathbf{K}}^p] < \infty$ for all $i \geq 1$, then there exists a constant C such that for all finite sequences F_i,

$$E\Big[\Big\|\sum_{i=1}^n F_i\Big\|_{\mathbf{K}}^p\Big] \leq C\sum_{i=1}^n E\Big[\|F_i\|_{\mathbf{K}}^p\Big].$$

Proof. Since $\mathfrak{X}$ is a Rademacher type p Banach space, there exists a constant C such that for all finite independent sequences x_i with $E[x_i] = 0$, $E[\|x_i\|^p] < \infty, 1 \leq i \leq n$

$$E\Big[\Big\|\sum_{i=1}^n x_i\Big\|^p\Big] \leq C\sum_{i=1}^n E\Big[\|x_i\|^p\Big], \tag{3.1}$$

thus

$$\begin{aligned}
E\Big[\Big\|\sum_{i=1}^n F_i\Big\|_{\mathbf{K}}^p\Big] &= E\Big[\sup_{x\in\sum_{i=1}^n F_i}\|x\|\Big]^p \\
&= E\Big[\|x_0\|^p\Big] (\textit{there exists } x_0 = \sum_{i=1}^n x_{0i} \in \sum_{i=1}^n F_i, x_{0i} \in F_i) \\
&= E\Big[\Big\|\sum_{i=1}^n x_{0i}\Big\|^p\Big] \\
&\leq C\sum_{i=1}^n E\Big[\|x_{0i}\|^p\Big] \ (\textit{by } (3.1)) \\
&\leq C\sum_{i=1}^n E\Big[\|F_i\|_{\mathbf{K}}^p\Big].
\end{aligned}$$

The result is proved. □

¿From Lemma 4, we obtain $\mathbf{K_k}(\mathfrak{X})$ has Rademacher type p property in $\|\cdot\|_{\mathbf{K}}$ sense. Furthermore, if $F_i \ominus E[F_i]$ exist for all $i \geq 1$, by Lemma 4, we can get

$$E\Big[\Big|d_H(\sum_{i=1}^n F_i, \sum_{i=1}^n E[F_i])\Big|^p\Big] \leq C\sum_{i=1}^n E\|F_i\|_{\mathbf{K}}^p. \tag{3.2}$$

Indeed,

$$\begin{aligned}
E\Big[\Big|d_H\Big(\sum_{i=1}^n F_i, \sum_{i=1}^n E[F_i]\Big)\Big|^p\Big] &= E\Big[\Big|d_H\Big(\sum_{i=1}^n((F_i \ominus E[F_i]) + E[F_i]), \sum_{i=1}^n E[F_i]\Big)\Big|^p\Big] \\
&= E\Big[\Big|d_H\Big(\sum_{i=1}^n(F_i \ominus E[F_i]) + \sum_{i=1}^n E[F_i], \sum_{i=1}^n E[F_i]\Big)\Big|^p\Big] \\
&\leq E\Big[\Big|d_H\Big(\sum_{i=1}^n(F_i \ominus E[F_i]), \{0\}\Big)\Big|^p\Big]
\end{aligned}$$

$$\leq C\sum_{i=1}^{n} E[\|F_i \ominus E[F_i]\|_{\mathbf{K}}^p] \ \ (by\ Lemma\ 4)$$
$$\leq C\sum_{i=1}^{n} E[\|F_i\|_{\mathbf{K}}^p],$$

where the first inequality comes from the property $d_H(A+D, B+D) \leq d_H(A,B)$ for $A, B, D \in \mathbf{K}(\mathfrak{X})$, for this property readers may refer to [14].

Lemma 5 Let $\{A_n : n \geq 1\}, \{B_n : n \geq 1\}, \{C_n : n \geq 1\}$ be compact convex subsets such that

$$A_n = B_n + C_n$$

and

$$d_H(A_n, A) \longrightarrow 0, \quad d_H(B_n, A) \longrightarrow 0.$$

Then

$$d_H(C_n, \{0\}) \longrightarrow 0.$$

Proof. Suppose that there exists $C \neq \{0\}$, such that

$$d_H(C_n, C) \longrightarrow 0.$$

Then

$$\begin{aligned} d_H(A_n, A+C) &= d_H(B_n + C_n, A + C) \\ &\leq d_H(B_n, A) + d_H(C_n, C) \longrightarrow 0. \end{aligned}$$

Hence, we have

$$d_H(A + C, A) \leq d_H(A_n, A + C) + d_H(A_n, A) \longrightarrow 0.$$

This means $A + C = A$, i.e. $C = \{0\}$, which contracts with supposition. □

Now we prove the Kronecker Lemma for sets.

Lemma 6 (**Kronecker Lemma**) Let $\{F_n : n \geq 1\}$ be a sequence of compact convex sets, $\{a_n : n \geq 1\}$ be real-valued sequence and $0 < a_n \uparrow \infty$, $\sum_{i=1}^{n} \frac{F_i}{a_i}$ convergent to F in the sense of d_H, then

$$d_H\Big(\frac{1}{a_n}\sum_{i=1}^{n} F_i, \{0\}\Big) \longrightarrow 0.$$

Proof. Let $a_0 = 0$, define $V_1 = \{0\}, V_n = \sum_{i=1}^{n-1} \frac{F_i}{a_i}$ for $n \geq 2, V = \sum_{i=1}^{\infty} \frac{F_i}{a_i}$, then

$$\lim_{n\to\infty} d_H(V_n, V) = 0,$$

furthermore, for any $\varepsilon > 0$, there exists n_0, such that $d_H(V_n, V) < \varepsilon$ for any $n \geq n_0$. Now we prove

$$V_{n+1} = \frac{1}{a_n}\sum_{i=1}^{n}(a_i - a_{i-1})V_i + \frac{1}{a_n}\sum_{i=1}^{n} F_i. \tag{3.3}$$

Indeed,

$$\begin{aligned}
& \frac{1}{a_n}\sum_{i=1}^{n}(a_i-a_{i-1})V_i+\frac{1}{a_n}\sum_{i=1}^{n}F_i \\
= & \frac{1}{a_n}\sum_{i=1}^{n}\Big((a_i-a_{i-1})\sum_{j=1}^{i-1}\frac{F_j}{a_j}\Big)+\frac{1}{a_n}\sum_{i=1}^{n}F_i \\
= & \frac{1}{a_n}\sum_{i=1}^{n}\sum_{j=1}^{i-1}\Big((a_i-a_{i-1})\frac{F_j}{a_j}\Big)+\frac{1}{a_n}\sum_{i=1}^{n}F_i \\
= & \frac{1}{a_n}\sum_{j=1}^{n}\sum_{i=j+1}^{n}\Big((a_i-a_{i-1})\frac{F_j}{a_j}\Big)+\frac{1}{a_n}\sum_{i=1}^{n}F_i \\
= & \frac{1}{a_n}\sum_{j=1}^{n}\frac{F_j}{a_j}\sum_{i=j+1}^{n}(a_i-a_{i-1})+\frac{1}{a_n}\sum_{i=1}^{n}F_i \\
= & \frac{1}{a_n}\sum_{j=1}^{n}\frac{F_j}{a_j}(a_n-a_j)+\frac{1}{a_n}\sum_{i=1}^{n}F_i \\
= & \frac{1}{a_n}\sum_{i=1}^{n}\frac{a_n}{a_i}F_i \\
= & V_{n+1}.
\end{aligned}$$

We also have

$$\begin{aligned}
d_H\Big(\frac{1}{a_n}\sum_{i=1}^{n}(a_i-a_{i-1})V_i,V\Big) &= d_H\Big(\frac{1}{a_n}\sum_{i=1}^{n}(a_i-a_{i-1})V_i,\frac{1}{a_n}\sum_{i=1}^{n}(a_i-a_{i-1})V\Big) \\
&\le \frac{1}{a_n}\sum_{i=1}^{n}(a_i-a_{i-1})d_H(V_i,V) \\
&\le \frac{1}{a_n}\sum_{i=1}^{n_0}(a_i-a_{i-1})d_H(V_i,V)+\frac{a_n+a_{n_0-1}}{a_n}\varepsilon.
\end{aligned}$$

That means

$$d_H\Big(\frac{1}{a_n}\sum_{i=1}^{n}(a_i-a_{i-1})V_i,V\Big)\longrightarrow 0.$$

This with (3.3) and Lemma 5 implies

$$d_H\Big(\frac{1}{a_n}\sum_{i=1}^{n}F_i,\{0\}\Big)\longrightarrow 0.$$

The result is proved. □

The following lemma plays an important role to obtain the main results.

Lemma 7 Let $\{V_n : n\ge 1\}$ be independent $\mathbf{K}_{kc}(\mathfrak{X})$-valued random variables in a real separable Rademacher type $p(1<p\le 2)$ Banach space $\mathfrak{X}$. Assume that $\{V_n : n\ge 1\}$ is stochastically dominated by a set-valued random variable V in the sense that (2.1) holds, and $V_i\ominus E[V_i]$ exist for all $i\ge 1$. Let $\{a_n : n\ge 1\}$ and $\{b_n : n\ge 1\}$ be constants satisfying

$0 < b_n \uparrow \infty$ and

$$\Big(\max_{1\le j\le n}\frac{b_j^p}{|a_j|^p}\Big)\sum_{j=n}^{\infty}\frac{|a_j|^p}{b_j^p}=O(n). \tag{3.4}$$

If

$$\sum_{n=1}^{\infty}P\{\|a_nV\|_{\mathbf{K}}>Db_n\}<\infty, \tag{3.5}$$

then

$$d_H\Big(\frac{1}{b_n}\sum_{j=1}^{n}a_jV_j,\ \frac{1}{b_n}\sum_{j=1}^{n}a_jE[V_jI_{\{\|V_j\|_{\mathbf{K}}\le D^2c_j\}}]\Big)\longrightarrow 0,\ \textit{a.e.}$$

Proof. Let $c_n=\frac{b_n}{|a_n|}, Y_n=V_nI_{\{\|V_n\|_{\mathbf{K}}\le D^2c_n\}}$. Then for $n\ge 1$,

$$\begin{aligned}
&\sup_{m>n}E\Big[\Big|d_H\Big(\sum_{j=1}^{m}\frac{a_j}{b_j}Y_j,\sum_{j=1}^{m}\frac{a_j}{b_j}E[Y_j]\Big)-d_H\Big(\sum_{j=1}^{n}\frac{a_j}{b_j}Y_j,\sum_{j=1}^{n}\frac{a_j}{b_j}E[Y_j]\Big)\Big|^p\Big]\\
&\le\sup_{m>n}E\Big[\Big|d_H\Big(\sum_{j=n+1}^{m}\frac{a_j}{b_j}Y_j,\sum_{j=n+1}^{m}\frac{a_j}{b_j}E[Y_j]\Big)\Big|^p\Big]\\
&\le\sup_{m>n}C\sum_{j=n+1}^{m}\frac{E[\|Y_j\|_{\mathbf{K}}^p]}{c_j^p}\ (\textit{by}\ (3.2))\\
&\longrightarrow 0\ \ (\textit{by Lemma}\ 1),
\end{aligned}$$

thus there exists S such that

$$E\Big[\Big|d_H\Big(\sum_{j=1}^{n}\frac{a_j}{b_j}Y_j,\sum_{j=1}^{n}\frac{a_j}{b_j}E[Y_j]\Big)-S\Big|^p\Big]\longrightarrow 0,$$

then

$$d_H\Big(\sum_{j=1}^{n}\frac{a_j}{b_j}Y_j,\sum_{j=1}^{n}\frac{a_j}{b_j}E[Y_j]\Big)\xrightarrow{P}S.$$

Since $\mathbf{K}_{kc}(\mathfrak{X})$ can be embedded as a closed cone in a real separable Banach space, Y_i can be considered to be an element of this Banach space. On the other hand, convergence in probability and almost every convergence are equivalent for sums of independent random elements (see Itô and Nisio[11]), so we have

$$d_H\Big(\sum_{j=1}^{n}\frac{a_j}{b_j}Y_j,\sum_{j=1}^{n}\frac{a_j}{b_j}E[Y_j]\Big)\longrightarrow S,\quad \textit{a.e.}.$$

Then by Kronecher Lemma(Lemma 6) we have

$$d_H\Big(\frac{1}{b_n}\sum_{j=1}^{n}a_jY_j,\frac{1}{b_n}\sum_{j=1}^{n}a_jE[Y_j]\Big)\longrightarrow 0,\ \textit{a.e.}. \tag{3.6}$$

Since (2.1) and (3.5) ensure that

$$\begin{aligned}\sum_{n=1}^{\infty} P\{V_n \neq Y_n\} &= \sum_{n=1}^{\infty} P\{\|V_n\|_{\mathbf{K}} > D^2 c_n\} \\ &\leq D\sum_{n=1}^{\infty} P\{\|V\|_{\mathbf{K}} > Dc_n\} \\ &< \infty,\end{aligned}$$

by the Borel-Cantelli Lemma we have $P\{\liminf_{n\to\infty}\{V_n = Y_n\}\} = 1$, then $P\{\lim_{n\to\infty}\{V_n = Y_n\}\} = 1$, so $\lim_{n\to\infty} d_H(V_n, Y_n) = 0$, a.e.. Then combining with triangular inequality and (3.6), we have

$$\begin{aligned}& d_H\Big(\frac{1}{b_n}\sum_{j=1}^{n} a_j V_j, \frac{1}{b_n}\sum_{j=1}^{n} a_j E[Y_j]\Big) \\ \leq\ & d_H\Big(\frac{1}{b_n}\sum_{j=1}^{n} a_j V_j, \frac{1}{b_n}\sum_{j=1}^{n} a_j Y_j\Big) + d_H\Big(\frac{1}{b_n}\sum_{j=1}^{n} a_j Y_j, \frac{1}{b_n}\sum_{j=1}^{n} a_j E[Y_j]\Big) \longrightarrow 0,\ a.e..\end{aligned}$$

The result is proved. □

Theorem 1 Let $\{V_n : n \geq 1\}$ be independent $\mathbf{K}_{kc}(\mathfrak{X})$-valued random variables in a real separable, Rademacher type $p(1 < p \leq 2)$ Banach space. Assume that $\{V_n : n \geq 1\}$ is stochastically dominated by set-valued random variable V in the sense that (2.1) holds, and $V_i \ominus E[V_i]$ exist for all $i \geq 1$. Let $\{a_n : n \geq 1\}$ and $\{b_n : n \geq 1\}$ be constants satisfying $0 < b_n \uparrow \infty$, $\frac{b_n}{a_n} \uparrow$,

$$\frac{b_n^p}{|a_n|^p}\sum_{j=n}^{\infty}\frac{|a_j|^p}{b_j^p} = O(n) \tag{3.7}$$

and

$$\frac{b_n}{|a_n|}\sum_{j=1}^{n}\frac{|a_j|}{b_j} = O(n). \tag{3.8}$$

If (3.5) is satisfied, then we obtain the SLLN

$$d_H\Big(\frac{1}{b_n}\sum_{j=1}^{\infty} a_j V_j, \frac{1}{b_n}\sum_{j=1}^{\infty} a_j E[V_j]\Big) \longrightarrow 0,\ a.e..$$

Proof. Let $c_n = \frac{b_n}{|a_n|}, Y_n = V_n I_{\{\|V_n\|_{\mathbf{K}} \leq D^2 c_n\}}$. Note that (3.8) ensures that $c_n \leq Cn, n \geq 1$, and so for all $j \geq 1$, by (2.1) and (3.5).

$$\begin{aligned}\sum_{n=1}^{\infty} P\{\|V_j\|_{\mathbf{K}} > CD^2 n\} &\leq D\sum_{n=1}^{\infty} P\{\|V\|_{\mathbf{K}} > CDn\} \\ &\leq D\sum_{n=1}^{\infty} P\{\|V\|_{\mathbf{K}} > Dc_n\} \\ &< \infty.\end{aligned}$$

By Borel-Cantelli Lemma, we have $P\{\|V_j\|_{\mathbf{K}} > CD^2 n \ i.o.\} = 0$, then $P\{\limsup_{n\to\infty} \|V_j\|_{\mathbf{K}} > CD^2 n\} = 0$. Thus $\|V_j\|_{\mathbf{K}}$ is bounded a.e., which means $E\|V_j\|_{\mathbf{K}} < \infty$.

Since the conditions of Lemma 7 are all satisfied, we have

$$d_H\Big(\frac{1}{b_n}\sum_{j=1}^{n} a_j V_j, \frac{1}{b_n}\sum_{j=1}^{n} a_j E[Y_j]\Big) \longrightarrow 0, \ a.e..$$

Hence it only needs to prove that

$$d_H\Big(\frac{1}{b_n}\sum_{j=1}^{n} a_j E[Y_j], \frac{1}{b_n}\sum_{j=1}^{n} a_j E[V_j]\Big) \longrightarrow 0, \ a.e..$$

Indeed,

$$\begin{aligned}
d_H\Big(\frac{1}{b_n}\sum_{j=1}^{n} a_j E[Y_j], \frac{1}{b_n}\sum_{j=1}^{n} a_j E[V_j]\Big) &\le \frac{1}{b_n}\sum_{j=1}^{n} d_H\Big(a_j E[Y_j], a_j E[V_j]\Big) \\
&= \frac{1}{b_n}\sum_{j=1}^{n} d_H\Big(a_j E[Y_j], a_j E[Y_j] + a_j E[V_j I_{\{\|V_j\|_{\mathbf{K}} > D^2 c_j\}}]\Big) \\
&\le \frac{1}{b_n}\sum_{j=1}^{n} d_H\Big(\{0\}, a_j E[V_j I_{\{\|V_j\|_{\mathbf{K}} > D^2 c_j\}}]\Big) \\
&= \frac{1}{b_n}\sum_{j=1}^{n} |a_j| \|E[V_j I_{\{\|V_j\|_{\mathbf{K}} > D^2 c_j\}}]\|_{\mathbf{K}}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{c_n} E[\|V_n\|_{\mathbf{K}} I_{\{\|V_n\|_{\mathbf{K}} > D^2 c_n\}}] &\le D^2 \sum_{n=1}^{\infty} \frac{1}{c_n} E[\|V\|_{\mathbf{K}} I_{\{\|V\|_{\mathbf{K}} > D c_n\}}] \ (by\ Lemma\ 2) \\
&= D^2 \sum_{n=1}^{\infty} \frac{1}{c_n} \sum_{j=n}^{\infty} E[\|V\|_{\mathbf{K}} I_{\{Dc_j < \|V\|_{\mathbf{K}} \le Dc_{j+1}\}}] \\
&\le D^2 \sum_{j=1}^{\infty} E[\|V\|_{\mathbf{K}} I_{\{Dc_j < \|V\|_{\mathbf{K}} \le Dc_{j+1}\}}] \sum_{n=1}^{j+1} \frac{1}{c_n} \\
&\le D^3 \sum_{j=1}^{\infty} c_{j+1} P\{Dc_j < \|V\|_{\mathbf{K}} \le Dc_{j+1}\} \frac{C(j+1)}{c_{j+1}} \quad (by\ (3.8)) \\
&= D^3 C \sum_{j=1}^{\infty} P\{Dc_j < \|V\|_{\mathbf{K}} \le Dc_{j+1}\}(j+1) \\
&\le C \sum_{j=1}^{\infty} j P\{Dc_j < \|V\|_{\mathbf{K}} \le Dc_{j+1}\} \ \ (by\ j+1 \le 2j) \\
&= C \sum_{j=1}^{\infty} \sum_{n=1}^{j} P\{Dc_j < \|V\|_{\mathbf{K}} \le Dc_{j+1}\} \\
&= C \sum_{n=1}^{\infty} \sum_{j=n}^{\infty} P\{Dc_j < \|V\|_{\mathbf{K}} \le Dc_{j+1}\}
\end{aligned}$$

$$= \quad C\sum_{n=1}^{\infty} P\{\|V\|_{\mathbf{K}} > Dc_n\} < \infty, \ (by\ (3.5))$$

then by Kronecker Lemma we have

$$\frac{1}{b_n}\sum_{j=1}^{n} |a_j| E[\|V_j\|_{\mathbf{K}} I_{\{\|V_j\|_{\mathbf{K}} > D^2 c_j\}}] \longrightarrow 0.$$

Then we have

$$d_H\Big(\frac{1}{b_n}\sum_{j=1}^{n} a_j V_j, \frac{1}{b_n}\sum_{j=1}^{n} a_j E[V_j]\Big) \longrightarrow 0, \ a.e..$$

The result is proved. □

Now we shall introduce the definition of Toeplitz sequence.

Definition 2 A double array $\{a_{nk} : n, k = 1, 2, \cdots\}$ of real numbers is said to be a Toeplitz sequence, if

(i) $\lim_{n\to\infty} a_{nk} = 0$ for each k,

(ii) $\sum_{k=1}^{\infty} |a_{nk}| \leq C$(constant) for each n.

For example, the following Toeplitz sequence

$$a_{nk} = \begin{cases} \frac{1}{n} & k = 1, \cdots, n \\ 0 & k > n \end{cases}$$

is the most simple one and often used. The following lemma is about Toeplitz sequence, we call it Toeplitz Lemma which will be used later.

Lemma 8(Toeplitz Lemma)(cf.[15]) Let $\{a_{nk} : n, k = 1, 2, \cdots\}$ be a Toeplitz sequence and $\{x_n : n \geq 1\}$ be a sequence of real-valued random variables,

(i) If $x_n \to 0$, then $\sum_{k=1}^{n} a_{nk} x_k \to 0$;

(ii) If $x_n \to x$ and $\sum_{k=1}^{n} a_{nk} \to 1$, then $\sum_{k=1}^{n} a_{nk} x_k \to x$.

Theorem 2 Let $\{V_n : n \geq 1\}$ be independent $\mathbf{K}_{kc}(\mathfrak{X})$-valued random variables in a real separable Rademacher type $p(1 < p \leq 2)$ Banach space. Suppose that $\{V_n : n \geq 1\}$ is stochastically dominated by a set-valued random variable V in the sense that (2.1) holds, and suppose that $E[\|V\|_{\mathbf{K}}] < \infty$, $V_i \ominus E[V_i]$ exist for all $i \geq 1$. Let $\{a_n : n \geq 1\}$ and $\{b_n : n \geq 1\}$ be constants satisfying $0 < b_n \uparrow \infty$, (3.4) and

$$\sum_{j=1}^{n} |a_j| = O(b_n). \tag{3.9}$$

If the series of (3.5) converges, then the SLLN is obtained

$$d_H\Big(\frac{1}{b_n}\sum_{j=1}^{n} a_j V_j, \frac{1}{b_n}\sum_{j=1}^{n} a_j E[V_j]\Big) \longrightarrow 0, \ a.e.$$

Proof. Let $c_n = \frac{b_n}{|a_n|}, Y_n = V_n I_{\{\|V_n\|_{\mathbf{K}} \leq D^2 c_n\}}$. Since the conditions of Lemma 7 are all satisfied, we have

$$d_H\Big(\frac{1}{b_n}\sum_{j=1}^n a_jV_j, \frac{1}{b_n}\sum_{j=1}^n a_jE[Y_j]\Big) \longrightarrow 0, \ a.e..$$

So it only needs to prove that

$$d_H\Big(\frac{1}{b_n}\sum_{j=1}^n a_jE[V_j], \frac{1}{b_n}\sum_{j=1}^n a_jE[Y_j]\Big) \longrightarrow 0, \quad a.e.$$

By (3.4) we can have that $c_n \to \infty$. By Lemma 2, $E[\|V\|_{\mathbf{K}}] < \infty$ and the dominated convergence theorem we have

$$\begin{aligned}\|E[V_nI_{\{\|V_n\|_{\mathbf{K}}>D^2c_n\}}]\|_{\mathbf{K}} &\le E[\|V_n\|_{\mathbf{K}}I_{\{\|V_n\|_{\mathbf{K}}>D^2c_n\}}] \\ &\le D^2E[\|V\|_{\mathbf{K}}I_{\{\|V\|_{\mathbf{K}}>Dc_n\}}] \\ &\to 0 \ \ (n\to\infty)\end{aligned}$$

Let $c_{nj} = \frac{a_j}{b_n}$, then for each j, $c_{nj} \to 0$ as $(n\to\infty)$. (3.9) means that $\sum\limits_{j=1}^n c_{nj} = \sum\limits_{j=1}^n \frac{|a_j|}{b_n} = O(1)$, then there exists a constant C such that

$$\sum_{j=1}^\infty c_{nj} = \sum_{j=1}^\infty \frac{|a_j|}{b_n} \le C.$$

Thus $\{c_{nj} : n, j = 1, 2, \cdots\}$ is a Teoplitz sequence. By Toeplitz Lemma we have

$$\begin{aligned}\frac{1}{b_n}\Big\|\sum_{j=1}^n a_jE[V_jI_{\{\|V_j\|_{\mathbf{K}}>D^2c_j\}}]\Big\|_{\mathbf{K}} &\le \frac{1}{b_n}\sum_{j=1}^n |a_j|\Big\|E[V_jI_{\{\|V_j\|_{\mathbf{K}}>D^2c_j\}}]\Big\|_{\mathbf{K}} \\ &\longrightarrow 0 \ (n\to\infty).\end{aligned}$$

Then we have

$$\begin{aligned}&d_H\Big(\frac{1}{b_n}\sum_{j=1}^n a_jE[V_j], \frac{1}{b_n}\sum_{j=1}^n a_jE[Y_j]\Big) \\ &= d_H\Big(\frac{1}{b_n}\sum_{j=1}^n a_jE[Y_j] + \frac{1}{b_n}\sum_{j=1}^n a_jE[V_jI_{\{\|V_j\|_{\mathbf{K}}>D^2c_j\}}], \frac{1}{b_n}\sum_{j=1}^n a_jE[Y_j]\Big) \\ &\le d_H\Big(\frac{1}{b_n}\sum_{j=1}^n a_jE[V_jI_{\{\|V_j\|_{\mathbf{K}}>D^2c_j\}}], \{0\}\Big) \\ &= \frac{1}{b_n}\Big\|\sum_{j=1}^n a_jE[V_jI_{\{\|V_j\|_{\mathbf{K}}>D^2c_j\}}\Big\|_{\mathbf{K}} \\ &\longrightarrow 0, \quad (n\to\infty).\end{aligned}$$

The result is proved. □

Theorem 3 Let $\{V_n : n \ge 1\}$ be independent $\mathbf{K}_{kc}(\mathfrak{X})$-valued random variables in a real separable Rademacher type $p (1 < p \le 2)$ Banach space. Suppose that $V_i \ominus E[V_i]$ exist for all $i \ge 1$ and

$$\sup_{n\ge1} E[\|V_n\|_{\mathbf{K}}^p] < \infty. \tag{3.10}$$

Let $\{a_n : n \geq 1\}$ and $\{b_n : n \geq 1\}$ be constants such that $0 < b_n \uparrow \infty$ and

$$\frac{a_n}{b_n} = O(n^{-1/p}(\log n)^{-1/q}) \quad \textit{for some } 0 < q < p. \tag{3.11}$$

Then the SLLN

$$d_H\Big(\frac{1}{b_n}\sum_{j=1}^n a_j V_j, \frac{1}{b_n}\sum_{j=1}^n a_j E[V_j]\Big) \longrightarrow 0, \quad a.e.. \tag{3.12}$$

Proof. Let $c_n = \frac{b_n}{|a_n|}, Y_n = V_n I_{\{\|V_n\|_{\mathbf{K}} \leq c_n\}}$. Then we have

$$\begin{aligned}\sum_{n=1}^\infty \frac{E[\|Y_n\|_{\mathbf{K}}^p]}{c_n^p} &\leq \sum_{n=1}^\infty \frac{E[\|V_n\|_{\mathbf{K}}^p]}{c_n^p} \\ &\leq \sum_{n=1}^\infty C\frac{1}{c_n^p} \quad (by\ (3.10)) \\ &\leq C\frac{1}{c_1^p} + C\sum_{n=2}^\infty \frac{1}{n(\log n)^{p/q}} \quad (by\ 3.11) \\ &< \infty,\end{aligned}$$

which implies (see the proof of Lemma 7)

$$d_H\Big(\frac{1}{b_n}\sum_{j=1}^n a_j Y_j, \frac{1}{b_n}\sum_{j=1}^n a_j E[Y_j]\Big) \longrightarrow 0, \quad a.e.. \tag{3.13}$$

Since

$$\begin{aligned}\sum_{n=1}^\infty P\{V_n \neq Y_n\} &= \sum_{n=1}^\infty P\{\|V_n\|_{\mathbf{K}} > c_n\} \\ &\leq \sum_{n=1}^\infty \frac{E[\|V_n\|_{\mathbf{K}}^p]}{c_n^p} \quad (by\ Markov\ inequality) \\ &< \infty,\end{aligned}$$

by the Borel-Cantellia Lemma we have

$$P\{\liminf_{n\to\infty}\{V_n = Y_n\}\} = 1,$$

Combine the above equality with (3.13), we can get

$$d_H\Big(\frac{1}{b_n}\sum_{j=1}^n a_j V_j, \frac{1}{b_n}\sum_{j=1}^n a_j E[Y_j]\Big) \longrightarrow 0, \quad a.e..$$

Next,

$$\begin{aligned}&d_H\Big(\sum_{n=1}^\infty \frac{1}{c_n}E[V_n I_{\{\|V_n\|_{\mathbf{K}} > c_n\}}], \{0\}\Big) \\ &= \Big\|\sum_{n=1}^\infty \frac{1}{c_n}E[V_n I_{\{\|V_n\|_{\mathbf{K}} > c_n\}}]\Big\|_{\mathbf{K}}\end{aligned}$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{c_n} E[\|V_n\|_{\mathbf{K}} I_{\{\|V_n\|_{\mathbf{K}} > c_n\}}]$$
$$= \sum_{n=1}^{\infty} P\{\|V_n\|_{\mathbf{K}} > c_n\} + \sum_{n=1}^{\infty} \frac{1}{c_n} \int_{c_n}^{\infty} P\{\|V_n\|_{\mathbf{K}} > t\} dt \quad (by\ Lemma\ 2)$$

$$\leq \sum_{n=1}^{\infty} P\{V_n \neq Y_n\} + \sum_{n=1}^{\infty} \frac{1}{c_n} \int_{c_n}^{\infty} \frac{E[\|V_n\|_{\mathbf{K}}^p]}{t^p} dt \ (by\ Markov\ inequality)$$
$$\leq C + C \sum_{n=1}^{\infty} \frac{1}{c_n} \int_{c_n}^{\infty} \frac{1}{t^p} dt$$
$$\leq C + C \sum_{n=1}^{\infty} \frac{1}{c_n^p}$$
$$< \infty,$$

and so by the Kronecker Lemma

$$d_H\Big(\frac{1}{b_n} \sum_{j=1}^{n} a_j E[V_j I_{\{\|V_j\|_{\mathbf{K}} > c_j\}}], \{0\}\Big) \longrightarrow 0.$$

Then we have

$$d_H\Big(\frac{1}{b_n} \sum_{j=1}^{n} a_j V_j, \frac{1}{b_n} \sum_{j=1}^{n} a_j E[V_j]\Big)$$
$$\leq d_H\Big(\frac{1}{b_n} \sum_{j=1}^{n} a_j V_j, \frac{1}{b_n} \sum_{j=1}^{n} a_j E[Y_j]\Big) + d_H\Big(\frac{1}{b_n} \sum_{j=1}^{n} a_j E[Y_j], \frac{1}{b_n} \sum_{j=1}^{n} a_j E[V_j]\Big)$$
$$= d_H\Big(\frac{1}{b_n} \sum_{j=1}^{n} a_j V_j, \frac{1}{b_n} \sum_{j=1}^{n} a_j E[Y_j]\Big)$$
$$+ d_H\Big(\frac{1}{b_n} \sum_{j=1}^{n} a_j E[Y_j], \frac{1}{b_n} \sum_{j=1}^{n} a_j E[Y_j] + \frac{1}{b_n} \sum_{j=1}^{n} a_j E[V_j I_{\{\|V_j\|_{\mathbf{K}} > c_j\}}]\Big)$$
$$\leq d_H\Big(\frac{1}{b_n} \sum_{j=1}^{n} a_j V_j, \frac{1}{b_n} \sum_{j=1}^{n} a_j E[Y_j]\Big) + d_H\Big(\{0\}, \frac{1}{b_n} \sum_{j=1}^{n} a_j E[V_j I_{\{\|V_j\|_{\mathbf{K}} > c_j\}}]\Big)$$
$$\longrightarrow 0, \quad a.e..$$

The result is proved. □

We can easily extend Lemma 1 in [1] to the case of set-valued random variables.

Lemma 9 Let V_0 and V be set-valued random variables such that V_0 is stochastically dominated by V in the sense that there exists a constant $D < \infty$ such that

$$P\{\|V_0\|_{\mathbf{K}} > t\} \leq DP\{\|DV\|_{\mathbf{K}} > t\}, \quad t \geq 0.$$

Then for all $p > 0$,

$$E[\|V_0\|_{\mathbf{K}}^p I_{\{\|V_0\|_{\mathbf{K}} \leq t\}}] \leq Dt^p P\{\|DV\|_{\mathbf{K}} > t\} + D^{p+1} E[\|V\|_{\mathbf{K}}^p I_{\{\|DV\|_{\mathbf{K}} \leq t\}}], \quad t \geq 0.$$

Then we can have the following theorem.

Theorem 4 Let $\{V_n : n \geq 1\}$ be independent $\mathbf{K}_{kc}(\mathfrak{X})$-valued random variables in a real separable, Rademacher type $p(1 < p \leq 2)$ Banach space. Suppose that $\{V_n : n \geq 1\}$ is stochastically dominated by a set-valued random variable V in the sense that (2.1) holds, and suppose that $E[\|V\|_{\mathbf{K}}^p] < \infty$ for some $1 \leq q < p$, $V_i \ominus E[V_i]$ exist for $i \geq 1$. Let $\{a_n : n \geq 1\}$ and $\{b_n : n \geq 1\}$ be constants satisfying $0 < b_n \uparrow \infty$, (3.9) and

$$\frac{a_n}{b_n} = O(n^{-1/q}). \tag{3.14}$$

Then we have

$$d_H\Big(\frac{1}{b_n}\sum_{j=1}^{n} a_j V_j, \frac{1}{b_n}\sum_{j=1}^{n} a_j E[V_j]\Big) \longrightarrow 0, \quad a.e..$$

Proof. Let $c_n = b_n/|a_n|, Y_n = V_n I_{\{\|V_n\|_{\mathbf{K}} \leq n^{1/q}\}}$. Now

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{E[\|Y_n\|_{\mathbf{K}}^p]}{c_n^p} &\leq D\sum_{n=1}^{\infty}\frac{n^{p/q}}{c_n^p}P\Big\{\|DV\|_{\mathbf{K}} > n^{\frac{1}{q}}\Big\} \\
&\quad + D^{p+1}\sum_{n=1}^{\infty}\frac{1}{c_n^p}E\Big[\|V\|_{\mathbf{K}}^p I_{\{\|DV\|_{\mathbf{K}} \leq n^{1/q}\}}\Big] \quad (by\ Lemma\ 9) \\
&\leq C + C\sum_{n=1}^{\infty} n^{-p/q}\sum_{k=1}^{n} E[\|V\|_{\mathbf{K}}^p] I_{\{(k-1)^{1/q} < \|DV\|_{\mathbf{K}} \leq k^{-1/q}\}} \\
&\quad (by\ (3.14)\ and\ E\|V\|_{\mathbf{K}}^p < \infty) \\
&= C + C\sum_{k=1}^{\infty} E\|V\|_{\mathbf{K}}^p I_{\{(k-1)^{1/q} < \|DV\|_{\mathbf{K}} \leq k^{-1/q}\}}\sum_{n=k}^{\infty} n^{-p/q} \\
&\leq C + C\sum_{k=1}^{\infty} E\Big[\|V\|_{\mathbf{K}}^p I_{\{(k-1)^{1/q} < \|DV\|_{\mathbf{K}} \leq k^{-1/q}\}}\Big] \\
&= C + CE[\|V\|_{\mathbf{K}}^p] < \infty.
\end{aligned}$$

The above inequality means (see the proof of Lemma 7)

$$d_H\Big(\frac{1}{b_n}\sum_{j=1}^{n} a_j Y_j, \frac{1}{b_n}\sum_{j=1}^{n} E[Y_j]\Big) \longrightarrow 0, \quad a.e.. \tag{3.15}$$

Now by (2.1) and $E[\|V\|_{\mathbf{K}}^p] < \infty$ and Markov inequality, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} P\{V_n \neq Y_n\} &= \sum_{n=1}^{\infty} P\{\|V_n\|_{\mathbf{K}} > n^{1/q}\} \\
&\leq D\sum_{n=1}^{\infty} P\{\|\|_{\mathbf{K}} > n^{1/q}\} \\
&\leq D\sum_{n=1}^{\infty}\frac{E[\|DV\|_{\mathbf{K}}^p]}{n^{p/q}} \\
&< \infty,
\end{aligned}$$

so by the Borel-Cantellia Lemma we have

$$P\{\liminf_{n\to\infty}[V_n = Y_n]\} = 1,$$

which combine with (3.15) can get

$$d_H\Big(\frac{1}{b_n}\sum_{j=1}^{n} a_j V_j, \frac{1}{b_n}\sum_{j=1}^{n} a_j E[Y_j]\Big) \longrightarrow 0, \quad a.e..$$

Next, by Lemma 2, $E[\|V\|_{\mathbf{K}}] < \infty$ and the dominated convergence theorem, we have

$$\begin{aligned} E\Big[\|V_n\|_{\mathbf{K}} I_{\{\|V_n\|_{\mathbf{K}} > n^{1/q}\}}\Big] &\leq D^2 E\Big[\|V\|_{\mathbf{K}} I_{\{\|DV\|_{\mathbf{K}} > n^{1/q}\}}\Big] \\ &\longrightarrow 0 \ \ (n \to \infty). \end{aligned}$$

$0 < b_n \uparrow \infty$ and (3.9) means that $\{\frac{a_j}{b_n} : j, n \geq 1\}$ is a Toeplitz sequence, by Toeplitz Lemma we can get

$$\begin{aligned} d_H\Big(\frac{1}{b_n}\sum_{j=1}^{n} a_j E[V_j I_{\{\|V_j\|_{\mathbf{K}} > j^{1/q}\}}], \{0\}\Big) &= \frac{1}{b_n}\Big\|\sum_{j=1}^{n} a_j E[V_j I_{\{\|V_j\|_{\mathbf{K}} > j^{1/q}\}}]\Big\|_{\mathbf{K}} \\ &\leq \frac{1}{b_n}\sum_{j=1}^{n} |a_j| E\Big[\|V_j\|_{\mathbf{K}} I_{\{\|V_j\|_{\mathbf{K}} > j^{1/q}\}}\Big] \\ &\longrightarrow 0 \end{aligned}$$

Then we have

$$\begin{aligned} & d_H\Big(\frac{1}{b_n}\sum_{j=1}^{n} a_j V_j, \frac{1}{b_n}\sum_{j=1}^{n} a_j E[V_j]\Big) \\ & \leq d_H\Big(\frac{1}{b_n}\sum_{j=1}^{n} a_j V_j, \frac{1}{b_n}\sum_{j=1}^{n} a_j E[Y_j]\Big) + d_H\Big(\frac{1}{b_n}\sum_{j=1}^{n} a_j E[Y_j], \frac{1}{b_n}\sum_{j=1}^{n} a_j E[V_j]\Big) \\ & = d_H\Big(\frac{1}{b_n}\sum_{j=1}^{n} a_j V_j, \frac{1}{b_n}\sum_{j=1}^{n} a_j E[Y_j]\Big) + d_H\Big(\frac{1}{b_n}\sum_{j=1}^{n} a_j E[Y_j], \frac{1}{b_n}\sum_{j=1}^{n} a_j E[Y_j] \\ & \quad + \frac{1}{b_n}\sum_{j=1}^{n} a_j E[V_j I_{\{\|V_j\|_{\mathbf{K}} > j^{1/q}\}}]\Big) \\ & \leq d_H\Big(\frac{1}{b_n}\sum_{j=1}^{n} a_j V_j, \frac{1}{b_n}\sum_{j=1}^{n} a_j E[Y_j]\Big) + d_H\Big(\{0\}, \frac{1}{b_n}\sum_{j=1}^{n} a_j E[V_j I_{\{\|V_j\|_{\mathbf{K}} > j^{1/q}\}}]\Big) \\ & \longrightarrow 0, \quad a.e.. \end{aligned}$$

The result is proved. □

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