PUTNAM'S INEQUALITY FOR CLASS A OPERATORS AND AN OPERATOR TRANSFORM BY CHŌ AND YAMAZAKI

S.M. Patel, M. Chō, K. Tanahashi and A. Uchiyama

Received July 17, 2007; revised September 19, 2007

Abstract. Let $T = U|T|$ be the polar decomposition of a bounded linear operator on a complex Hilbert space $\mathcal{H}$. T is called a class A operator if $|T|^2 \le |T^2|$. Recently, M. Chō and T. Yamazaki found an interesting operator transform from a class A operator T to a hyponormal operator $\hat{T}$. In this paper we obtain a more suitable form for $\hat{T}$, and prove Putnam's inequality for a class A operator, i.e., $\| \, |T^2| - |T|^2 \, \| \le \| \, |T(1,1)| - |T(1,1)^*| \, \| \le \frac{1}{\pi}\text{meas } (\sigma(T))$ where $T(1,1) = |T|U|T|$ denotes the generalized Aluthge transform. Also, we study related results.

1 Introduction Let $\mathcal{H}$ be a complex Hilbert space and $B(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. Let $T = U|T|$ be the polar decomposition of $T \in B(\mathcal{H})$. In this paper, we consider following classes of operators.

(1) p-hyponormal : $(TT^*)^p \le (T^*T)^p$, where $p > 0$.
(2) class A : $|T|^2 \le |T^2|$.
(3) class $A(s,t)$: $|T^*|^{2t} \le (|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}}$, where $s, t > 0$.
(4) paranormal : $\|Tx\|^2 \le \|T^2x\|\|x\|$ for $x \in \mathcal{H}$.
(5) normaloid : $\|T\| = r(T)$ (spectral radius of T).

For $p = 1$ and $p = 1/2$, p-hyponormal operator turns out to be hyponormal and semihyponormal, respectively. It is well known that every p-hyponormal operator is a class $A(s,t)$ operator for any $0 < s, t$, a class $A(s,t)$ operator is class $A(s', t')$ if $s \le s', t \le t'$, the class A coincides with $A(1,1)$ and a class A operator is paranormal. (see ([5], [6], [7], [8], [9], [12].)

The well known operator transform $\tilde{T} = |T|^{1/2}U|T|^{1/2}$ introduced by A. Aluthge [1] is now known as the Aluthge transform in the literature. A further extension of $\tilde{T}$ called the generalized Aluthge transform is defined as $T(s,t) = |T|^sU|T|^t$. (We remark class $A(s,t)$ can be definded as $|T|^{2t} \le |T(s,t)|^{\frac{t}{s+t}}$.) These transforms have wide variety of applications for p-hyponormal operators as well as the detailed investigation of several classes of non-hyponormal operators. On the other hand M. Chō and T. Yamazaki [3] revealed some spectral properties of class A operators with the help of the new operator transform as follows:

DEFINITION. Let $T = U|T|$ and $|T||T^*| = W| \, |T||T^*| \, |$ be the polar decompositions of T and $|T||T^*|$. The operator transform $\hat{T}$ of T is defined as $\hat{T} = WU|T^2|^{1/2}$.

This transform $\hat{T}$ may seem complicated, but it is very interesting. By using this transform, M. Chō and T. Yamazaki [3] proved that a class A operator has Bishop's property (β), and also proved the following.

Theorem A [3]. Let T be a class A operator. Then the following assertions hold.
(i) $\hat{T}$ is hyponormal.
(ii) $\sigma(T) = \sigma(\hat{T})$.

2000 *Mathematics Subject Classification.* 47B20.
Key words and phrases. Putnam inequality, class A operator, class $A(s,t)$ operator.

(iii) For $\mu \in \mathbb{C}$ and a sequence $\{x_n\}$ of unit vectors,

$$(T-\mu)x_n \to 0 \Longleftrightarrow (\hat{T}-\mu)x_n \to 0.$$

In Section 2 we prove this transform has a close relation to the generalized Aluthge transform $T(1,1) = |T|U|T|$ and we obtain a more suitable form for $\hat{T}$. Also we consider the normality condition on $\hat{T}$.

Putnam's inequality [13] is famous in operator theory. It means that if T is hyponormal, i.e., $T^*T - TT^* \geq 0$, then

$$\|T^*T - TT^*\| \leq \frac{1}{\pi}\text{meas } (\sigma(T)).$$

This inequality was extended for p-hyponormal operators by M. Chō and M. Itoh [2], and for log-hyponormal operator by Tanahashi [14]. In Section 3 we prove Putnam's inequality for class A operator T, i.e.,

$$\|\ |T^2| - |T|^2\ \| \leq \|\ |T(1,1)| - |T(1,1)^*|\ \| \leq \frac{1}{\pi}\text{meas } (\sigma(T))$$

where $T(1,1) = |T|U|T|$ denotes the generalized Aluthge transform. Also, we prove Putnam's inequality for class $A(s,t)$ operators for $0 < s, t \leq 1$.

Section 4 is mainly concerned with the polar decomposition of the product of two operators. M. Ito, T. Yamazaki and M. Yanagida [10] proved an interesting polar decomposition of the product of two operators and obtained characterizations of binormal operator T (i.e., $|T|$ commutes with $|T^*|$). In this section, we rewrite the polar decomposition of the product of two operators and prove related results.

2 Normality

First we make a simple formula of the operator transform defined by M. Chō and T. Yamazaki [3]. Let $T \in B(\mathcal{H})$ and [ran T] be the closure of the range of T.

Theorem 2.1. *Let $T = U|T|$ and $T(1,1) = |T|U|T| = V|T(1,1)|$ be the polar decompositions of T and $T(1,1)$. Then $\hat{T}$ has the polar decomposition given by $\hat{T} = V|T(1,1)|^{1/2}$.*

Proof. As noted in (2.5) of [3],

$$\hat{T}|T(1,1)|^{1/2} = \hat{T}|T^2|^{1/2} = |T|T = |T|U|T| = V|T(1,1)|.$$

Then $\hat{T}y = V|T(1,1)|^{1/2}y$ for $y \in [\text{ran } |T(1,1)|^{1/2}]$. On the other hand, $\hat{T}x = WU|T(1,1)|^{1/2}x = 0$ and $V|T(1,1)|^{1/2}x = 0$ for $x \in \ker |T(1,1)|^{1/2}$. This implies $\hat{T} = V|T(1,1)|^{1/2}$ and

$$\ker \hat{T} = \ker |T(1,1)|^{1/2} = \ker |T(1,1)| = \ker V.$$

This completes the proof. □

In [11], the first author established that a p-hyponormal operator is normal if its Aluthge transform is normal. Various extensions of this result can be found in [14], [15]; among the recent ones, we quote the following from [12].

Theorem B [12]. Let T be a class $A(s,t)$ operator. Then the following assertions hold.
(i) T is quasinormal $\Longleftrightarrow T(s,t)$ is quasinormal.
(ii) T is normal $\Longleftrightarrow T(s,t)$ is normal.

We consider the analogous situations for $\hat{T}$.

Theorem 2.2. *Let T be a class A operator. Then the following assertions hold.*
(i) *T is quasinormal $\Longleftrightarrow$ $\hat{T}$ is quasinormal.*
(ii) *T is normal $\Longleftrightarrow$ $\hat{T}$ is normal.*

Proof. (i) ($\Longrightarrow$) Let T be quasinormal. Then $T(1,1) = |T|U|T| = U|T|^2$. Hence $\hat{T} = V|T(1,1)|^{1/2} = U|T| = T$.

($\Longleftarrow$) Let $\hat{T} = V|T(1,1)|^{1/2}$ be quasinormal. Then V commutes with $|T(1,1)|^{1/2}$. Hence V commutes with $|T(1,1)|$ and this implies $T(1,1)$ is quasinormal. Thus T is quasinormal by Theorem B.

(ii) ($\Longrightarrow$) (i) implies $T = \hat{T}$.

($\Longleftarrow$) T is quasinormal by (i). Since $\hat{T}$ is normal,

$$|T(1,1)| = V|T(1,1)|V^* = |T(1,1)^*|.$$

$T(1,1)$ is quasinormal by Theorem A and $\ker T(1,1) = \ker T(1,1)^*$. Hence $T(1,1)$ is normal and this implies T is normal by Theorem B. □

We know that an operator T is quasinormal if and only if $T = \tilde{T}$. This prompts the corresponding result for $\hat{T}$.

Theorem 2.3. *$T = \hat{T}$ if and only if T is quasinormal.*

Proof. ($\Longleftarrow$) $T = \hat{T}$ by Theorem 2.2.

($\Longrightarrow$) By the assumption, $V|T(1,1)|^{1/2} = U|T|$. Then the uniqueness of the polar decomposition shows that $V = U$ and $|T(1,1)|^{1/2} = |T|$. Since

$$|T|V|T(1,1)|^{1/2} = |T|U|T| = V|T(1,1)|,$$

we have $|T|Vy = V|T(1,1)|^{1/2}y$ for $y \in [\text{ran } |T(1,1)|^{1/2}]$. On the otherhand $|T|Vx = 0$ and $V|T(1,1)|^{1/2}x = 0$ for $x \in \ker |T(1,1)|^{1/2} = \ker V$. Hence $|T|V = V|T(1,1)|^{1/2} = V|T|$. □

M. Fujii, S. Izumino and R. Nakamoto [4] showed the following characterization of normaloid operators via the Aluthge transform $\tilde{T} = |T|^{1/2}U|T|^{1/2}$ as follows:

Theorem C [4]. An operator T is normaloid if and only if $\tilde{T}$ is normaloid and $\|T\| = \|\tilde{T}\|$.

The following theorem shows that analogus result holds for $\hat{T}$.

Theorem 2.4. *An operator T is normaloid if and only if $\hat{T}$ is normaloid and $\|T\| = \|\hat{T}\|$.*

Proof. Suppose T is normaloid. Let $T = U|T|$ and $T(1,1) = |T|U|T| = V|\ |T|U|T|\ |$ be the polar decompositions of T and $T(1,1)$. Then $\|T\| = r$ for some $re^{i\theta} \in \sigma(T)$. Select a sequence $\{x_n\}$ of unit vectors such that $(|T| - r)x_n \to 0$ and $(U - e^{i\theta})x_n \to 0$. Then $(U - e^{i\theta})^*x_n \to 0$. Hence $(T(1,1) - r^2e^{i\theta})x_n = (|T|U|T| - r^2e^{i\theta})x_n \to 0$ and $(T(1,1)^* - r^2e^{-i\theta})x_n \to 0$, and so $(|T(1,1)| - r^2)x_n \to 0$ and $(V - e^{i\theta})x_n \to 0$. Hence $re^{i\theta} \in \sigma(\hat{T})$. In particular, $\|T\| \leq r(\hat{T}) \leq \|\hat{T}\|$. Since

$$\begin{aligned}\|\hat{T}\| &= \|V|T(1,1)|^{1/2}\| \leq \|\ |\ |T|U|T|\ |^{\frac{1}{2}}\| \\ &\leq \|\ |T|\ \| = \|T\|,\end{aligned}$$

it follows that $r(\hat{T}) = \|\hat{T}\| = \|T\|$.

Now assume the converse. Choose $re^{i\theta} \in \sigma(\hat{T})$ such that $r = \|\hat{T}\| = \|T\|$. Then there exists a sequence $\{x_n\}$ of unit vectors for which $(|T(1,1)|^{1/2} - r)x_n \to 0$ and $(V - e^{i\theta})x_n \to 0$.

This clearly implies $(T(1,1) - r^2e^{i\theta})x_n \to 0$. Therefore $r^2e^{i\theta} \in \sigma(U|T|^2)$ and hence $r^2 \leq \|U|T|^2\| \leq \|T\|^2 = \|\hat{T}\|^2 = r^2$. This shows that $U|T|^2$ is normaloid. Choose a sequence $\{y_n\}$ of unit vectors such that $(U - e^{i\theta})y_n \to 0$ and $(|T|^2 - r^2)y_n \to 0$. Consequently $(T - re^{i\theta})y_n \to 0$ and so $re^{i\theta} \in \sigma(T)$. Since $\|T\| = r$, we conclude that T is normaloid. □

Theorem 2.5. *Let T be a class $A(s,t)$ operator with $\ker T \subset \ker T^*$. If $\hat{T}$ is normal, then so is T.*

Proof. Assume $s \geq t$. The normality of $\hat{T}$ implies that $|\,|T|U|T|\,|$ commutes with V and $\ker V = \ker V^*$. Since $|T|U|T| = V|\,|T|U|T|\,|$, it is clear that $|T|U|T|$ is normal or $|T|U^*|T|^2U|T| = |T|U|T|^2U^*|T|$. Then $UU^*|T|^2UU^* = U^2|T|^2U^{*2}$ and hence $U^*|T|^2U = U^*U^2|T|^2U^{*2}U$. Since $\ker U = \ker T \subset \ker T^* = \ker U^*$, we have $[\text{ran } UU^*] \subset [\text{ran } U^*U]$. Hence $U^*UUU^* = UU^*$, and so $U^*U^2 = U$. Therefore

$$U^*|T|^2U = U|T|^2U^*. \tag{2.1}$$

If $U^*x = 0$, then (2.1) gives $|T|Ux = 0$ and $U^2x = 0$. Hence $Ux \in \ker U \subset \ker U^*$. This implies $U^*Ux =$ and so $Ux = 0$. Thus we see $\ker U = \ker U^*$. Hence U is normal. This fact along with (2.1) shows that $U^*|T|^{2s}U = U|T|^{2s}U^*$ and therefore $T(s,s) = |T|^sU|T|^s$ is normal. Since $s \geq t$, T is a class $A(s,s)$ operator. Now the normality of T follows from Theorem B. □

Theorem 2.6. *Let T be a class $A(s,t)$ operator. If $\hat{T}$ is selfadjoint, then so is T.*

Proof. Assume $s \geq t$. Then T is both a class $A(s,s)$ operator and a class $A(2s,2s)$ operator. Equivalently, both $U|T|^s$ and $U|T|^{2s}$ are class A operators. Let $z = re^{i\theta}$ be a non-zero complex number in Bdry $\sigma(U|T|^{2s})$. Then there exists a sequence $\{x_n\}$ of unit vectors such that $(U|T|^{2s} - z)x_n \to 0$. Since $U|T|^{2s}$ is a class A operator, $(U|T|^{2s} - z)^*x_n \to 0$ by [15]. Therefore $(U - e^{i\theta})x_n \to 0$ and $(|T|^{2s} - r)x_n \to 0$. In turn, this implies $(|T|U|T| - r^{1/s}e^{i\theta})x_n \to 0$. Since $\hat{T}$ is selfadjoint, $|T|U|T| = V|\,|T|U|T|\,| = V|T(1,1)|$ is quasinormal. Then $(|T(1,1)| - r^{1/s})x_n \to 0$ and $(V - e^{i\theta})x_n \to 0$ imply $(\hat{T} - r^{1/2s}e^{i\theta})x_n \to 0$. Since $\hat{T}$ is selfadjoint, $r^{1/2s}e^{i\theta}$ and hence z is real. Thus $\sigma(T(s,s)) = \sigma(U|T|^{2s}) \subset \mathbb{R}$. Since T is a class $A(s,s)$ operator, it follows that $T(s,s)$ is selfadjoint, and T is normal by Theorem B. Consequently, $T = \hat{T}$ by Theorem 2.3. □

Corollary 2.7. *Let T be a class $A(s,t)$ operator with real spectrum. Then T is self-adjoint.*

Proof. Let $s \geq t$. Then the operator $S = U|T|^s$ is of class A. Let $z = re^{i\theta}$ be a non-zero complex number in the boundary of $\sigma(S)$. Then there exists a sequence $\{x_n\}$ of unit vectors such that $(|T|^s - r)x_n \to 0$ and $(U - e^{i\theta})x_n \to 0$ and therefore $(T - r^{1/s}e^{i\theta})x_n \to 0$. Since $\sigma(T) \subset \mathbb{R}$, $r^{1/s}e^{i\theta}$ and hence $re^{i\theta}$ is real. Thus $\sigma(S) \subset \mathbb{R}$. Since $\hat{S}$ is hyponormal and $\sigma(\hat{S}) = \sigma(S) \subset \mathbb{R}$ by Theorem A, it follows that $\hat{S}$ is selfadjoint by [2]. Then Theorem 2.6 implies that S is self-adjoint. Thus T is self-adjoint. □

3 Putnam's inequality In this section, we extend Putnam's inequality for class A and $A(s,t)$ operators for $0 < s,t \leq 1$. The following result due to M. Ito and T. Yamazaki [9] is essential.

Theorem D [9]. Let $0 \leq A, B \in B(\mathcal{H})$ and $0 < p, r \in \mathbb{R}$. Then

$$B^r \leq (B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}}$$

implies

$$A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}.$$

Theorem 3.1. *Let T be a class $A(s,t)$ operator for $0 < s,t \leq 1$ and $T(s,t) = |T|^sU|T|^t$. Then*

$$\begin{aligned} \| \, |T(s,t)|^{\frac{2\min\{s,t\}}{s+t}} - |T|^{2\min\{s,t\}} \, \| &\leq \| \, |T(s,t)|^{\frac{2\min\{s,t\}}{s+t}} - |T(s,t)^*|^{\frac{2\min\{s,t\}}{s+t}} \, \| \\ &\leq \frac{\min\{s,t\}}{\pi} \iint_{\sigma(T)} r^{2\min\{s,t\}-1} dr d\theta. \end{aligned}$$

Moreover, if meas $(\sigma(T)) = 0$*, then T is normal.*

Proof. Assume $0 < t \leq s$. Since T is class $A(s,t)$, we have

$$|T^*|^{2t} \leq \left(|T^*|^t|T|^{2s}|T^*|^t\right)^{\frac{t}{s+t}}$$

and

$$|T|^{2s} \geq \left(|T|^s|T^*|^{2t}|T|^s\right)^{\frac{s}{s+t}} = |T(s,t)^*|^{\frac{2s}{s+t}}$$

by Theorem D. Also, we have

$$\begin{aligned} U|T|^{2t}U^* &\leq \left(U|T|^tU^*|T|^{2s}U|T|^tU^*\right)^{\frac{t}{s+t}} \\ &= U\left(|T|^tU^*|T|^{2s}U|T|^t\right)^{\frac{t}{s+t}} U^* \end{aligned}$$

and

$$|T|^{2t} \leq \left(|T|^tU^*|T|^{2s}U|T|^t\right)^{\frac{t}{s+t}} = |T(s,t)|^{\frac{2t}{s+t}}.$$

Hence

$$|T(s,t)^*|^{\frac{2t}{s+t}} \leq |T|^{2t} \leq |T(s,t)|^{\frac{2t}{s+t}}$$

by Löwner-Heinz's inequality. This implies that $T(s,t)$ is $\frac{t}{s+t}$-hyponormal. Hence

$$\begin{aligned} \| \, |T(s,t)|^{\frac{2t}{s+t}} - |T|^{2t} \, \| &\leq \| \, |T(s,t)|^{\frac{2t}{s+t}} - |T(s,t)^*|^{\frac{2t}{s+t}} \, \| \\ &\leq \frac{t}{\pi(s+t)} \iint_{\sigma(T(s,t))} \rho^{\frac{2t}{s+t}-1} d\rho d\theta \end{aligned}$$

by [2]. Since $0 < s,t \leq 1$, T is class A. Hence we have

$$\sigma(T(s,t)) = \{r^{s+t}e^{i\theta} | re^{i\theta} \in \sigma(T) \}$$

by [16, Theorem 5]. By taking $r^{s+t}e^{i\theta} = \rho e^{i\theta} \in \sigma(T(s,t))$, we have

$$\frac{t}{\pi(s+t)} \iint_{\sigma(T(s,t))} \rho^{\frac{2t}{s+t}-1} d\rho d\theta = \frac{t}{\pi} \iint_{\sigma(T)} r^{2t-1} dr d\theta.$$

The proof of the case $0 < s \leq t$ is similar.

If meas $(\sigma(T)) = 0$, then $|T(s,t)| = |T(s,t)^*|$. Hence $T(s,t)$ is normal. Thus T is normal by Theorem B. $\square$

Corollary 3.2. *Let T be a class A operator. Then*

$$\| \, |T^2| - |T|^2 \, \| \leq \| \, |T(1,1)| - |T(1,1)^*| \, \| \leq \frac{1}{\pi} \text{meas } (\sigma(T)).$$

Moreover, if meas $(\sigma(T)) = 0$*, then T is normal.*

Proof. Since T is class $A(1,1)$, we have

$$\begin{aligned}|T|^2 \leq |T^2| &= (T^*T^*TT)^{1/2}\\ &= \left(|T|U^*|T|^2U|T|\right)^{1/2} = |T(1,1)|.\end{aligned}$$

Hence

$$\begin{aligned}\| \, |T^2| - |T|^2 \, \| &\leq \| \, |T(1,1)| - |T(1,1)^*| \, \| \\ &\leq \frac{1}{\pi}\iint_{\sigma(T)} r dr d\theta = \frac{1}{\pi}\text{meas }(\sigma(T)).\end{aligned}$$

by Theorem 3.1. □

Remark. It seems an interesting problem whether Putnam's inequality holds for the case $1 < s$ or $1 < t$. If $0 < s, t \leq 1$, a class $A(s,t)$ operator $T = U|T|$ is class A, so $(T-\lambda)x_n \to 0(\lambda \neq 0, \|x_n\| = 1)$ implies $(T-\lambda)^* x_n \to 0$ by [16, Lemma 2]. Hence we have $\sigma(T(s,t)) = \sigma(U|T|^{s+t}) = \{r^{s+t}e^{i\theta} : re^{i\theta} \in \sigma(T)\}$ by [16, Theorem 5]. However it is an open problem that a class $A(s,t)$ operator T with $1 < s$ or $1 < t$ has such property.

4 Polar decomposition

M. Ito, T. Yamazaki and M. Yanagida [10] obtained an interesting polar decomposition of the product of two operators as follows:

Theorem E [10]. Let $T = U|T|, S = V|S|$ and $|T||S^*| = W \mid |T||S^*| \mid$ be the polar decompositions. Then $TS = UWV|TS|$ is also the polar decomposition.

In this section we consider another form of the polar decomposition of the products of two operators and prove related results.

Theorem 4.1. *Let $T = U|T|, S = V|S|$ and $|T|V|S| = W| \; |T|V|S| \;|$ be the polar decompositions of T, S and $|T|V|S|$, respectively. Then $|T||S^*| = WV^* \mid |T||S^*| \mid$ is the polar decomposition of $|T||S^*|$.*

Proof. Since

$$|T||S^*| = |T|V|S|V^* = W| \; |T|V|S| \; |V^*,$$

we have

$$\begin{aligned}(|T||S^*|)^*(|T||S^*|) &= V \mid |T|V|S| \mid W^*W \mid |T|V|S| \mid V^*\\ &= V \mid |T|V|S| \mid^2 V^*.\end{aligned}$$

Hence $\mid |T||S^*| \mid = V \mid |T|V|S| \mid V^*$ and

$$\begin{aligned}|T||S^*| &= W \mid |T|V|S| \mid V^*\\ &= WV^*V \mid |T|V|S| \mid V^* = WV^* \mid |T||S^*| \mid.\end{aligned}$$

[Claim 1] WV^* is a partial isometry.

Let $Vx = 0$. Then $|S|x = 0$. Hence $|T|V|S|x = 0$ and $Wx = 0$. Then $\ker V^*V = \ker V \subset \ker W = \ker W^*W$ and [ran W^*W] $\subset$ [ran V^*V]. Hence $V^*VW^*W = W^*W$. Therefore $WV^*(WV^*)^*WV^* = W(V^*VW^*W)V^* = WW^*WV^* = WV^*$.

[Claim 2] $\ker WV^* = \ker \mid |T||S^*| \mid$.

$$\begin{aligned} x \in \ker WV^* &\iff V^*x \in \ker W = \ker|\,|T|V|S|\,| \\ &\iff |\,|T||S^*|\,|x = V|\,|T|V|S|\,|V^*x = 0 \\ &\iff x \in \ker|\,|T||S^*|\,|. \end{aligned}$$

□

Theorem 4.2. *Let $T = U|T|, S = V|S|$ and $|T|V|S| = W|\,|T|V|S|\,|$ be the polar decompositions of T, S and $|T|V|S|$, respectively. Then $TS = UW|TS|$ is the polar decomposition of TS.*

Proof. It is clear that $TS = U|T|V|S| = UW|TS|$.

[Claim 1] UW is a partial isometry.

Let $Ux = 0$. Then $|S|V^*|T|x = 0$ and $W^*x = 0$. Hence [ran WW^*] $\subset$ [ran U^*U] and $U^*UWW^* = WW^*$. Then

$$UW(UW)^*UW = UWW^*U^*UW = UWW^*W = UW.$$

[Claim 2] $\ker UW = \ker|TS|$.

Let $x \in \ker UW$. Then $Wx \in \ker|T|$ and $(|T|V|S|)^*Wx = 0$. Hence $W^*Wx = 0$ and $Wx = 0$. Then $|T|V|S|x = 0$ and $TSx = 0$. Hence $|TS|x = 0$. Conversely, if $|TS|x = 0$, then $TSx = 0$. Hence $|T|V|S|x = 0$ and $Wx = 0$. Thus $x \in \ker UW$. □

Theorem 4.3. *Let T be a class A operator and $T = U|T|$ and $|T||T^*| = W|\,|T||T^*|\,|$ be the polar decompositions of T and $|T||T^*|$. Then UW is a quasinormal partial isometry.*

Proof. As shown in the proof of Theorem 1.2 of [7], $T(1,1) = |T|U|T|$ is semi-hyponormal with the polar decomposition $U^*UWU|\,|T|U|T|\,|$. Then $\ker|T|U|T| \subset \ker|T|U^*|T|$ or $\ker U^*UWU \subset \ker U^*W^*U^*U$. Let $U^*UWUx = 0$. Then $U^*W^*U^*Ux = 0$ and so $UU^*W^*U^*Ux = 0$. Since $\ker UU^* \subset \ker|\,|T||T^*|\,| = \ker W$, $WW^*U^*Ux = 0$ and so $W^*U^*Ux = 0$. Then $0 = |T^*||T|U^*Ux = |T^*||T|x$ and $W^*x = 0$. Thus $\ker U^*UWU \subset \ker W^*$ or [ran (WW^*)] $\subset$ [ran $U^*W^*U^*UWU$]. Then $U^*W^*U^*UWUWW^* = WW^*$ and so $U^*W^*U^*UWUW = W$. Since $\ker UU^* \subset \ker W^*W$, $UU^*W^*W = W^*W$ and so $UU^*W^* = W^*$. Hence

$$\begin{aligned} UW &= UU^*W^*U^*UWUW \\ &= W^*U^*UWUW = (UW)^*(UW)^2. \end{aligned}$$

If $Ux = 0$, then $|T^*||T|x = 0$ or $W^*x = 0$. Therefore [ran WW^*] $\subset$ [ran U^*U]. This implies $U^*UWW^* = WW^*$ and so $U^*UW = W$. Hence $UW(UW)^*UW = UWW^*U^*UW = UWW^*W = UW$. This completes the proof. □

Theorem 4.4. *Let $T = U|T|$ be binormal, i.e., $|T|$ commutes with $|T^*|$, and* $\ker T = \ker T(s,t)$. *Then $T(s,t)$ has the polar decomposition given by $T(s,t) = U|T(s,t)|$.*

Proof. Since T is binormal, an application of Theorem 2.3 of [10] shows that $T(s,t) = U^*U^2|T(s,t)|$ is the polar decomposition of $T(s,t)$. Consequently, U^*U^2 is a partial isometry and so U^2 turns out to be a partial isometry. If $U^2x = 0$ then $U^*U^2x = 0$ and therefore $T(s,t)x = 0$. That $Ux = 0$ follows from the kernel condition. Hence $\ker U^2 = \ker U$. Thus we have two projections U^*U and $U^{*2}U^2$ acting on the same range space. This means $U^*U = U^{*2}U^2$. Since U is a contraction, we see that for all $x \in \mathcal{H}$,

$$\begin{aligned}\|U^*U^2x - Ux\|^2 &= \langle U^*U^2x, U^*U^2x\rangle - 2\langle U^2x, U^2x\rangle + \langle Ux, Ux\rangle \\ &\leq \|Ux\|^2 - \|U^2x\|^2 = 0.\end{aligned}$$

Hence $U^*U^2 = U$. Therefore $T(s,t) = U|T(s,t)|$. Since $\ker T = \ker |T(s,t)|$, the result follows. □

Theorem 4.5. *Let $T = U|T|$ and $|T|U|T| = V|\ |T|U|T|\ |$ be the polar decompositions of T and $|T|U|T|$, respectively. Then $(T^*)\hat{}$ has the polar decomposition given by $(T^*)\hat{} = UV^*U^*|\ |T^*|U^*|T^*|\ |^{1/2}$.*

Proof. In view of Theorem 2.1, it is enough to show that the operator $|T^*|U^*|T^*|$ has the polar decomposition given by $|T^*|U^*|T^*| = UV^*U^*|\ |T^*|U^*|T^*|\ |$. Now

$$\begin{aligned}|T^*|U^*|T^*| &= U|T|U^*|T|U^* = U(|\ |T|U|T|\ |V^*)U^* \\ &= UV^*(V|\ |T|U|T|\ |V^*)U^* = UV^*|\ |T|U^*|T|\ |U^* \\ &= UV^*U^*[U(|T|U|T|^2U^*|T|)^{1/2}U^*] \\ &= UV^*U^*[U|T|U|T|^2U^*|T|U^*]^{1/2} \\ &= UV^*U^*|T^{*2}| = UV^*U^*|\ |T^*|U^*|T^*|\ |.\end{aligned}$$

Next we show that UV^*U^* is a partial isometry. Since $\ker U \subset \ker V$ and $\ker U \subset \ker V^*$, one can show that $U^*UV = V$ and $U^*UV^* = V^*$. Therefore $(UV^*U^*)(UV^*U^*)^*(UV^*U^*) = UV^*VV^*U^* = UV^*U^*$. This means that UV^*U^* is a partial isometry. Finally,

$$\begin{aligned}UV^*U^*x = 0 &\iff U^*UV^*U^*x = 0 \\ &\iff V^*U^*x = 0 \\ &\iff |T|U^*|T|U^*x = 0 \\ &\iff |T^{*2}|^2x = 0.\end{aligned}$$

Thus we have shown that $\ker UV^*U^* = \ker |T^{*2}|$, which finishes the proof. □

Acknowledgement. The authors would like to express their sincere thanks to the referee for many helpful comments.

References

[1] A. Aluthge, *On p-hyponormal operators for* $0 < p < 1$, Integr. Equat. Oper. Th., **13**(1990), 307–315.

[2] M. Chō and M. Itoh, *Putnam's inequality for p-hyponormal operators,* Proc. Amer. Math. Soc., **123** (1995) 2435–2440.

[3] M. Chō and T. Yamazaki, *An operator transform from Class A to the class of hyponormal operators and its applications,* Integr. Equat. Oper. Th., **53** (2005) 497–508.

[4] M. Fujii, S. Izumino and R. Nakamoto, *Class of operators determined by the Heinz-Kato-Furuta inequality and Holder-MaCarthy inequality,* Nihonkai Math. J., **5** (1994) 61–67.

[5] M. Fujii, D. Jung, S.H. Lee, M.Y. Lee and R. Nakamoto, *Some classes of operators related to paranormal and log-hyponormal operators,* Math. Japonica, **51** (2000) 395–402.

[6] T. Furuta, *On the class of paranormal operators,* Proc. Japan Acad., **43** (1967) 594–598.

[7] M. Ito, *Some classes of operators associated with generalized Aluthge transformation*, SUT J. Mathematics, **1**(1999), 149–165.

[8] M. Ito, *Several properties on class A including p-hyponormal and log-hyponormal operators*, Math. Inequalities and Appl., **2**(1999), 569–578.

[9] M. Ito and T. Yamazaki, *Relations between two operator inequalities* $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$ *and* $A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}$ *and their applications,* Integr. Equat. Oper. Th., **44**(2002), 442–450.

[10] M. Ito, T. Yamazaki, and M. Yanagida, *On the polar decomposition of the product of two operators and its applications,* Integr. Equat. Oper. Th., **49**(2004), 461–472.

[11] S. M. Patel, *A note on p-hyponormal operators for* $0 < p < 1$, Integral Equations and Operator Theory, **21**(1995), 498–503.

[12] S.M. Patel, K. Tanahashi, A. Uchiyama and M. Yanagida, *Quasi-normality and Fuglede-Putnam theorem for* $A(s,t)$ *operators*, Nihonkai Math. J., **17**(2006), 49–67.

[13] C. R. Putnam, *An inequality for the area of hyponormal spectra*, Math. Z., **116**(1970), 323–330.

[14] K. Tanahashi, *Putnam's inequality for log-hyponormal operators,* Integr. Equat. Oper. Th., **48**(2004), 103–114.

[15] A. Uchiyama, *Weyl's theorem for class A operators,* Mathematical Inequalities and Applications, **1**(2001), 143–140.

[16] A. Uchiyama, K. Tanahashi and J.I. Lee, *Spectrum of class* $A(s,t)$ *operators,* Acta Sci. Math. (Szeged), **70**(2004), 279–287.

Department of Mathematics, Tohoku Pharmaceutical University, Sendai 981-8558 Japan

E-mail ; tanahasi@tohoku-pharm.ac.jp