EVERY NONEMPTY OPEN SET OF THE DIGITAL N-SPACE IS EXPRESSIBLE AS THE UNION OF FINITELY MANY NONEMPTY REGULAR OPEN SETS *

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Abstract. In this paper, it is proved that every nonempty open set of the digital n-space $(\mathbf{Z}^n, \kappa^n)$ is expressible as the union of finitely many nonempty regular open sets.

1 Introduction and a main theorem The aim of this paper is to prove the following property on the digital n-space (cf.[8]):

Theorem 1.1 *Every nonempty open set of the digital n-space $(\mathbf{Z}^n, \kappa^n)$ is expressible as the union of finitely many nonempty regular open sets.*

For $n = 1$ (resp. $n = 2$), the digital n-space is called as the *digital line*, so called the *Khalimsky line*, (resp. the *digital plane*), cf. [10], [12], [11], [6], [14], [9], [8]. The digital line is the set of the integers, $\mathbf{Z}$, equipped with the topology κ having $\{\{2m-1, 2m, 2m+1\} | m \in \mathbf{Z}\}$ as a subbase. This is denoted by $(\mathbf{Z}, \kappa)$. Thus, a set U is open in $(\mathbf{Z}, \kappa)$ if and only if whenever $x \in U$ is an even integer, then $x-1, x+1 \in U$. For an odd integer x, the singleton $\{x\}$ is open; for an even integer x, the singleton $\{x\}$ is closed in $(\mathbf{Z}, \kappa)$. It is observed that the digital line is not T_1, but all non-closed singletons are open. In 1977, Dunham [7] proved that every topological space where all non-closed singletons are open, is a $T_{1/2}$-space. Such $T_{1/2}$-spaces arises from a different angle ,i.e. the investigation of the generalized closed sets. The initiation of the generalized closed sets was done by Levine [13]; a subset A in a topological space is called generalized closed if the closure of A contains the every open super set of A. He studied their most fundamental properties. The spaces in which the concepts of the generalized closed sets and closed sets coincides are called $T_{1/2}$-spaces. Ganster and Dontchev [4] introduced a new separation axiom $T_{3/4}$, as the class of topological spaces where every δ-generalized closed sets is δ-closed [16]. They showed that the class of $T_{3/4}$-spaces is properly placed between the classes of $T_{1/2}$- and T_1-spaces and also that the digital line is a $T_{3/4}$-spaces but not T_1. The digital plane $(\mathbf{Z}^2, \kappa^2)$ is the topological product of two same digital lines $(\mathbf{Z}, \kappa)$, that is, $\kappa^2 = \kappa \times \kappa$. The digital plane is not $T_{1/2}$. More topological properties in the digital line and the digital plane are investigated (cf.[8], [9], [2], [14], [6], [3]). The digital line and the digital plane have the following property: every open set is expressible as the union of finitely many regular open sets [8, Theorems A, C]. In the present paper, we prove the same property above for the digital n-space (Theorem 1.1) and corollaries.

In Section 3, we construct regular open sets induced from a given open set of the digital n-space (cf. Theorem 3.2). The theorem is proved using lemmas in Section 2. In Section 4, Theorem 1.1 is proved; the proof shows an explicit construction of finitely many regular

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open sets in the digital n-space. The digital 2-space is a topological model of of a computer screen and the points are the pixels.

2 Lemmas for digital n-spaces In this section we construct regular open sets induced from a given open set of the digital n-space (cf.Theorem 3.2 in Section 3 below). We prepare some notations and definitions (Definition 2.1, Definition 2.4). We prove lemmas needed later (i.e.,Lemma 2.2, Lemma 2.3, Lemma 2.5).

We recall first the following: for the digital line $(\mathbf{Z}, \kappa)$, $U(2y) := \{2y-1, 2y, 2y+1\}$ is the smallest open set containing $2y$ and $U(2y+1) := \{2y+1\}$ is the smallest open set containing $2y+1$, where $y \in \mathbf{Z}$.

Let $n \geq 1$ be an integer. The *digital n-space* is the topological product of n-copies of the digital line $(\mathbf{Z}, \kappa)$ and this topological space is denoted by $(\mathbf{Z}^n, \kappa^n)$ (eg.[12, Definition 4], [8]). Let $x = (x_1, x_2, ..., x_n)$ be a point of $(\mathbf{Z}^n, \kappa^n)$. For the point x, the following subset $U^n(x) := \prod_{i=1}^n U(x_i)$ is called the smallest open neighbourhood of x in $(\mathbf{Z}^n, \kappa^n)$, where $U(x_i)$ is the smallest open neighbourhood of x_i in the i-th component space $(\mathbf{Z}, \kappa)$ of the digital n-space $(\mathbf{Z}^n, \kappa^n)$. It is shown that, for any open set V containing a point x of $(\mathbf{Z}^n, \kappa^n), x \in U^n(x) \subset V$ hold; moreover, if W is any open set containing a point x such that $W \subset U^n(x)$, then $W = U^n(x)$ holds. Note that for a point $y \in \mathbf{Z}^n$, $y = (2a_1, 2a_2, \ldots, 2a_n)$ if and only if $\{y\}$ is closed in $(\mathbf{Z}^n, \kappa^n)$, where $a_i \in \mathbf{Z}(1 \leq i \leq n)$. For a closed singleton $\{y\}$, where $y = (2a_1, 2a_2, ..., 2a_n)$, we have that $U^n(y) = \prod_{i=1}^n U(2a_i) = \prod_{i=1}^n \{2a_i - 1, 2a_i, 2a_i + 1\}$. Moreover, for a point $z \in \mathbf{Z}^n, z = (2b_1+1, 2b_2+1, \ldots, 2b_n+1)$ if and only if $\{z\}$ is open in $(\mathbf{Z}^n, \kappa^n)$, where $b_i \in \mathbf{Z}(1 \leq i \leq n)$. For an open singleton $\{z\}$, where $z = (2b_1+1, 2b_2+1, \ldots, 2b_n+1)$, $U^n(z) = \prod_{i=1}^n U(2b_i+1) = \prod_{i=1}^n \{2b_i+1\} = \{z\}$.

We introduce the following definition:

Definition 2.1 (i) (cf.[8, Section 6]) In $(\mathbf{Z}^n, \kappa^n)$, we define the following sets $(\mathbf{Z}^n)_{\kappa^n}$ and $(\mathbf{Z}^n)_{\mathcal{F}^n}$:

$(\mathbf{Z}^n)_{\kappa^n} := \{x \in \mathbf{Z}^n | \{x\}$ is open in $(\mathbf{Z}^n, \kappa^n)\}$;

$(\mathbf{Z}^n)_{\mathcal{F}^n} := \{x \in \mathbf{Z}^n | \{x\}$ is closed in $(\mathbf{Z}^n, \kappa^n)\}$.

For a subset A of $(\mathbf{Z}^n, \kappa^n)$, define the subsets A_{κ^n} and $A_{\mathcal{F}^n}$ by $A_{\kappa^n} := A \cap (\mathbf{Z}^n)_{\kappa^n}$ and $A_{\mathcal{F}^n} := A \cap (\mathbf{Z}^n)_{\mathcal{F}^n}$, respectively.

(ii) For a subset A of $(\mathbf{Z}^n, \kappa^n)$ and an integer a with $0 \leq a \leq n$, in general we define the following sets $A_{mix(a)}$:

$A_{mix(0)} := \{(z_1, z_2, \ldots, z_n) \in A |\ z_i$ is odd for each i with $1 \leq i \leq n\}$;

$A_{mix(a)} := \{(z_1, z_2, \ldots, z_n) \in A |\ \#\{i | z_i$ is even $(1 \leq i \leq n)\} = a\}$, where $1 \leq a \leq n$ and $\#B$ is the cardinality of a finite set B.

It is shown that $A_{mix(0)} = A_{\kappa^n} = \{(2y_1+1, 2y_2+1, \ldots, 2y_n+1) \in A | y_i \in \mathbf{Z}(1 \leq i \leq n)\} = A \cap (\mathbf{Z}^n)_{mix(0)}$ and $A_{mix(n)} = A_{\mathcal{F}^n} = \{(2y_1, 2y_2, \ldots, 2y_n) \in A | y_i \in \mathbf{Z}(1 \leq i \leq n)\} = A \cap (\mathbf{Z}^n)_{\mathcal{F}^n}$ for any subset A of $(\mathbf{Z}^n, \kappa^n)$. Sometimes, $A_{mix(a)}$ is denoted by $(A)_{mix(a)}$.

In order to prove Theorem 3.2 we prepare the following Lemma 2.2–Lemma 2.5.

Lemma 2.2 *Let $a \in \mathbf{Z}$ with $0 \leq a \leq n$. If $y \in (U^n(z))_{mix(a)}$ for a point $z \in (\mathbf{Z}^n)_{mix(a)}$, then $y = z$.*

Proof. Suppose that $y \in (U^n(z))_{mix(a)}$, where $z \in (\mathbf{Z}^n)_{mix(a)}$. Let $z = (z_1, z_2, \ldots, z_n)$ and $y = (y_1, y_2, \ldots, y_n)$. We prove that $y = z$ considering the following three cases:

Case 1. $a = 0$: For $a = 0, z \in (\mathbf{Z}^n)_{\kappa^n}$ and so $\{z\}$ is open, i.e., $U^n(z) = \{z\}$. Then, we have that $(U^n(z))_{mix(0)} = \{z\}$, and so $y = z$. **Case 2.** $a = n$: For this case, $U^n(z) = \prod_{i=1}^n U(z_i) = \prod_{i=1}^n \{z_i - 1, z_i, z_i + 1\}$ because $z_i (1 \leq i \leq n)$ are even. Then, we have that $(U^n(z))_{mix(n)} = (U^n(z))_{\mathcal{F}^n} = \{z\}$ and so $y = z$. **Case 3**. $1 \le a \le n-1$: Since $a = \#\ \{i|z_i \text{ is even } (1 \le i \le n)\}$ and $z \in (\mathbf{Z}^n)_{mix(a)}$, we can set $\{i|z_i \text{ is even } (1 \le i \le n)\} = \{e(1), e(2), \ldots, e(a)\}$ and $\{j|z_j \text{ is odd } (1 \le j \le n)\} = \{o(1), o(2), \ldots, o(b)\}$, where $b = n - a$ and $e(s), o(t) \in \{i \in \mathbf{Z}|1 \le i \le n\}(1 \le s \le a, 1 \le t \le b)$. Then, we have that $z_{o(1)}, z_{o(2)}, \ldots, z_{o(b)}$ are odd and $z_{e(1)}, z_{e(2)}, \ldots, z_{e(a)}$ are even. It is noted that, in $(\mathbf{Z}, \kappa)$, $U(z_{e(s)}) = \{z_{e(s)} - 1, z_{e(s)}, z_{e(s)} + 1\}$ holds for each integer s with $1 \le s \le a$, because $z_{e(s)}$ is even. Moreover, we have that $U(z_{o(t)}) = \{z_{o(t)}\}$ holds for each integer t with $1 \le t \le b$, because $z_{o(t)}$ is odd. Since $y \in U^n(z) = \prod_{i=1}^n U(z_i)$ and $y \in (\mathbf{Z}^n)_{mix(a)}$, we have that, for each $o(t)$-th component of z (i.e., $z_{o(t)}(1 \le t \le b)), z_{o(t)} = y_{o(t)}$ holds. It is shown that y_j is even for any interger $j \notin \{o(1), o(2), \ldots, o(b)\}$, because $y \in (\mathbf{Z}^n)_{mix(a)}$. Then, we have that, for such component y_j of y, y_j is expressible as $y_j = z_j, z_j - 1$ or $z_j + 1$, where $j \notin \{o(1), o(2), \ldots, o(b)\}$. Since $j \in \{e(1), e(2), \ldots, e(a)\}$, we have that $z_j - 1$ and $z_j + 1$ are odd and so $y_j = z_j$. Thus, we conclude that $y_{e(s)} = z_{e(s)}$ for each integer s with $1 \le s \le a$.

Therefore, we have that $y = z$ for all cases. □

Lemma 2.3 *Let* $x = (x_1, x_2, \ldots, x_n) \in (\mathbf{Z}^n)_{mix(a')}$ *and* $y = (y_1, y_2, \ldots, y_n) \in (\mathbf{Z}^n)_{mix(a)}$, *where* a' *and* a *are integers such that* $a' \le a, 1 \le a' \le n$ *and* $1 \le a \le n$. *Suppose that* $U^n(x) \cap U^n(y)$ *contains exactly the* $2^{a'}$ *open singletons, say* $\{q^{(1)}\}, \{q^{(2)}\}, \ldots, \{q^{(2^{a'})}\}$. *Then the following properties hold.*

(i) $\{q^{(1)}, q^{(2)}, \ldots, q^{(2^{a'})}\} = (U^n(x))_{\kappa^n} = (U^n(x) \cap U^n(y))_{\kappa^n} \subset (U^n(y))_{\kappa^n}$.

(ii) $\{i|x_i$ *is even* $(1 \le i \le n)\} \subset \{i|y_i$ *is even* $(1 \le i \le n)\}$.

(ii)' *If* $a' = a$ *especially, then* $\{i|x_i$ *is even* $(1 \le i \le n)\} = \{i|y_i$ *is even* $(1 \le i \le n)\}$.

(iii) $x \in U^n(y)$ *holds.*

(iii)' *If* $a' = a$ *especially, then* $x = y$.

Proof . **(i)** We note that there exist exactly 2^a (resp. $2^{a'}$) open singletons, say $\{q^{''(i)}\}(1 \le i \le 2^a)$(resp. $\{q^{'(i)}\}(1 \le i \le 2^{a'})$) in $U^n(y)$ (resp. $U^n(x)$). Using assumptions, we may choice the open singletons such that $q^{(i)} = q^{'(i)} = q^{''(i)} \in U^n(x) \cap U^n(y)$ for each integer i with $1 \le i \le 2^{a'}$. Thus, we show that $\{q^{(1)}, q^{(2)}, \ldots, q^{(2^{a'})}\} = (U^n(x))_{\kappa^n} = (U^n(x) \cap U^n(y))_{\kappa^n}$. It is shown that, in general, $A_{\kappa^n} \subset B_{\kappa^n}$ holds if $A \subset B$. Then, we have that $(U^n(x) \cap U^n(y))_{\kappa^n} \subset (U^n(y))_{\kappa^n}$ holds.

(ii) First assume that $a = n$, i.e., $\{j|y_j \text{ is even } (1 \le j \le n)\} = \{s \in \mathbf{Z}|1 \le s \le n\}$. Then, the proof of (ii) is obvious. We consider the case of $a < n$. Let $\{i|x_i \text{ is even } (1 \le i \le n)\} = \{e'(1), e'(2), \ldots, e'(a')\}, \{j|x_j \text{ is odd } (1 \le j \le n)\} = \{o'(1), o'(2), \ldots, o'(n-a')\}, \{i|y_i \text{ is even } (1 \le i \le n)\} = \{e(1), e(2), \ldots, e(a)\}$ and $\{j|y_j \text{ is odd } (1 \le j \le n)\} = \{o(1), o(2), \ldots, o(n-a)\}$. In above, we assume that $e'(t_1) < e'(t_2), o'(t_1) < o'(t_2), e(t_1) < e(t_2)$ and $o(t_1) < o(t_2)$ for integers t_1, t_2 with $t_1 < t_2$. We denote $q^{(\alpha)} = (2m_1^{(\alpha)} + 1, 2m_2^{(\alpha)} + 1, \ldots, 2m_n^{(\alpha)} + 1)$ for each integer α with $1 \le \alpha \le 2^{a'}$ where $m_i^{(\alpha)} \in \mathbf{Z}(1 \le i \le n)$. We proceed the proof of (ii). Let $d \in \{i|x_i \text{ is even } (1 \le i \le n)\} = \{e'(1), e'(2), \ldots, e'(a')\}$. We claim that y_d is even. For the integer d, there exists a unique integer k' such that $d = e'(k')$, where $1 \le k' \le a'$, and so $x_d = x_{e'(k')}$ is even. We note that $U^n(x) = \prod_{i=1}^n U(x_i), U(x_{e'(s)}) = \{x_{e'(s)}-1, x_{e'(s)}, x_{e'(s)}+1\}$ and $U(x_{o'(t)}) = \{x_{o'(t)}\}$ hold, where $1 \le s \le a'$ and $1 \le t \le n-a'$. Since $q^{(\alpha)} \in (U^n(x))_{mix(0)} = (U^n(x))_{\kappa^n}$ for each integer α with $1 \le \alpha \le 2^{a'}$, we have that for the d-th component of $q^{(\alpha)}(1 \le \alpha \le 2^{a'})$,

$(*)$ $2m_d^{(\theta)} + 1 = x_{e'(k')} - 1$ and $2m_d^{(\theta')} + 1 = x_{e'(k')} + 1$ for some distinct integers $\theta, \theta' \in \{1, 2, \ldots, 2^{a'}\}$.

Now, we suppose that y_d is odd. We note that $U^n(y) = \prod_{i=1}^n U(y_i)$,$U(y_{e(s)}) = \{y_{e(s)} - 1, y_{e(s)}, y_{e(s)} + 1\}$ and $U(y_{o(t)}) = \{y_{o(t)}\}$ hold, where $1 \le s \le a$ and $1 \le t \le n-a$. Then, for the odd integer y_d there exists a unique integer $y_{o(k)}(1 \leq k \leq n-a)$ such that $y_d = y_{o(k)}$. By considering the d-th component of $q^{(\alpha)} \in (U^n(y))_{\kappa^n}$, we have that

$(**)$ $2m_d^{(\alpha)} + 1 = y_{o(k)} = y_d$ for any integer α with $1 \leq \alpha \leq 2^{a'}$.

Using $(*)$ and $(**)$, we have that $2m_d^{(\theta)}+1 = x_{e'(k')}-1 = y_d$ and $2m_d^{(\theta')}+1 = x_{e'(k')}+1 = y_d$, and hence $-1 = 1$ holds in $\mathbf{Z}$. This is a contradiction. Therefore, we conclude that y_d is even and so $\{i|x_i \text{ is even } (1 \leq i \leq n)\} \subset \{i|y_i \text{ is even } (1 \leq i \leq n)\}$.

(ii)' Since $a \leq a'$ and $a' \leq a$, (ii)' is proved by using (ii).

(iii) Let x_d (resp. y_d) be the d-th component of x (resp. y). We claim that $x_d \in U(y_d)$ for each integer d with $1 \leq d \leq n$. We recall that $q^{(\alpha)} \in (U^n(x) \cap U^n(y))_{\kappa^n}$ and $q^{(\alpha)} = (2m_1^{(\alpha)} + 1, 2m_2^{(\alpha)} + 1, \ldots, 2m_n^{(\alpha)} + 1)$, for each integer α with $1 \leq \alpha \leq 2^{a'}$.

Case 1. x_d and y_d are both odd: Since the point $q^{(1)} \in (U^n(x))_{\kappa^n} = (\prod_{i=1}^n U(x_i))_{\kappa^n}$ and $U(x_d) = \{x_d\}$, we have that $x_d = 2m_d^{(1)} + 1$. Since $q^{(1)} \in (U^n(y))_{\kappa^n}$, we have that $y_d = 2m_d^{(1)} + 1$. Thus we have that $x_d = y_d$,i.e., $x_d \in U(y_d) = \{y_d\}$. **Case 2**. x_d is odd and y_d is even: Since $q^{(1)} \in (U^n(x) \cap U^n(y))_{\kappa^n}$, we have $x_d = 2m_d^{(1)} + 1$ and $2m_d^{(1)} + 1 \in \{y_d + 1, y_d - 1\}$. Thus we have that $x_d = y_d + 1$ or $x_d = y_d - 1$, i.e., $x_d \in U(y_d) = \{y_d - 1, y_d, y_d + 1\}$. **Case 3**. x_d is even: Using (ii), it is obtained that y_d is also even. Since $q^{(\alpha)} \in (U^n(x) \cap U^n(y))_{\kappa^n}$ for each integer α with $1 \leq \alpha \leq 2^{a'}$, we have that $2m_d^{(1)} + 1 \in \{y_d + 1, y_d - 1\}$ and also $2m_d^{(1)} + 1 \in \{x_d + 1, x_d - 1\}$. If $x_d + 1 = 2m_d^{(1)} + 1 = y_d + 1$, then $x_d = y_d$. If $x_d - 1 = 2m_d^{(1)} + 1 = y_d - 1$, then $x_d = y_d$. If $x_d + 1 = 2m_d^{(1)} + 1 = y_d - 1$ holds,then there is a contradiction. Indeed, $x_d < 2m_d^{(1)} + 1 < y_d$ and $x_d - 1$ is odd. We use the following notation: let $\{e'(1), e'(2), \ldots, e'(a')\} := \{i|x_i$ is even $(1 \leq i \leq n)\}$ and $\{o'(1), o'(2), \ldots, o'(n-a')\} := \{j|x_j$ is odd $(1 \leq j \leq n)\}$ (if $a' < n$). Define the following point $w = (w_1, w_2, \ldots, w_n)$ by $w_{e'(i)} := x_{e'(i)} - 1$ for each integer i with $1 \leq i \leq a'$ and $w_{o'(j)} := x_{o'(j)}$ for each integer j with $1 \leq j \leq n - a'$ (if $a' < n$). Then, we have that $w \in (U^n(x))_{\kappa^n}; w \notin (U^n(y))_{\kappa^n}$ because $w_d = x_d - 1 \notin \{y_d + 1, y_d - 1\}$ and $U(y_d) = \{y_d - 1, y_d, y_d + 1\}$. This contradicts (i). Similarly, the following case, $x_d - 1 = 2m_d^{(1)} + 1 = y_d + 1$, does not occur. Indeed, take the point $w = (w_1, w_2, \ldots, w_n)$, where $w_{e'(i)} := x_{e'(i)} + 1$ for each integer i with $1 \leq i \leq a'$ and $w_{o'(j)} := x_{o'(j)}$ for each integer j with $1 \leq j \leq n-a'$ (if $a' < n$). Then, it is shown that $w \in (U^n(x))_{\kappa^n}; w \notin (U^n(y))_{\kappa^n}$. This contradicts (i). Thus, for this case , we can show that $x_d \in U(y_d) = \{y_d - 1, y_d, y_d + 1\}$.

Therefore, it follows from Case 1 through Case 3 that $x_d \in U(y_d)$ for each integer d with $1 \leq d \leq n$, that is, $x \in U^n(y)$ holds. **(iii)'** Let x_d (resp. y_d) be the d-th component of x (resp. y). By (ii)', Case 2 in the proof of (iii) does not occur. Therefore, it is shown that $x_d = y_d$ holds for any integer d with $1 \leq d \leq n$ and hence $x = y$. □

Definition 2.4 For an n-tuple $(k_1, k_2, \ldots, k_n)$ of integers $k_i \in$\{0,1,2,3,4,5\} $(1 \leq i \leq n)$, we define the following set $U(k_1, k_2, \ldots, k_n)$:

$U(k_1, k_2, \ldots, k_n) := \bigcup\{U^n((k_1 + 6m_1, k_2 + 6m_2, \ldots, k_n + 6m_n))|m_1, m_2, \ldots, m_n \in \mathbf{Z}\}$, where $U^n((k_1+6m_1,k_2+6m_2, \ldots, k_n+6m_n))$ is the smallest open neighbourhood of a point $(k_1 + 6m_1, k_2 + 6m_2, \ldots, k_n + 6m_n)$ in $(\mathbf{Z}^n, \kappa^n)$.

Lemma 2.5 (i) *Assume that there exist two distinct points $q^{(1)}$ and $q^{(2)}$ satisfying the following property:*

$(*)$ $q^{(1)}, q^{(2)} \in (U^n(x))_{\kappa^n}$ *for a point* $x \in (\mathbf{Z}^n)_{mix(a')}$,*where* $1 \leq a' \leq n$.

Then, $Q_i^{(1)} - Q_i^{(2)} \in \{0, 2, -2\}$ *for each integer i with* $1 \leq i \leq n$, *where $Q_i^{(\alpha)}$ is the i-th component of the point* $q^{(\alpha)}(\alpha = 1, 2)$.

(ii) *Let* $(k_1, k_2, \ldots, k_n) \in (\mathbf{Z}^n)_{mix(a)}$*, where* $k_i \in \{0,1,2,3,4,5\}(1 \le i \le n)$ *and* $a \in \mathbf{Z}$ *with* $1 \le a \le n$*. If there exist points* $y^{(1)}, y^{(2)} \in (U(k_1, k_2, \ldots, k_n))_{mix(a)}$ *such that* $q^{(\alpha)} \in U^n(y^{(\alpha)})$ *for each integer* α*=1,2, where* $q^{(1)}$ *and* $q^{(2)}$ *satisfy the assumption* $(*)$ *of* (i) *above, then* $y^{(1)} = y^{(2)}$ *holds.*

(iii) *Let* $(k_1, k_2, \ldots, k_n) \in (\mathbf{Z}^n)_{\mathcal{F}^n}$*, where* $k_i \in \{0,1,2,3,4,5\}(1 \le i \le n)$*. If there exist points* $y^{(1)}, y^{(2)} \in (U(k_1, k_2, \ldots, k_n))_{\mathcal{F}^n}$ *and a point* $q \in (U^n(y^{(1)}) \cap U^n(y^{(2)}))_{\kappa^n}$*, then* $y^{(1)} = y^{(2)}$*.*

Proof. **(i)** Let $x = (x_1, x_2, \ldots, x_n)$ and $q^{(\alpha)} = (Q_1^{(\alpha)}, Q_2^{(\alpha)}, \ldots, Q_n^{(\alpha)})$ for $\alpha = 1,2$, where $x_i \in \mathbf{Z}$ and $Q_i^{(\alpha)} \in \mathbf{Z}(1 \le i \le n)$. From assumption, we can set that, for $a' < n$, $\{i | x_i$ is even $(1 \le i \le n)\} = \{v(1), v(2), \ldots, v(a')\}(v(1) < v(2) < \ldots < v(a'))$ and $\{i | x_i$ is odd $(1 \le i \le n)\} = \{v'(1), v'(2), \ldots, v'(n-a')\}(v'(1) < v'(2) < \ldots < v'(n-a'))$. Then, $Q_{v(s)}^{(\alpha)} \in \{x_{v(s)}+1, x_{v(s)}-1\}(1 \le s \le a')$ and $Q_{v'(t)}^{(\alpha)} = x_{v'(t)}(1 \le t \le n-a')$, because $q^{(\alpha)} \in (U^n(x))_{\kappa^n}$ and $x \in (\mathbf{Z}^n)_{mix(a')}$. Thus, considering the differences of any i-components of $q^{(1)}$ and $q^{(2)}$, we have that $Q_i^{(1)} - Q_i^{(2)} \in \{0, 2, -2\}(1 \le i \le n)$ for $a' < n$. For the case of $a' = n$, $\{i | x_i$ is even $(1 \le i \le n)\} = \{i \in \mathbf{Z} | 1 \le i \le n\}$ and $\{i | x_i$ is odd $(1 \le i \le n)\} = \emptyset$ hold. Then, we have that $Q_i^{(\alpha)} \in \{x_i - 1, x_i + 1\}(1 \le i \le n)$, because $U^n(x) = \prod_{i=1}^n \{x_i - 1, x_i, x_i + 1\}(1 \le i \le n)$ and $q^{(\alpha)} \in (U^n(x))_{\kappa^n}$. Thus, we have also that $Q_i^{(1)} - Q_i^{(2)} \in \{0, 2, -2\}(1 \le i \le n)$ for $a' = n$.

(ii) We prove that $y^{(1)} = y^{(2)}$ considering the following two cases. There exist points $(k_1 + 6m_1^{(\alpha)}, k_2 + 6m_2^{(\alpha)}, \ldots, k_n + 6m_n^{(\alpha)}) \in (\mathbf{Z}^n)_{mix(a)}$, where $m_j^{(\alpha)} \in \mathbf{Z}(1 \le j \le n)$ and $\alpha = 1,2$, such that $y^{(\alpha)} \in (U^n((k_1 + 6m_1^{(\alpha)}, k_2 + 6m_2^{(\alpha)}, \ldots, k_n + 6m_n^{(\alpha)})))_{mix(a)}$. Using Lemma 2.2, we conclude that $y^{(\alpha)} = (k_1 + 6m_1^{(\alpha)}, k_2 + 6m_2^{(\alpha)}, \ldots, k_n + 6m_n^{(\alpha)})$ for each integer $\alpha = 1,2$.

Case 1. $a < n$: Since $y^{(\alpha)} \in (\mathbf{Z}^n)_{mix(a)}$, we can assume that $\{i | k_i$ is even $(1 \le i \le n)\} = \{e(1), e(2), \ldots, e(a)\}$ $(e(1) < e(2) < \ldots < e(a))$ and $\{j | k_j$ is odd $(1 \le j \le n)\} = \{o(1), o(2), \ldots, o(b)\}(o(1) < o(2) < \ldots < o(b))$, where $e(i), o(j) \in \mathbf{Z}(1 \le i \le a, 1 \le j \le b)$ and b is a positive integer with $a + b = n$. Then, we recall that $U^n(y^{(\alpha)}) = \prod_{i=1}^n U(k_i + 6m_i^{(\alpha)})$, $U(k_{e(s)} + 6m_{e(s)}^{(\alpha)}) = \{k_{e(s)} - 1 + 6m_{e(s)}^{(\alpha)}, k_{e(s)} + 6m_{e(s)}^{(\alpha)}, k_{e(s)} + 6m_{e(s)}^{(\alpha)} + 1\}$ and $U(k_{o(t)} + 6m_{o(t)}^{(\alpha)}) = \{k_{o(t)} + 6m_{o(t)}^{(\alpha)}\}$ hold for each integers s and t with $1 \le s \le a$ and $1 \le t \le b$. Since $q^{(\alpha)} \in (U^n(y^{(\alpha)}))_{\kappa^n}$ by assumptions of (ii) and $y^{(\alpha)} \in (\mathbf{Z}^n)_{mix(a)}$, we have

(1) $Q_{o(t)}^{(\alpha)} = k_{o(t)} + 6m_{o(t)}^{(\alpha)}$ for each integers t, α with $1 \le t \le b$ and $\alpha \in \{1,2\}$;

(2) $Q_{e(s)}^{(\alpha)} \in \{k_{e(s)} + 1 + 6m_{e(s)}^{(\alpha)}, k_{e(s)} - 1 + 6m_{e(s)}^{(\alpha)}\}$ for each integers s, α with $1 \le s \le a$ and $\alpha \in \{1,2\}$.

Then, for the values of $Q_{o(t)}^{(1)} - Q_{o(t)}^{(2)}(1 \le t \le b)$, using (1) and (i), we have the following possible two cases:

(1.1) $k_{o(t)} + 6m_{o(t)}^{(1)} - (k_{o(t)} + 6m_{o(t)}^{(2)}) = 0$;

(1.2) $k_{o(t)} + 6m_{o(t)}^{(1)} - (k_{o(t)} + 6m_{o(t)}^{(2)}) \in \{2, -2\}$.

If there exists an $o(t)$-th component satisfying (1.2), then $0 \equiv 2 \bmod 6$ or $0 \equiv -2 \bmod 6$ hold. Thus, this case does not occur. By (1.1), it is shown that $m_{o(t)}^{(1)} = m_{o(t)}^{(2)}$ for each integer t with $1 \le t \le b$.

For the values of $Q_{e(s)}^{(1)} - Q_{e(s)}^{(2)}(1 \le s \le a)$, using (2) and (i), we have the following possible six cases:

(2.1) $k_{e(s)} + 1 + 6m_{e(s)}^{(1)} - (k_{e(s)} + 1 + 6m_{e(s)}^{(2)}) = 0$;

(2.2) $k_{e(s)}+1+6m_{e(s)}^{(1)}-(k_{e(s)}+1+6m_{e(s)}^{(2)})\in\{2,-2\}$;
(2.3) $k_{e(s)}+1+6m_{e(s)}^{(1)}-(k_{e(s)}-1+6m_{e(s)}^{(2)})\in\{0,-2\}$;
(2.4) $k_{e(s)}+1+6m_{e(s)}^{(1)}-(k_{e(s)}-1+6m_{e(s)}^{(2)})=2$;
(2.5) $k_{e(s)}-1+6m_{e(s)}^{(1)}-(k_{e(s)}-1+6m_{e(s)}^{(2)})=0$;
(2.6) $k_{e(s)}-1+6m_{e(s)}^{(1)}-(k_{e(s)}-1+6m_{e(s)}^{(2)})\in\{2,-2\}$.

If there exists an $e(s)$-th component satisfying (2.2), (2.3) or (2.6), then $0\equiv 2 \bmod 6$, $0\equiv -2 \bmod 6$ or $0\equiv 4 \bmod 6$. Thus, these cases don't occur. By (2.1), (2.4) and (2.5), it is shown that $m_{e(s)}^{(1)}=m_{e(s)}^{(2)}$ for each integer s with $1\le s\le a$. Therefore, we conclude that $y^{(1)}=y^{(2)}$ for Case 1 ($a<n$).

Case 2. $a=n$: For this case, $\{i|k_i \text{ is even } (1\le i\le n)\}=\{i\in \mathbf{Z}|1\le i\le n\}$ and $\{i|k_i \text{ is odd } (1\le i\le n)\}=\emptyset$ hold and so $U^n(y^{(\alpha)})=\prod_{i=1}^{n}\{k_i+6m_i^{(\alpha)}-1, k_i+6m_i^{(\alpha)}, k_i+6m_i^{(\alpha)}+1\}$ holds. Since $q^{(\alpha)}\in(U^n(y^{(\alpha)}))_{\kappa^n}$, we have that

(2)' $Q_i^{(\alpha)}\in\{k_i+1+6m_i^{(\alpha)},\ k_i-1+6m_i^{(\alpha)}\}(1\le i\le n)$ for each integer $\alpha\in\{1,2\}$.

Using (2)' and (i), we have the possible six cases on the values of $Q_i^{(1)}-Q_i^{(2)}(1\le i\le n)$ (cf. (2.1)–(2.6) in Case 1 above). Similarly, we conclude that $y^{(1)}=y^{(2)}$ for Case 2 ($a=n$).

(iii) Let $y^{(1)}=(y_1^{(1)},y_2^{(1)},\dots,y_n^{(1)})$, $y^{(2)}=(y_1^{(2)},y_2^{(2)},\dots,y_n^{(2)})$ and $q=(q_1,q_2,\dots,q_n)$. Since $(k_1,k_2,\dots,k_n)\in(\mathbf{Z}^n)_{\mathcal{F}^n}$, the integers $k_i(1\le i\le n)$ are even and so $(k_1+6m_1,k_2+6m_2,\dots,k_n+6m_n)\in(\mathbf{Z}^n)_{\mathcal{F}^n}$ for any integers $m_i(1\le i\le n)$. Using Lemma 2.2 for $a=n$, we have that $y_i^{(1)}=k_i+6m_i^{(1)}$ and $y_i^{(2)}=k_i+6m_i^{(2)}$ for some integers $m_i^{(1)}$ and $m_i^{(2)}$ with $1\le i\le n$. Then $U^n(y^{(1)})=\prod_{i=1}^{n}\{k_i-1+6m_i^{(1)},k_i+6m_i^{(1)},k_i+1+6m_i^{(1)}\}$ and $U^n(y^{(2)})=\prod_{i=1}^{n}\{k_i-1+6m_i^{(2)},k_i+6m_i^{(2)},k_i+1+6m_i^{(2)}\}$. Thus we have that $q_i\in\{k_i-1+6m_i^{(1)},k_i+1+6m_i^{(1)}\}\cap\{k_i-1+6m_i^{(2)},k_i+1+6m_i^{(2)}\}(1\le i\le n)$. We have the following possible four cases for the integer q_i. If there exists an integer i with $1\le i\le n$ such that $q_i=k_i-1+6m_i^{(1)}=k_i+1+6m_i^{(2)}$ or $q_i=k_i+1+6m_i^{(1)}=k_i-1+6m_i^{(2)}$, then $0\equiv 2 \bmod 6$ or $0\equiv -2 \bmod 6$. Thus, these cases don't occur. We have the following cases: $q_i=k_i-1+6m_i^{(1)}=k_i-1+6m_i^{(2)}$ or $q_i=k_i+1+6m_i^{(1)}=k_i+1+6m_i^{(2)}$. Therefore, we prove that $m_i^{(1)}=m_i^{(2)}$ for each i with $1\le i\le n$ and so $y^{(1)}=y^{(2)}$. □

For a subset E of a topological space (X,τ), the intersection of all open sets of (X,τ) containing E is called the *kernel of* E and it is denoted by $Ker(E)$. Namely $Ker(E):=\bigcap\{U|E\subset U, U\in\tau\}$.

Lemma 2.6 (i) *For every point x of $(\mathbf{Z}^n,\kappa^n)$, $Ker(\{x\})=U^n(x)$ holds and $Ker(\{x\})$ is open.*

(ii) *For every family $\{A_i|i\in\mathbf{N}\}$ of subsets of the digital n-space $(\mathbf{Z}^n,\kappa^n)$, we have the equality $Cl(\bigcup\{A_i|i\in\mathbf{N}\})=\bigcup\{Cl(A_i)|i\in\mathbf{N}\}$, where $\mathbf{N}$ is the set of all natural numbers.*

Proof. **(i)** It is known that, for any open set V containing $x, x\in U^n(x)\subset V$ and $U^n(x)$ is open in $(\mathbf{Z}^n,\kappa^n)$. Then, $Ker(\{x\})=U^n(x)$ holds. **(ii)** It is easily obtained that $\bigcup\{Cl(A_i)|i\in\mathbf{N}\}\subset Cl(\bigcup\{A_i|i\in\mathbf{N}\})$ holds. We prove only the converse implication: $Cl(\bigcup\{A_i|i\in\mathbf{N}\})\subset\bigcup\{Cl(A_i)|i\in\mathbf{N}\}$. Let x be any point such that $x\notin\bigcup\{Cl(A_i)|i\in\mathbf{N}\}$. Then, for each $i\in\mathbf{N}$, $x\notin Cl(A_i)$ and so there exists an open set U_i containing x such that $U_i\cap A_i=\emptyset$ and so $(\bigcap\{U_i|i\in\mathbf{N}\})\cap(\bigcup\{A_i|i\in\mathbf{N}\})=\emptyset$. It follows from definition that $x\in Ker(\{x\})\subset\bigcap\{U_i|i\in\mathbf{N}\}$. Therefore, using assumption and (i), there exists an open set $U^n(x)$ containing x such that $U^n(x)\cap(\bigcup\{A_i|i\in\mathbf{N}\})=\emptyset$ and so $x\notin Cl(\bigcup\{A_i|i\in\mathbf{N}\})$. □

3 Regular open sets induced by a given open set For a given open set of $(\mathbf{Z}^n, \kappa^n)$, we construct regular open sets relating the open set, cf. Theorem 3.2 below. We recall the notation in Section 2:

(i) (cf.Section 2) $U^n(x) := \prod_{i=1}^n U(x_i)$, where $x = (x_1, x_2, \ldots, x_n) \in \mathbf{Z}^n$;

(ii) (cf.Definition 2.4) For an n-tuple $(k_1, k_2, \ldots, k_n)$ of integers $k_i \in \{0,1,2,3,4,5\}(1 \le i \le n)$, $U(k_1, k_2, \ldots, k_n) := \bigcup\{U^n((k_1+6m_1, k_2+6m_2, \ldots, k_n+6m_n))|m_1, m_2, \ldots, m_n \in \mathbf{Z}\}$.

(iii) (cf.Definition 2.1) For a subset E of $(\mathbf{Z}^n, \kappa^n)$ and an integer a' with $1 \le a' \le n$,

$E_{\kappa^n} := \{x \in E|\{x\}$ is open in $(\mathbf{Z}^n, \kappa^n)\} = \{(x_1, x_2, \ldots, x_n) \in E|x_i$ is odd for each integer i with $1 \le i \le n\}$;

$E_{mix(a')} := \{(x_1, x_2, \ldots, x_n) \in E|\#\{i|x_i$ is even $(1 \le i \le n)\} = a'\}$. Sometimes, the set $E_{mix(n)}$ is denoted by $E_{\mathcal{F}^n}$.

Definition 3.1 For a subset E of $(\mathbf{Z}^n, \kappa^n), U^n(E) := \bigcup\{U^n(x)|x \in E\}$.

Then, $U^n(E_{\kappa^n}) = E_{\kappa^n}$, where $E_{\kappa^n} = E \cap (\mathbf{Z}^n)_{\kappa^n}$ (cf. Definition 2.1); $U^n(E_{mix(a')}) = \bigcup\{U^n(x)|x \in E_{mix(a')}\}$.

Theorem 3.2 *Let V be a nonempty open subset of the digital n-space $(\mathbf{Z}^n, \kappa^n)$ and $(k_1, \ldots, k_n)$ an n-tuple of integers $k_i \in \{0,1,2,3,4,5\}(1 \le i \le n)$.*

(i) A subset $U^n((V \cap U(k_1, k_2, \ldots, k_n))_{mix(a)})$ is regular open, where $a = \#\{i|k_i \in \{0,2,4\}(1 \le i \le n)\}$ and $1 \le a \le n$.

(ii) A subset $U^n((V \cap U(k_1, k_2, \ldots, k_n))_{\kappa^n})$ is regular open and $U^n((V \cap U(k_1, k_2, \ldots, k_n))_{\kappa^n}) = (V \cap U(k_1, k_2, \ldots, k_n))_{\kappa^n}$, where $k_i \in \{1,3,5\}(1 \le i \le n)$.

Proof. Put $V^1 = V \cap U(k_1, k_2, \ldots, k_n)$.

(i) We show that $Int(Cl(U^n((V^1)_{mix(a)}))) = U^n((V^1)_{mix(a)})$. Since $U^n(A)$ is open for any subset A in general, we prove only the following implication: $Int(Cl(U^n((V^1)_{mix(a)}))) \subset U^n((V^1)_{mix(a)})$.

Let $x \in Int(Cl(U^n((V^1)_{mix(a)})))$. We claim that $x \in U^n((V^1)_{mix(a)})$.

Case 1. $x \in (\mathbf{Z}^n)_{\kappa^n}$:

$\{x\} \subset Cl(U^n((V^1)_{mix(a)}))$ holds and so $x \in U^n((V^1)_{mix(a)})$, because $\{x\}$ is an open set containing x.

Case 2. $x \in (\mathbf{Z}^n)_{mix(a')}$, where $a' \in \mathbf{Z}$ with $1 \le a' \le n$:

There exists the basic open neighbourhood $U^n(x)$ of x such that

$U^n(x) \subset Cl(U^n((V^1)_{mix(a)}))$. Set $x = (x_1, x_2, \ldots, x_n)$. Assume $\{i|x_i$ is even $(1 \le i \le n)\} = \{e(1), e(2), \ldots, e(a')\}(e(1) < e(2) < \ldots < e(a'))$. We note that $U(x_{e(s)}) = \{x_{e(s)} - 1, x_{e(s)}, x_{e(s)} + 1\}$ where s is an integer with $1 \le s \le a'$. Then we can take the exactly $2^{a'}$ open singletons $\{q^{(\alpha)}\} \subset U^n(x) = \prod_{i=1}^n U(x_i)(1 \le \alpha \le 2^{a'})$ and so $q^{(\alpha)} \in U^n((V^1)_{mix(a)})$. For each point $q^{(\alpha)}(1 \le \alpha \le 2^{a'})$, there exists a point $y^{(\alpha)} \in (V^1)_{mix(a)}$ such that $q^{(\alpha)} \in U^n(y^{(\alpha)})$. We have that $y^{(\alpha)} \in (U(k_1, k_2, \ldots, k_n))_{mix(a)}, q^{(\alpha)} \in U^n(y^{(\alpha)})$, $q^{(\alpha)} \in (U^n(x))_{\kappa^n}$ and $x \in (\mathbf{Z}^n)_{mix(a')}$ for each integer α with $1 \le \alpha \le 2^{a'}(1 \le a' \le n)$. Since $a = \#\{i|k_i \in \{0,2,4\}(1 \le i \le n)\}$, we have that $(k_1, k_2, \ldots, k_n) \in (\mathbf{Z}^n)_{mix(a)}$. Thus we can use Lemma 2.5(ii) for the points $y^\alpha(1 \le \alpha \le 2^{a'})$. By using Lemma 2.5(ii) repeatedly, it is shown that $y^{(1)} = y^{(2)} = y^{(3)} = \ldots = y^{(2^{a'})}$. Then, we have that $q^{(\alpha)} \in U^n(y^{(1)})$ and $q^{(\alpha)} \in U^n(x)$ for each integer α with $1 \le \alpha \le 2^{a'}$. Since $y^{(1)} \in (\mathbf{Z}^n)_{mix(a)}, U^n(y^{(1)})$ contains exactly 2^a open singletons and $q^{(\alpha)} \in U^n(y^{(1)})(1 \le \alpha \le 2^{a'})$. Thus, we have that $2^{a'} \le 2^a$. Then, $U^n(y^{(1)}) \cap U^n(x)$ contains exactly the $2^{a'}$ open singletons $\{q^{(1)}\}, \{q^{(2)}\}, \{q^{(3)}\}, \ldots, \{q^{(2^{a'})}\}$. Then, using Lemma 2.3(iii), we have that $x \in U^n(y^{(1)})$. Since $y^{(1)} \in (V^1)_{mix(a)}$, we conclude that $x \in U^n((V^1)_{mix(a)})$.

By Case 1 and Case 2, we prove that $U^n((V^1)_{mix(a)})$ is regular open.

(ii) We prove that a subset $U^n((V^1)_{\kappa^n})$ is regular open, where $\{i \mid k_i \in \{1,3,5\}\ (1 \leq i \leq n)\ \} = \{i \in \mathbf{Z} \mid 1 \leq i \leq n\}$. Let x be a point of $Int(Cl(U^n((V^1)_{\kappa^n})))$. We claim that $x \in U^n((V^1)_{\kappa^n})$ for the following cases.

Case 1. $x \in (\mathbf{Z}^n)_{\kappa^n} : \{x\} \subset Cl(U^n((V^1)_{\kappa^n}))$ holds and so $x \in U^n((V^1)_{\kappa^n})$, because $\{x\}$ is an open set containing x.

Case 2. $x \in (\mathbf{Z}^n)_{mix(a')}$, where a' is an integer with $1 \leq a' \leq n$: It is proved that this case does not occur under our assumptions. There exists the smallest open neighbourhood $U^n(x)$ of x such that $U^n(x) \subset Cl(U^n((V^1)_{\kappa^n}))$. Put $x = (x_1, x_2, \ldots, x_n)$, where $x_i \in \mathbf{Z}(1 \leq i \leq n)$. Assume that $\{i | x_i \text{ is even } (1 \leq i \leq n)\} = \{e(1), e(2), \ldots, e(a')\}$. We note that $U(x_{e(s)}) = \{x_{e(s)} - 1, x_{e(s)}, x_{e(s)} + 1\}$ where s is an integer with $1 \leq s \leq a'$. Then we can take the exactly $2^{a'}$ open singletons $\{q^{(\alpha)}\} \subset U^n(x) = \prod_{i=1}^n U(x_i)(1 \leq \alpha \leq 2^{a'})$ such that $q^{(\alpha)} \in U^n((V^1)_{\kappa^n})$. Put $q^{(\alpha)} = (Q_1^{(\alpha)}, Q_2^{(\alpha)}, \ldots, Q_n^{(\alpha)})$, where $Q_i^{(\alpha)} \in \mathbf{Z}(1 \leq i \leq n)$. Then, we have that, by Lemma 2.5(i), for distinct integers α and β with $1 \leq \alpha \leq 2^{a'}$ and $1 \leq \beta \leq 2^{a'}$,

$(*)$ $\quad Q_i^{(\alpha)} - Q_i^{(\beta)} \in \{0, 2, -2\}$ for each integer i with $1 \leq i \leq n$.

For each point $q^{(\alpha)}$, there exists a point $y^{(\alpha)} \in (V^1)_{\kappa^n}$ such that $q^{(\alpha)} \in U^n(y^{(\alpha)}) = \{y^{(\alpha)}\}$. Thus, we have that $y^{(\alpha)} = q^{(\alpha)}$ for each integer α. Since $y^{(\alpha)} \in (U(k_1, k_2, \ldots, k_n))_{\kappa^n}$, there exist a point $z^{(\alpha)} \in \mathbf{Z}^n$ such that $y^{(\alpha)} \in (U^n(z^{(\alpha)}))_{\kappa^n}$ and $z^{(\alpha)} = (k_1 + 6m_1^{(\alpha)}, k_2 + 6m_2^{(\alpha)}, \ldots, k_n + 6m_n^{(\alpha)})$ for some integers $m_i^{(\alpha)}(1 \leq i \leq n)$. Since $k_i \in \{1,3,5\}$ for every i with $1 \leq i \leq n$, by assumption, we have that $z^{(\alpha)} \in (\mathbf{Z}^n)_{mix(0)} = (\mathbf{Z}^n)_{\kappa^n}$. Using Lemma 2.2 for the points $y^{(\alpha)}$ and $z^{(\alpha)}$, we have that $y^{(\alpha)} = z^{(\alpha)}$. Using $(*)$ above, the difference between each i-components of $z^{(\alpha)}$ and $z^{(\beta)}(\alpha \neq \beta)$ is $0, 2$ or -2, because $y^{(\alpha)} = q^{(\alpha)} = z^{(\alpha)}$ for every α. If there exist i-components of two points $z^{(\alpha)}$ and $z^{(\beta)}$ such that $k_i + 6m_i^{(\alpha)} - (k_i + 6m_i^{(\beta)}) \in \{2, -2\}$, then we have that $0 \equiv 2 \bmod 6$ or $0 \equiv -2 \bmod 6$ and so this case does not occur. Thus, we conclude that, for all i with $1 \leq i \leq n, k_i + 6m_i^{(\alpha)} - (k_i + 6m_i^{(\beta)}) = 0$ hold (i.e., $m_i^{(\alpha)} = m_i^{(\beta)}(1 \leq i \leq n)$) and so $q^{(\alpha)} = q^{(\beta)}$ for $\alpha \neq \beta$. This contradicts the definition of the open singletons $\{q^{(\alpha)}\}(1 \leq \alpha \leq 2^{a'})$. Then, Case 2 does not occur.

By Case 1 and Case 2, we proved that $U^n((V^1)_{\kappa^n})$ is regular open. It is obvious that, for a point $w \in \mathbf{Z}^n, w \in U^n((V^1)_{\kappa^n})$ if and only if $w \in (V^1)_{\kappa^n}$. Thus we have the desired equality: $U^n((V^1)_{\kappa^n}) = (V^1)_{\kappa^n}$. □

The following examples suggest that every nonempty open set of the digital n-space can be expressible as the union of finitely many nonempty regular open sets (cf. Example 3.3, Example 4.2(ii) below).

Example 3.3 **(i)** Throughout this example (i), assume that $k_j = 1$ for each integer j with $2 \leq j \leq n$. Let $V := H \times (\prod_{j=2}^n \{k_j\})$, where $H = \{2c + 1 | c \in \mathbf{Z}\}$. The set V is open; it is not regular open. Indeed, it is shown that $Int(Cl(V)) = Int(\mathbf{Z} \times (\prod_{j=2}^n \{k_j - 1, k_j, k_j + 1\})) = \mathbf{Z} \times (\prod_{j=2}^n \{k_j\})$ and so $Int(Cl(V)) \neq V$; V is not regular open. We have that $(V \cap U(1, k_2, \ldots, k_n))_{\kappa^n} = V \cap U(1, k_2, \ldots, k_n) = \{(6m_1 + 1, k_2, \ldots, k_n) | m_1 \in \mathbf{Z}\}$. Similarly we have the following:

$(V \cap U(3, k_2, \ldots, k_n))_{\kappa^n} = \{(6m_1 + 3, k_2, \ldots, k_n) | m_1 \in \mathbf{Z}\}$;
$(V \cap U(5, k_2, \ldots, k_n))_{\kappa^n} = \{(6m_1 + 5, k_2, \ldots, k_n) | m_1 \in \mathbf{Z}\}$;
$(V \cap U(0, k_2, \ldots, k_n))_{\kappa^n} = (V \cap U(1, k_2, \ldots, k_n))_{\kappa^n} \cup (V \cap U(5, k_2, \ldots, k_n))_{\kappa^n}$;
$(V \cap U(2, k_2, \ldots, k_n))_{\kappa^n} = (V \cap U(1, k_2, \ldots, k_n))_{\kappa^n} \cup (V \cap U(3, k_2, \ldots, k_n))_{\kappa^n}$;
$(V \cap U(4, k_2, \ldots, k_n))_{\kappa^n} = (V \cap U(3, k_2, \ldots, k_n))_{\kappa^n} \cup (V \cap U(5, k_2, \ldots, k_n))_{\kappa^n}$.

We observe that $V = \{(6m_1+1, k_2, \ldots, k_n)|m_1 \in \mathbf{Z}\} \cup \{(6m_1+3, k_2, \ldots, k_n)|m_1 \in \mathbf{Z}\} \cup \{(6m_1+5, k_2, \ldots, k_n)|m_1 \in \mathbf{Z}\} = \bigcup\{(V \cap U(k_1, k_2, \ldots, k_n))_{\kappa^n}|k_1 \in \{1,3,5\}\}$. Therefore, V is expressible as the union of three regular open sets $(V \cap U(1, k_2, \ldots, k_n))_{\kappa^n}$, $(V \cap U(3, k_2, \ldots, k_n))_{\kappa^n}$ and $(V \cap U(5, k_2, \ldots, k_n))_{\kappa^n}$ (cf. Theorem 3.2(ii)). This suggests Lemma 4.1 below and so Theorem 1.1, i.e., *every open set of the digital n-space can be expressible as the union of finitely many nonempty regular open sets.*

(ii) Throughout this example (ii), assume that $k_j = 2$ for each integer j with $2 \le j \le n$. Let $V := \bigcup\{U^n((4c, k_2, \ldots, k_n))|c \in \mathbf{Z}\}$. Then, V is open; it is not regular open. Indeed, we have that $Int(Cl(V)) = \mathbf{Z} \times (\prod_{j=2}^n (Int A_j)) = \mathbf{Z} \times (\prod_{j=2}^n B_j) \ne V$, where $A_j := \{0,1,2,3,4\}$ and $B_j := U(2) = \{1,2,3\}(2 \le j \le n)$. Since $V \cap U(0, k_2, \ldots, k_n) = \bigcup[\bigcup\{(\{4c-1, 4c, 4c+1\} \cap \{6m_1-1, 6m_1, 6m_1+1\}) \times \prod_{j=2}^n (B_j \cap \{1+6m_j, 2+6m_j, 3+6m_j\})|c \in \mathbf{Z}\}\ |m_i \in \mathbf{Z}(1 \le i \le n)]$, it is shown that $(V \cap U(0, k_2, \ldots, k_n))_{mix(n)} = \{(12s, k_2, \ldots, k_n)|s \in \mathbf{Z}\}$. Thus, we show that $U^n((V \cap U(0, k_2, \ldots, k_n))_{mix(n)}) = \bigcup\{U^n((12s, k_2, \ldots, k_n))|s \in \mathbf{Z}\}$. Similarly, we show that

$U^n((V \cap U(2, k_2, \ldots, k_n))_{mix(n)}) = \bigcup\{U^n((12s+8, k_2, \ldots, k_n))|s \in \mathbf{Z}\}$;

$U^n((V \cap U(4, k_2, \ldots, k_n))_{mix(n)}) = \bigcup\{U^n((12s+4, k_2, \ldots, k_n))|s \in \mathbf{Z}\}$.

Since V is expressible as $V = \bigcup\{U^n((12s, k_2, \ldots, k_n))|s \in \mathbf{Z}\} \cup (\bigcup\{U^n((12s+4, k_2, \ldots, k_n))|s \in \mathbf{Z}\}) \cup (\bigcup\{U^n((12s+8, k_2, \ldots, k_n))|s \in \mathbf{Z}\})$, V is expressible as the union of three regular open sets $U^n((V \cap U(0, k_2, \ldots, k_n))_{mix(n)})$, $U^n((V \cap U(2, k_2, \ldots, k_n))_{mix(n)})$ and $U^n((V \cap U(4, k_2, \ldots, k_n))_{mix(n)})$ (cf.Theorem 3.2(i)). This also suggests Lemma 4.1 below and so Theorem 1.1, i.e.,*every nonempty open set of the digital n-space can be expressible as the union of finitely many nonempty regular open sets.*

4 The proof of Theorem 1.1

In this section, using Lemma 4.1 below and Theorem 3.2, we prove that every nonempty open set of $(\mathbf{Z}^n, \kappa^n)$ is expressible as the union of finitely many nonempty regular open sets (Theorem 1.1).

Lemma 4.1 *Let V be a nonempty open set in $(\mathbf{Z}^n, \kappa^n)$. Then, $V = \bigcup\{V_a|a \in \mathbf{Z}, 0 \le a \le n\}$ holds,*

where $V_0 := \bigcup\{(V \cap U(k_1, k_2, \ldots, k_n))_{\kappa^n}|k_i \in \{1,3,5\}(1 \le i \le n)\}$ and

$V_a := \bigcup\{U^n((V \cap U(k_1, k_2, \ldots, k_n))_{mix(a)})|k_i \in \{0,1,2,3,4,5\}(1 \le i \le n), a = \#\{i|k_i \in \{0,2,4\}(1 \le i \le n)\}\}$ for integer a with $1 \le a \le n$.

Proof. First we show that $V \subset \bigcup\{V_a|a \in \mathbf{Z}, 0 \le a \le n\}$. Let $x = (x_1, x_2, \ldots, x_n) \in V$, where $x_i \in \mathbf{Z}(1 \le i \le n)$. For the integer $x_i(1 \le i \le n)$, there exists a unique integer $k_i \in \{0,1,2,3,4,5\}$ such that $x_i = k_i + 6m_i$ for some integers $m_i(1 \le i \le n)$. We note that $\#\{i|x_i \text{ is even } (1 \le i \le n)\} = \#\{i|k_i \in \{0,2,4\}(1 \le i \le n)\}$. Since $x \in V$ and V is open in $(\mathbf{Z}^n, \kappa^n)$, there exists the smallest open neighbourhood $U^n(x)$ of x such that $U^n(x) \subset V$. Thus, we have that, for $x \in V$, $x \in V \cap U^n((k_1+6m_1, k_2+6m_2, \ldots, k_n+6m_n)) \subset V \cap U(k_1, k_2, \ldots, k_n)$.

Case 1. $x \in (\mathbf{Z}^n)_{mix(a)}$, where $1 \le a \le n$: Then, $a = \#\{i\ |k_i \in \{0,2,4\}\ (1 \le i \le n)\}$ holds. It is shown that, for $x \in V_{mix(a)}$, $x \in (V \cap U(k_1, k_2, \ldots, k_n))_{mix(a)} \subset U^n((V \cap U(k_1, k_2, \ldots, k_n))_{mix(a)})$. Thus, we have that

$(*)$ $V_{mix(a)} \subset \bigcup\{U^n((V \cap U(k'_1, k'_2, \ldots, k'_n))_{mix(a)})|k'_i \in \{0,1,2,3,4,5\}(1 \le i \le n), a = \#\{i|k'_i \in \{0,2,4\}(1 \le i \le n)\}\}$, (i.e., $V_{mix(a)} \subset V_a$).

Case 2. $x \in (\mathbf{Z}^n)_{\kappa^n}$: By an argument similar to that in Case 1, it is shown that

$(**) V_{\kappa^n} \subset \bigcup\{U^n((V \cap U(k'_1, k'_2, \ldots, k'_n))_{\kappa^n})|k'_i \in \{1,3,5\}(1 \le i \le n)\}$ (i.e., $V_{\kappa^n} \subset V_0$).

Using $(*)$ and $(**)$, we have that $V = V_{\kappa^n} \cup (\bigcup\{V_{mix(a)}|a \in \mathbf{Z}, 1 \le a \le n\}) \subset \bigcup\{V_j|j \in \mathbf{Z}, 0 \le j \le n\}$. Finally, we claim that $\bigcup\{V_j|j \in \mathbf{Z}, 0 \le j \le n\} \subset V$. Let $x \in \bigcup\{V_j|j \in \mathbf{Z}, 0 \le j \le n\}$. Then, there exists an integer a such that $0 \le a \le n$ and $x \in V_a$. If $1 \le a \le n$, then there exists a point $z \in (V \cap U(k_1, k_2, \ldots, k_n))_{mix(a)} \subset V$ such that $x \in U^n(z)$ for some integers $k_i \in \{0,1,2,3,4,5\}(1 \leq i \leq n)$ with $a = \#\{i | k_i \in \{0,2,4\}(1 \leq i \leq n)\}$. If $a = 0$, then there exists a point $z \in (V \cap U(k_1, k_2, \ldots, k_n))_{\kappa^n} \subset V$ such that $x \in U^n(z)$ for some integers $k_i \in \{1,3,5\}(1 \leq i \leq n)$. For both cases, we have that $x \in U^n(z)$ and $z \in V$. Since $U^n(z)$ is the smallest open neighbourhood of $z, U^n(z) \subset V$ and so $x \in V$. We conclude that $\bigcup\{V_j | j \in \mathbf{Z}, 0 \leq j \leq n\} \subset V$ holds and hence $V = \bigcup\{V_j | j \in \mathbf{Z}, 0 \leq j \leq n\}$. □

Proof of Theorem 1.1: Let V be any nonempty open set in $(\mathbf{Z}^n, \kappa^n)$. By Lemma 4.1, it is obtained that $V = \bigcup\{V_a | a \in \mathbf{Z}, 0 \leq a \leq n\}$ holds,

where $V_0 := \bigcup\{(V \cap U(k_1, k_2, \ldots, k_n))_{\kappa^n} | k_i \in \{1,3,5\}(1 \leq i \leq n)\}$ and $V_a := \bigcup\{U^n((V \cap U(k_1, k_2, \ldots, k_n))_{mix(a)}) | k_i \in \{0,1,2,3,4,5\}(1 \leq i \leq n), a = \#\{i | k_i \in \{0,2,4\}(1 \leq i \leq n)\}$ for integer a with $1 \leq a \leq n$.

We note that the set $V_a, 1 \leq a \leq n$, is a finite union of the following subsets: $U^n((V \cap U(k_1, k_2, \ldots, k_n))_{mix(a)})$, where $k_i \in \{0,1,2,3,4,5\}(1 \leq i \leq n)$. Using Theorem 3.2, the set $U^n((V \cap U(k_1, k_2, \ldots, k_n))_{mix(a)})$ is regular open, where $a = \#\{i | k_i \in \{0,2,4\}(1 \leq i \leq n)\}(1 \leq a \leq n)$, and the set $(V \cap U(k_1, k_2, \ldots, k_n))_{\kappa^n}$ is regular open, where $k_i \in \{1,3,5\}(1 \leq i \leq n)$. Therefore, every open set is expressible as the union of finitely many nonempty regular open sets in $(\mathbf{Z}^n, \kappa^n)$. □

Example 4.2 (i) In Example 3.3, it is shown that the open sets V in Example 3.3 are expressible as the union of finitely many regular open sets.

(ii) Let $V := (\mathbf{Z}^n)_{\kappa^n}$ be the set of all open singletons of $(\mathbf{Z}^n, \kappa^n)$ (cf. Definition 2.1(i)). The open set V above is expressible as the union of finitely many regular open sets as follows. By Lemma 4.1 or Proof of Theorem 1.1, V is expressible as $V = \bigcup\{(V \cap U(k_1, k_2, \ldots, k_n))_{\kappa^n} | k_j \in \{1,3,5\}(1 \leq j \leq n)\}$, where $(V \cap U(k_1, k_2, \ldots, k_n))_{\kappa^n} = (U(k_1, k_2, \ldots, k_n))_{\kappa^n} = \{(k_1 + 6m_1, k_2 + 6m_2, \ldots, k_n + 6m_n) | m_i \in \mathbf{Z}(1 \leq i \leq n)\}(k_j \in \{1,3,5\}(1 \leq j \leq n))$. Namely, the open set V is expressible as the union of 3^n regular open sets $\{(k_1 + 6m_1, k_2 + 6m_2, \ldots, k_n + 6m_n) | m_i \in \mathbf{Z}(1 \leq i \leq n)\}$, where $k_i \in \{1,3,5\}(1 \leq i \leq n)$. We note that the set V is not regular open. It is claimed that $Cl(V) = \mathbf{Z}^n$ and so $Int(Cl(V)) = \mathbf{Z}^n \neq V$. Indeed, first let x be a point such that $x \in (\mathbf{Z}^n)_{mix(a)}$ and $1 \leq a \leq n-1$ (i.e., $1 \leq \#\{i | x_i \text{ is even } (1 \leq i \leq n)\} \leq n-1$, where $x = (x_1, x_2, \ldots, x_n)$). Put $\{i | x_i \text{ is even } (1 \leq i \leq n)\} = \{e(1), e(2), \ldots, e(a)\}$ and $\{j | x_j \text{ is odd } (1 \leq j \leq n)\} = \{o(1), o(2), \ldots, o(n-a)\}$. Define a point $y = (y_1, y_2, \ldots, y_n) \in (\mathbf{Z}^n)_{\kappa^n}$ as follows:

$y_{e(s)} := x_{e(s)} + 1, y_{o(t)} := x_{o(t)}$, where $1 \leq s \leq a$ and $1 \leq t \leq n-a$.

Then, we show that $x \in \prod_{j=1}^{n}\{y_j - 1, y_j, y_j + 1\} = \prod_{j=1}^{n} Cl(\{y_j\}) \subset Cl(V)$. Namely, we show that $(\mathbf{Z}^n)_{mix(a)} \subset Cl(V)$, where $1 \leq a \leq n-1$. Finally let x be a point such that $x \in (\mathbf{Z}^n)_{mix(n)}$ (i.e., x_i is even for all i with $1 \leq i \leq n$). Define a point $y = (y_1, y_2, \ldots, y_n) \in (\mathbf{Z}^n)_{\kappa^n}$ as follows:

$y_i := x_i + 1$ for all integers i. Then, it is similarly shown that $x \in Cl(V)$, i.e., $(\mathbf{Z}^n)_{mix(n)} \subset Cl(V)$. Therefore, we show that $\mathbf{Z}^n = Cl(V)$, because $\mathbf{Z}^n = (\mathbf{Z}^n)_{\kappa^n} \cup (\bigcup\{(\mathbf{Z}^n)_{mix(a)} | 1 \leq a \leq n-1\}) \cup (\mathbf{Z}^n)_{mix(n)} \subset Cl(V)$, and so $Int(Cl(V)) = \mathbf{Z}^n \neq V$. Namely V is not regular open. The set $(\mathbf{Z}^n)_{\kappa^n}$ is called as the *open screen* [12, p.178] for $n = 2$.

As a corollary, we have [8, Theorem A] for the digital line (resp. [8, Theorem C] for the digital plane). Moreover we have the following corollaries.

For a topological space (X, τ), let τ_δ be the family of all δ-open sets of (X, τ). Namely, for a subset A of (X, τ), $A \in \tau_\delta$ if and only if A is the union of regular open sets (e.g. [4, p.16]) and τ_δ is a topology of X [16, lemma 3]. The following family of subsets of (X, τ), denoted by $\tau_{f\delta}$, is used in Corollay 4.3 below:

$\tau_{f\delta} := \{U \in P(X) | U$ is the union of finitely many regular open sets of $(X, \tau)\}$; clearly $\tau_{f\delta} \subset \tau_\delta \subset \tau$ hold in general.

Corollary 4.3 *For the digital n-space* $(\mathbf{Z}^n, \kappa^n)$, $\kappa^n = (\kappa^n)_{f\delta} = (\kappa^n)_\delta$ *hold.* □

Corollary 4.4 *Let* $f : (\mathbf{Z}^n, \kappa^n) \to (\mathbf{Z}^n, \kappa^n)$ *be a function. The following properties are equivalent:*

(1) $f : (\mathbf{Z}^n, \kappa^n) \to (\mathbf{Z}^n, \kappa^n)$ *is continuous;*

(2) *for each* $x \in \mathbf{Z}^n$ *and each open neighborhood* U *of* $f(x)$*, there exists a regular open subset* V *such that* $x \in V$ *and* $f(V) \subset U$. □

Remark 4.5 **(i)** The notion of a π-set was introduced by Zaǐcev[17]. Zolotarev [18] proved that in metric space every closed set is a π-set [18, Theorem 1](i.e., every closed set is the intersection of finitely many regular closed sets). The digital n-space is not a metric space, because it is not T_1. By Theorem 1.1, it is shown that in the digital n-space every closed set is π-set.

(ii) The proof of Theorem 1.1 shows explicitly a construction of the union of **finitely many** regular open sets of $(\mathbf{Z}^n, \kappa^n)$ (cf. Theorem 3.2, Definitions 3.1,2.4,2.1).

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