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QUOTIENT HYPER MV-ALGEBRAS

Shokoofeh Ghorbani, Esfandiar Eslami and Abbas Hasankhani

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Abstract. In this paper we define the concepts of good H-congruence relations, weak positive implicative subsets, positive implicative subsets, weak maximal hyper MV-filters and maximal hyper MV-filters in hyper MV-algebras. We then use some of the above notions to define quotient hyper MV-algebras. We state and prove some related theorems with appropriate results. In particular, we define the notion of the fundamental relation with its basic properties.

1 Introduction The hyper algebraic structure theory was introduced in 1934 [3] by Marty at 8th Congress of Scandinavian Mathematicians. Since then many researchers have worked on this area. Recently in [2] we introduced and studied hyper MV-algebras.
In the next section some preliminary theorems are stated from [2] which are needed in this paper. In section 3 we define (weak) positive implicative subsets and (weak) maximal hyper MV-filters and obtain some results about them. In section 4 we define the concept of quotient hyper MV-algebra and we prove the first, second and third isomorphism theorem. In section 5, we define the notion of the fundamental relation on a hyper MV-algebra and we show that it is the smallest good H-congruence relation such that the related quotient hyper MV-algebra is an MV-algebra.

2 Preliminaries **Definition 2.1.**[2] A hyper MV-algebra is a non-empty set endowed with a hyper operation "$\oplus$", a unary operation "$*$" and a constant 0 satisfying the following axioms:
(hMV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
(hMV2) $x \oplus y = y \oplus x$,
(hMV3) $(x^*)^* = x$,
(hMV4) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$,
(hMV5) $0^* \in x \oplus 0^*$,
(hMV6) $0^* \in x \oplus x^*$,
(hMV7) if $x << y$ and $y << x$, then $x = y$,
for all $x, y, z \in M$, where $x << y$ is defined by $0^* \in x^* \oplus y$.
For every $A, B \subseteq M$, we define $A << B$ if and only if there exist $a \in A$ and $b \in B$ such that $a << b$. We define $0^* := 1$ and $A^* = \{a^* : a \in A\}$.

Definition 2.2.[2] Let $< M, \oplus, *, 0 >$ be a hyper MV-algebra. Then for all $x, y, z \in M$ and for all non-empty subset A, B and C of M the following hold:
(1) $A \oplus (B \oplus C) = (A \oplus B) \oplus C$,
(2) $0 << x$,
(3) $x << x$,
(4) If $x << y$, then $y^* << x^*$ and $A << B$ implies $B^* << A^*$,

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(5) $x << 1$,
(6) $A << A$,
(7) $A \subseteq B$ implies $A << B$,
(8) $x << x \oplus y$ and $A << A \oplus B$,
(9) $(A^*)^* = A$,
(10) $0 \oplus 0 = \{0\}$,
(11) $x \in x \oplus 0$,
(12) if $y \in x \oplus 0$, then $y << x$,
(13) if $x \oplus 0 = y \oplus 0$, then $x = y$.

A hyper MV-algebra $< M, \oplus, *, 0 >$ is called nontrivial if $M \neq \{0\}$. It is clear that a hyper MV-algebra is nontrivial if and only if $0 \neq 1$. In this paper, we consider nontrivial hyper MV-algebras.

Definition 2.3.[2] Let F be a non-empty subset of a hyper MV-algebra $< M, \oplus, *, 0 >$. Then F is called a weak hyper MV-filter of M , if
(whF1)$1 \in F$,
(whF2) if $F \subseteq x^* \oplus y$ and $x \in F$, then $y \in F$ for all $x, y \in M$.

Definition 2.4.[2] Let F be a non-empty subset of a hyper MV-algebra $< M, \oplus, *, 0 >$. Then F is called a hyper MV-filter of M , if
(hF1)$1 \in F$,
(hF2) if $F << x^* \oplus y$ and $x \in F$, then $y \in F$ for all $x, y \in M$.

Proposition 2.5. Let $< M, \oplus, *, 0 >$ be a hyper MV-algebra. Then $\{1\}$ is a hyper MV-filter of M .

Proof: Let $x, y \in M$ such that $1 << x^* \oplus y$ and $x \in \{1\}$. So $x = 1$ and then $x^* = 0$. Thus $1 << 0 \oplus y$. Therefore there exists $a \in 0 \oplus y$ such that $1 << a$. Thus $1 = a$ and we have $1 \in 0 \oplus y$, i.e., $1 << y$ which implies $y = 1 \in \{1\}$.

Proposition 2.6.[2] Let $< M, \oplus, *, 0 >$ be a hyper MV-algebra and F be a hyper MV-filter of M. If $x << y$ and $x \in F$, then $y \in F$.

Proposition 2.7.[2] Let $< M, \oplus, *, 0 >$ be a hyper MV-algebra and F be a hyper MV-filter of M. Then F is a weak hyper MV-filter of M.

Definition 2.8.[2] Let $< M, \oplus, *, 0 >$ be a hyper MV-algebra and S be a non-empty subset of M. If S is a hyper MV-algebra with respect to the hyper operation " $\oplus$ " and unary operation " $*$ " on M , we say that S is a hyper MV-subalgebra of M .

Definition 2.9.[2] Let M_1 and M_2 be two hyper MV-algebras. A mapping $f\colon M_1 \to M_2$ is said to be a homomorphism, if
(i) $f(0) = 0$,
(ii)$f(x \oplus y) = f(x) \oplus f(y)$,
(iii)$f(x^*) = (f(x))^*$.
Clearly if f is aa homomorphism, then$f(1) = 1$.

If f is one to one (or onto), then we say that f is a monomorphism (or epimorphism), and if f is both one to one and onto, then we say that f is an isomorphism and in this case we say that M_1 and M_2 are isomorphic and it is denoted by $M_1 \cong M_2$.

Theorem 2.10.[2] Let$f : M_1 \to M_2$ be a homomorphism of hyper MV-algebras. Then
(1) if S is a hyper MV-subalgebra of M_1, then $f(S)$ is a hyper MV-subalgebra of M_2.
(2) if S is a hyper MV-subalgebra of M_2, then $f^{-1}(S)$ is a hyper MV-subalgebra of M_1.
(3) if F is a (weak) hyper MV-filter of M_2, then $f^{-1}(F)$ is a (weak) hyper MV-filter of M_1.
(4) Let $\ker f = \{x \in M_1 : f(x) = 1\}$. Then $\ker f$ is a hyper MV-filter of M_1, consequently $\ker f$ is a weak hyper MV-filter of M_1.
(5) f is one to one if and only if $\ker f = \{1\}$.
(6) if f is onto and F is a hyper MV-filter of M_1 which contains $\ker f$, then $f(F)$ is a hyper MV-filter of M_2.

3 Positive implicative subsets and maximal hyper MV-filters

Definition 3.1. Let F be a non-empty subset of a hyper MV-algebra $< M, \oplus, *, 0 >$ such that
(i) $1 \in F$,
(ii) if $F \subseteq (x^* \oplus y) \oplus z$ and $F \subseteq y^* \oplus z$ then $F << x^* \oplus z$ for all $x, y, z \in M$. Then F is called a weak positive implicative subset of M.

Proposition 3.2. Let $\{F_\alpha\}_{\alpha \in \Gamma}$ be a family of weak positive implicative subsets of a hyper MV-algebra $< M, \oplus, *, 0 >$. Then $\bigcup_{\alpha \in \Gamma} F_\alpha$ is a weak positive implicative subset of M.

Proof: The proof is straightforward.

Remark 3.3. Let $\{F_\alpha\}_{\alpha \in \Gamma}$ be a family of weak positive implicative subset of a hyper MV-algebra $< M, \oplus, *, 0 >$. Then $\bigcap_{\alpha \in \Gamma} F_\alpha$ may not be a weak positive implicative subset of M . Consider the following example.

Example 3.4. Let $M = \{0, b, 1\}$. Consider the following tables:

$\oplus$	0	b	1
0	$\{0\}$	$\{0, b\}$	$\{1\}$
b	$\{0, b\}$	$\{0, b, 1\}$	$\{0, b, 1\}$
1	$\{1\}$	$\{0, b, 1\}$	$\{1\}$

$*$	0	b	1
	1	b	0

Then $< M, \oplus, *, 0 >$ is a hyper MV-algebra. $F_1 = \{0, 1\}$ and $F_2 = \{b, 1\}$ are weak positive implicative subsets of M. But $F = F_1 \cap F_2$ is not a weak positive implicative subset of M, since $F \subseteq (1^* \oplus b) \oplus b$ and $F \subseteq b^* \oplus b$ but it is not true that $F << (1^* \oplus b)$.

Definition 3.5. Let F be a non-empty subset of a hyper MV-algebra $< M, \oplus, *, 0 >$ such that
(i) $1 \in F$,
(ii) if $F << (x^* \oplus y) \oplus z$ and $F << y^* \oplus z$, then $F << x^* \oplus z$ for all $x, y, z \in M$. Then F is called a positive implicative subset of M .

Proposition 3.6. Let F be a positive implicative subset of a hyper MV-algebra $< M, \oplus, *, 0 >$. Then F is a weak positive implicative subset of M.

Proof: The proof follows from Proposition 2.2(7).

Remark 3.7. Let $\{F_\alpha\}_{\alpha\in\Gamma}$ be a family of positive implicative subsets of a hyper MV-algebra $< M, \oplus, *, 0 >$. Then $\bigcup_{\alpha\in\Gamma} F_\alpha$ may not be a positive implicative subset of M. Consider the following example.

Example 3.8. Let $M = \{0, a, b, 1\}$. Consider the following tables:

$\oplus$	0	a	b	1
0	$\{0\}$	$\{0, a\}$	$\{0, b\}$	$\{0, a, b, 1\}$
a	$\{0, a\}$	$\{0, a\}$	$\{0, a, b, 1\}$	$\{0, a, b, 1\}$
b	$\{0, b\}$	$\{0, a, b, 1\}$	$\{0, b\}$	$\{0, a, b, 1\}$
1	$\{0, a, b, 1\}$	$\{0, a, b, 1\}$	$\{0, a, b, 1\}$	$\{0, a, b, 1\}$

$*$	0	a	b	1
	1	b	a	0

Then $< M, \oplus, *, 0 >$ is a hyper MV-algebra.Consider hyper MV-filters $F_1 = \{b, 1\}$ and $F_2 = \{a, 1\}$ which are positive implicative subsets of M. But $F = F_1 \cup F_2$ is not a positive implicative subset of M, since $F << (1^* \oplus a) \oplus 0$ and $F << a^* \oplus 0$ but it is not true that$F << 1^* \oplus 0$.

Remark 3.9. Let $< M, \oplus, *, 0 >$ be a hyper MV-algebra.

(1) If F is a weak positive implicative subset of M then F may not be a positive implicative subset of M . Consider Example 3.8. Then $F = \{a, b, 1\}$ is a weak positive implicative subset of M . But F is not a positive implicative subset of M, since $F << (1^* \oplus b) \oplus 0$ and $F << b^* \oplus 0$ but it is not true that $F << 1^* \oplus 0$.

(2)A positive implicative subset of M may not be a hyper MV-filter of M. Consider Example 3.8. Then $F = \{0, a, 1\}$ is a positive implicative subset of M. But F is not a hyper MV-filter of M . since $F << 0^* \oplus b$ and $0 \in F$ but $b \notin F$.

(3) A hyper MV-filter of M may not be a positive implicative subset of M. Consider Example 3.4. Then $F = \{1\}$ is a hyper MV-filter of M. But F is not a positive implicative subset of M, since $F << (1^* \oplus b) \oplus b$ and $F << b^* \oplus b$ but it is not true that $F << 1^* \oplus b$.

(4)A weak positive implicative subset of M may not be a weak hyper MV-filter of M . Consider Example 3.8. Then $F = \{0, a, 1\}$ is a weak positive implicative subset of M. But F is not a weak hyper MV-filter of M, since $F \subseteq 0^* \oplus b$ and $0 \in F$ but $b \notin F$.

(5) A weak hyper MV-filter of M may not be a weak positive implicative subset of M. Consider Example 3.4. Then $F = \{1\}$ is a weak hyper MV-filter of M. But F is not a positive implicative subset of M, since $F \subseteq (1^* \oplus b) \oplus b$ and $F \subseteq b^* \oplus b$ but it is not true that$F << 1^* \oplus b$.

Definition 3.10. Let F be a proper (weak) hyper MV-filter of hyper MV-algebra $< M, \oplus, *, 0 >$. F is called a (weak) maximal hyper MV-filter of M, if $F \subseteq J \subseteq M$ for some (weak)hyper MV-filter of M, then $F = J$ or $J = M$.

Example 3.11. Consider Example 3.8. Then $F = \{a, b, 1\}$ is a weak maximal hyper MV-filter of M.

Remark 3.12. If F is a maximal hyper MV-filter of M, then F may not be a weak maximal hyper MV-filter of M . Consider the following example.

Example 3.13. Let $M = \{0, a, b, c, 1\}$. Consider the following tables:

$\oplus$	0	a	b	c	1
0	$\{0\}$	$\{0,a\}$	$\{0,a,b\}$	$\{0,c\}$	$\{0,a,b,c,1\}$
a	$\{0,a\}$	$\{0,a\}$	$\{0,a,b,c,1\}$	$\{0,a,c\}$	$\{0,a,b,c,1\}$
b	$\{0,a,b\}$	$\{0,a,b,c,1\}$	$\{0,a,b,1\}$	$\{0,a,b,c\}$	$\{0,a,b,c,1\}$
c	$\{0,c\}$	$\{0,a,c\}$	$\{0,a,b,c\}$	$\{0,a,b,c,1\}$	$\{0,a,b,c,1\}$
1	$\{0,a,b,c,1\}$	$\{0,a,b,c,1\}$	$\{0,a,b,c,1\}$	$\{0,a,b,c,1\}$	$\{0,a,b,c,1\}$

$*$	0	a	b	c	1
	1	b	a	c	0

Then $< M, \oplus, *, 0 >$ is a hyper MV-algebra. $F = \{b, 1\}$ is a maximal hyper MV-filter of M. But it is not a weak maximal hyper MV-filter of M because $J = \{a, b, 1\}$ is a weak maximal hyper MV-filter of M and $F \subset J \subset M$.

Theorem 3.14. Let $< M, \oplus, *, 0 >$ be a hyper MV-algebra. Then every proper hyper MV-filter is contained in a maximal hyper MV-filter of M .

Proof: Let F be a proper hyper MV-filter of M and Σ be the collection of all proper hyper MV-filter J of M such that $F \subseteq J$. Then $F \in \Sigma$ and $\Sigma \neq \emptyset$. Σ is ordered set by $\subseteq$. Let $C = \{J_\alpha : \alpha \in \Gamma\}$ be a chain in Σ. We can show that $J = \bigcup_{\alpha \in \Gamma} J_\alpha$ is a hyper MV-filter of M. It is clear that $F \subseteq J$. If $J = M$ then $0 \in J$. Thus there exists $\alpha \in \Gamma$ such that $0 \in J_\alpha$. Hence $J_\alpha = M$ by Proposition 2.6, this is a contradiction. Therefore $J \neq M$ and then $J \in C$. Now the proof follows by Zorn's Lemma.

Corollary 3.15. Let $< M, \oplus, *, 0 >$ be a hyper MV-algebra. Then M has a maximal hyper MV-filter.

Proof: The proof follows from Proposition 2.5 and Theorem 3.14.

Theorem 3.16. Let $f : M_1 \to M_2$ be an epimorphism of hyper MV-algebras. Then
(1) if F is a maximal hyper MV-filter of M_1 which contains $\ker f$, then $f(F)$ is a maximal hyper MV-filter of M_2.

(2) if F is a maximal hyper MV-filter of M_2, then $f^{-1}(F)$ is a maximal hyper MV-filter of M_1 which contains $\ker f$.
(3) The map $F \longmapsto f(F)$ is one to one corresponding between the maximal hyper MV-filters of M_1 containing $\ker f$ and maximal hyper MV-filters of M_2 .

Proof: (1) Let $f(F) = M_2$. Since $F \neq M_1$, there exists $x \in M_1$ such that $x \notin F$. So $f(x) \in M_2 = f(F)$. Thus there exists some $a \in F$ such that $f(x) = f(a)$ and then

$1 = f(1) \in f(x) \oplus f(a)^* = f(x \oplus a^*)$. So there exists $t \in a^* \oplus x$ such that $f(t) = 1$, i.e., $t \in \ker f \subseteq F$. Then $F << a^* \oplus x$. Thus $x \in F$ which is a contradiction. Hence $f(F) \neq M_2$. Since F is a hyper MV-filter of M_1 which contains $\ker f$, then $f(F)$ is a hyper MV-filter of M_2 by Theorem 2.10 (6). Let J be a hyper MV-filter of M_2 such that $f(F) \subseteq J \subseteq M_2$. By Theorem 2.10 (3), $f^{-1}(J)$ is a hyper MV-filter of M_1 such that $F \subseteq f^{-1}(J) \subseteq f^{-1}(M_2) = M_1$. Thus $f^{-1}(J) = F$ or $f^{-1}(J) = M_1$. Hence $J = f(F)$ or $J = f(M_1) = M_2$, i.e., $f(F)$ is a maximal hyper MV-filter of M_2.

(2) Let $f^{-1}(F) = M_1$. Since $F \neq M_2$, there exists $y \in M_2$ such that $y \notin F$. So $f^{-1}(y) \in M_1 = f^{-1}(F)$. Thus there exists some $x \in M_1$ such that $y = f(x) \in F$ which is a contradiction. Hence $f^{-1}(F) \neq M_1$. Similarly to the part (1), we can show that $f^{-1}(F)$ is a maximal hyper MV-filter ofM_1 .

(3) The proof is straightforward by (1), (2).

Remark 3.17. Let $f : M_1 \to M_2$ be an epimorphism of hyper MV-algebras. If F is a weak maximal hyper MV-filter of M_1 which contains $\ker f$, then $f(F)$ may not be a weak maximal hyper MV-filter of M_2 . Consider the following example.

Example 3.18. Let $< M_1, \oplus_1, *_1, 0 >$ be a hyper MV-algebra which is defined in Example 3.8. Then $F = \{a, b, 1\}$ is a weak maximal hyper MV-filter of M_1 . Let $M_2 = \{0, 1\}$ .Consider the following tables:

$\oplus_2$	0	1
0	{0}	{0,1}
1	{0,1}	{0,1}

$*_2$	0	1
	1	0

Then $< M_2, \oplus_2, *_2, 0 >$ is a hyper MV-algebra. Define$f : M_1 \to M_2$ by $f(0) = 0$, $f(a) = 0$, $f(b) = 1$ and $f(1) = 1$. Then$f$ is an epimorphism and $\ker f = \{b, 1\}$. But $f(F) = \{0, 1\}$ is not a weak maximal hyper MV-filter of M_2 .

4 Quotient hyper MV-algebras

Let $< M, \oplus, *, 0 >$ be a hyper MV-algebra. From now on, we will show it by M .

Definition 4.1.[1] Let $\sim$ be an equivalence relation on M and $A, B \subseteq M$. Then
(i)$A \sim B$ if there exist $a \in A$ and $b \in B$such that $a \sim b$,
(ii) $A \approx B$ if for all $a \in A$ there exists $b \in B$ such that $a \sim b$ and for all $b \in B$ there exists $a \in A$ such that $a \sim b$.

Proposition 4.2. Let $\sim$ be an equivalence relation on M and $A, B \subseteq M$.
(i) if $A \approx B$ and $B \approx C$, then $A \approx C$ (see[1]),
(ii) if $A \approx B$ and $B \sim C$, then $A \sim C$,
(iii) if $A \sim B$ and $B \approx C$, then $A \sim C$.

Proof: The proof is clear.

Definition 4.3. Let $\sim$ be an equivalence relation on M such that for all $x, y, z \in M$
(i) if $x \sim y$, then $x^* \sim y^*$,
(ii) if $x \sim y$ and $z \sim w$, then $x \oplus z \approx y \oplus w$.
Then $\sim$ is called a hyper congruence (H- congruence) relation on M.

Proposition 4.4. Let $\sim$ be an equivalence relation on M. Then $\sim$ is a H-congruence relation on M if and only if for all $x, y, z \in M$
(1) if $x \sim y$, then $x^* \sim y^*$,
(2) if $x \sim y$, then $x \oplus z \approx y \oplus z$.

Proof: The proof is straightforward.

Lemma 4.5. Let $\sim$ be a H-congruence relation on M. If $x \oplus y^* \approx y \oplus y^*$, then $x^* \oplus y \sim \{1\}$.

Proof: Since $1 \in y \oplus y^*$, then $y \oplus y^* \sim \{1\}$. Using Proposition 4.2 (ii), we have $x^* \oplus y \sim \{1\}$.

Definition4.6. Let $\sim$ be a H-congruence relation on M such that

$$x^* \oplus y \sim \{1\} \text{ and } y^* \oplus x \sim \{1\} \text{ imply that } x \sim y \ .$$

Then $\sim$ is called a good H-congruence relation on M.

Remark 4.7. Let $\sim$ be a H-congruence relation on M . Then $\sim$ may not be a good H-congruence relation on M . Consider the following example.

Example 4.8. Let $M = \{0, a, b, 1\}$. Consider the following tables:

$\oplus$	0	a	b	1
0	$\{0\}$	$\{0, a\}$	$\{0, b\}$	$\{0, b, 1\}$
a	$\{0, a\}$	$\{0, a, b, 1\}$	$\{0, a, b\}$	$\{0, a, b, 1\}$
b	$\{0, b\}$	$\{0, a, b\}$	$\{0, b, 1\}$	$\{0, b, 1\}$
1	$\{0, b, 1\}$	$\{0, a, b, 1\}$	$\{0, b, 1\}$	$\{0, b, 1\}$

$*$	0	a	b	1
	1	a	b	0

Then $< M, \oplus, *, 0 >$ is a hyper MV-algebra. Define

$\sim = \{ (0,0), (a,a),\ (b,b),\ (1,1),\ (1,b),\ (b,1),\ (0,b),\ (b,0)\ ,\ (0,1),\ (1,0)\}$.

Then $\sim$ is a H-congruence relation on M. We have $a^* \oplus b \sim \{1\}$ and $b^* \oplus a \sim \{1\}$ but a and b are not related with respect $\sim$. Hence $\sim$ is not a good H-congruence relation on M .

Lemma 4.9. Let $\sim$ be a H-congruence relation on M . We denote the set of all equivalence class of $\sim$ by $\frac{M}{\sim}$ and for all $[x], [y] \in \frac{M}{\sim}$ define

$$[x]\overline{\oplus}[y] = \{[t] : t \in x \oplus y\} \text{ and } [x]^{\overline{*}} = [x^*]$$

Then "$\overline{\oplus}$ " and " $\overline{*}$ " are well defined.

Proof: Let $[x_1] = [x_2]$ and $[y_1] = [y_2]$. Then $x_1 \sim x_2$ and $y_1 \sim y_2$. Since $\sim$ is a H-congruence relation on M, we have $x_1 \oplus y_1 \approx x_2 \oplus y_2$.
Let $[t] \in [x_1]\overline{\oplus}[y_1]$. Then $[t] = [s]$ for some $s \in x_1 \oplus y_1$. From $x_1 \oplus y_1 \approx x_2 \oplus y_2$, we get that $s \sim u$ for some $u \in x_2 \oplus y_2$. Thus $[s] = [u]$ and then $[t] = [u]$. Therefore

$[x_1]\overline{\oplus}[y_1] \subseteq [x_2]\overline{\oplus}[y_2]$. Similarly, we have $[x_2]\overline{\oplus}[y_2] \subseteq [x_1]\overline{\oplus}[y_1]$. Hence " $\overline{\oplus}$ " is well defined. Let $[x_1] = [x_2]$. Then $x_1 \sim x_2$. Since $\sim$ is a H-congruence relation on M, we get $x_1^* \sim x_2^*$. Thus$[x_1^*] = [x_2^*]$. Hence " $\overline{*}$" is well defined.

Definition 4.10. Let $\sim$ be a good H-congruence relation on M. We define the relation $<<$ on $\frac{M}{\sim}$ by $[x] << [y]$ if and only if $[1] \in [x]^{\overline{*}}\overline{\oplus}[y]$.

Theorem 4.11. Let $\sim$ be a good H-congruence relation on M. Then $< \frac{M}{\sim}, \overline{\oplus}, \overline{*}, [0] >$ is a hyper MV-algebra and it is called the quotient hyper MV-algebra of M respect to $\sim$.

Proof: We show that $< \frac{M}{\sim}, \overline{\oplus}, \overline{*}, [0] >$ satisfies all conditions of Definition 2.1.

(1)
$$\begin{aligned}([x]^{\overline{*}}\overline{\oplus}[y])^{\overline{*}}\overline{\oplus}[y] &= \{[t] : t \in x^* \oplus y\}^{\overline{*}}\overline{\oplus}[y] \\ &= \{[t^*] : t \in x^* \oplus y\}\overline{\oplus}[y] \\ &= \{[s] : s \in t^* \oplus y \text{ } where \text{ } t \in x^* \oplus y\} \\ &= \{[s] : s \in (x^* \oplus y)^* \oplus y\} \\ &= \{[s] : s \in (y^* \oplus x)^* \oplus x\} = ([y]^{\overline{*}}\overline{\oplus}[x])\overline{\oplus}[x]\end{aligned}$$

(hMV7) Let $[x] << [y]$ and $[y] << [x]$. Since $[x] << [y]$, we have $[1] \in [x]^{\overline{*}}\overline{\oplus}[y]$. Then $[1] = [t]$ for some $t \in x^* \oplus y$. This implies that $x^* \oplus y \sim \{1\}$. Similarly, from $[y] << [x]$, we get that $y^* \oplus x \sim \{1\}$. Since $\sim$ is a good H-congruence relation on M, we have $x \sim y$. Hence $[x] = [y]$. The proof of the other conditions are clear.

Example 4.12. Let $M = \{0, a, b, 1\}$. Consider the following tables:

$\oplus$	0	a	b	1
0	$\{0\}$	$\{0, a\}$	$\{0, b\}$	$\{0, a, b, 1\}$
a	$\{0, a\}$	$\{0, a\}$	$\{0, a, b, 1\}$	$\{0, a, b, 1\}$
b	$\{0, b\}$	$\{0, a, b, 1\}$	$\{0, a, b, 1\}$	$\{0, a, b, 1\}$
1	$\{0, a, b, 1\}$	$\{0, a, b, 1\}$	$\{0, a, b, 1\}$	$\{0, a, b, 1\}$

$*$	0	a	b	1
	1	b	a	0

Then $< M, \oplus, *, 0 >$ is a hyper MV-algebra. Define

$$\sim = \{ \ (0, 0), (a, a), \ (b, b), \ (1, 1), \ (1, b), \ (b, 1), \ (0, a), \ (a, 0) \ \}.$$

Then $\sim$ is a good H-congruence relation on M. We can show that $[1] = [b] = \{1, b\}$, $[0] = [a] = \{0, a\}$ and $\frac{M}{\sim} = \{[0], [1]\}$.
Consider the following tables:

$\overline{\oplus}$	[0]	[1]
[0]	{[0]}	{[0],[1]}
[1]	{[0],[1]}	{[0],[1]}

$\overline{*}$	[0]	[1]
	[1]	[0]

$< \frac{M}{\sim}, \overline{\oplus}, \overline{*}, [0] >$ is a hyper MV-algebra.

Theorem 4.13. If $\sim$ is a good H-congruence relation on M, then $F = [1]$ is a hyper MV-filter of M.

Proof: It is clear that $1 \in F$. Let $F << x^* \oplus y$ and $x \in F$. So there exist $a \in F$ and $b \in x^* \oplus y$ such that $a << b$. Hence $1 \in a^* \oplus b$ and then $a^* \oplus b \sim \{1\}$. Since $a \in F$, we have $a \sim 1$. We get that $b^* \oplus a \approx b^* \oplus 1$ because $\sim$ is a H-congruence relation on M. By (hMV5) $1 \in b^* \oplus 1$, so we have $b^* \oplus 1 \sim \{1\}$. Then $b^* \oplus a \sim \{1\}$ by Proposition 4.2 (ii). We conclude that $a \sim b$ because $\sim$ is a good H-congruence relation on M. So $x^* \oplus y \sim b$, $a \sim b$ and $a \sim 1$ imply that $x^* \oplus y \sim \{1\}$. $x \in F$ imply that $x \sim 1$. Thus $y^* \oplus x \approx y^* \oplus 1$. Since $1 \in y^* \oplus 1$, we obtained $y^* \oplus 1 \sim \{1\}$. Then by Proposition 4.2 (ii) $y^* \oplus x \sim \{1\}$. We conclude that $x \sim y$. Hence $y \sim 1$ and then $y \in F$.

Theorem 4.14. Let $\sim$ be a good H-congruence relation on M. Then $\pi : M \to \frac{M}{\sim}$ defined by $\pi(x) = [x]$ is an epimorphism and $\ker \pi = F$ (π is called canonical homomorphism).

Proof: The proof is easy.

Theorem 4.15. Let $\sim$ be an equivalence relation on M, $F = [1]$ and $\Delta_M = \{(x, x) : x \in M\}$. Then
(i) if $x << y$, then $[x] << [y]$,
(ii) If $\sim = \Delta_M$, then $\frac{M}{\sim} \cong M$,
(iii) If $\sim = M \times M$, then $\frac{M}{\sim} = \{F\}$,
(iv) If $\sim = \Delta_M$, then we have $[x] << [y]$ if and only if $x << y$.

Proof: (i) The proof is clear.
(ii) Let $\sim = \Delta_M$. Then $[x] = \{x\}$. It is clear that π is an isomorphism.
(iii) Let $\sim = M \times M$. Then $[x] = M = [1]$ for all $x \in M$. So $\frac{M}{\sim} = \{F\}$.
(iv) Let $[x] << [y]$. Then $[1] \in [x]^* \overline{\oplus} [y]$. So $[t] = [1]$ for some $t \in x^* \oplus y$. Thus we have $\{1\} = \{t\}$ and then $1 \in x^* \oplus y$. Hence $x << y$. The converse of (iv) is obvious.

Notation: (i) The set of all hyper MV- filters of M is denoted by $HF(M)$,
(ii) The set of all hyper MV- filters of M containing F is denoted by $HF(M, F)$.

Theorem 4.16. Let $\sim$ be a good H-congruence relation on M and $F = [1]$. Then there exists a one to one corresponding between $HF(M, F)$ and $HF(\frac{M}{\sim})$.

Proof: Consider canonical homomorphism $\pi : M \to \frac{M}{\sim}$. If J is a hyper MV-filter of M containing F, then by Proposition 2.10 (3,6), the function $\varphi : HF(M, F) \to HF(\frac{M}{\sim})$ by $J \longmapsto \pi(J)$ is onto. Now, we show that φ is one to one. Let $A, B \in HF(M, F)$ and $\pi(A) = \pi(B)$. Assume that $x \in B$. Then $[x] \in \pi(B) = \pi(A)$. Thus there exists $y \in A$ such that $[x] = [y]$. So $x \sim y$. Since $\sim$ is a H-congruence relation on M, we have $x \oplus y^* \approx y \oplus y^*$. Using Lemma 4.5, we have $x \oplus y^* \sim \{1\}$. Then there exists $t \in x \oplus y^*$ such that $t \sim 1$, i.e., $t \in F$. Since $F \subseteq A$, we have $A << x \oplus y^*$. Thus $x \in A$. Hence $B \subseteq A$. Similarly, we can show that $A \subseteq B$. Therefore $A = B$ and φ is one to one.

Notation: Let $\sim$ be a good H-congruence relation on M, $F = [1]$ and J be a hyper MV-filter of M such that $F \subseteq J$. The set of all $[x]$, where $x \in J$ defined by $\frac{J}{\sim}$,i.e., $\frac{J}{\sim} = \{[x] : x \in J\}$.

Corollary 4.17. Let $\sim$ be a good H-congruence relation on M and $F = [1]$. Then $A \in HF(\frac{M}{\sim})$ if and only if there exists some $B \in HF(M, F)$ such that $A = \frac{B}{\sim}$.

Proof: It follows from Theorem 4.16.

Theorem 4.18. Let M_1 and M_2 be two hyper MV-algebras and let $\sim$ be a good H-congruence relation on M_1 and $F = [1]$. If $f : M_1 \to M_2$ is a homomorphism of hyper MV-algebras such that $F \subseteq \ker f$, then f induces a unique homomorphism $\overline{f} : \frac{M_1}{\sim} \to M_2$ by $\overline{f}([x]) := f(x)$ such that $Im\overline{f} = Imf$ and $\ker \overline{f} = \frac{\ker f}{\sim}$. Moreover $\overline{f}$ is an isomorphism if and only if f is surjective and $F = \ker f$.

Proof: First we show that $\overline{f}$ is well defined. Let $[x] = [y]$. Then $x \sim y$ which implies that $x \oplus y^* \approx y \oplus y^*$. Using Lemma 4.5, we have $x \oplus y^* \sim \{1\}$. Then there exists $t \in x \oplus y^*$ such that $t \sim 1$, i.e., $t \in F \subseteq \ker f$. So

$$1 = f(t) \in f(x \oplus y^*) = f(x) \oplus f(y^*) = f(x) \oplus f(y)^*.$$

Hence $f(x) << f(y)$. Similarly , we can show that $f(y) << f(x)$. Therefore $f(y) = f(x)$ by (hMV7) and hence $\overline{f}$ is well defined.
It is clear that $\overline{f}$ is a homomorphism. We have

$$\begin{aligned} [x] \in \ker \overline{f} & \text{ if and only if } \overline{f}\ ([x]) = 1 \\ & \text{ if and only if } f(x) = 1 \\ & \text{ if and only if } x \in Ker\ f. \end{aligned}$$

Hence $\ker \overline{f} = \frac{\ker f}{\sim}$. It is clear that $Im\overline{f} = Imf$. So $\overline{f}$ is a surjective homomorphism if and only if f is a surjective homomorphism.

Let $\overline{f}$ be one to one and $x \in \ker f$. Then $[x] \in \ker \overline{f}$, i.e., $x \sim 1$ and we have $x \in F$. Hence $\ker f \subseteq F$ and then $\ker f = F$.

Conversely, let $\ker f = F$ and $[x] \in \ker \overline{f}$. Then $x \in \ker f$ which implies that $x \sim 1$. Thus $[x] = [1]$. Hence $\ker \overline{f} = \{[1]\}$ and $\overline{f}$ is one to one by Theorem 2.10 (5).

Corollary 4.19. (First isomorphism theorem) Let M_1 and M_2 be two hyper MV-algebras and let $\sim$ be a good H-congruence relation on M_1 such that $F = [1]$. If $f : M_1 \to M_2$ is a homomorphism of hyper MV-algebras and $F = \ker f$, then $\frac{M_1}{\sim} \cong Im\ f$.

Proof: The proof follows from Theorem 4.18.

Definition 4.20. Let $\emptyset \neq S \subseteq M$, then the intersection of all hyper MV-filters containing S is called the hyper MV-filter generated by S and it denoted by $< S >$.

Corollary 4.21. (Second isomorphism theorem) let $\sim_1$ and $\sim_2$ be two good H-congruence relations on M. Then $\frac{M}{\sim_2}$ is homomorphic image of $\frac{M}{\sim_1 \cap \sim_2}$.

Proof: First, we define a relation $\sim_3$ on M as follows:

$$x \sim_3 y \text{ if and only if } x \sim_1 y \text{ and } x \sim_2 y.$$

Then we can show that $\sim_3$ is a H-good congruence relations on M. Hence $\frac{M}{\sim_1 \cap \sim_2}$ is a hyper MV-algebra by Theorem 4.11. Define $f : \frac{M}{\sim_1 \cap \sim_2} \to \frac{M}{\sim_2}$ by $f([x]_3) = [x]_2$. Clearly$f$ is an epimorphism.

Corollary 4.22. Let M_1 and M_2 be two hyper MV-algebras and let $\sim_1$ and $\sim_2$ be two good H-congruence relations on M_1 and M_2 respectively such that $F = [1]_1$ and $J = [1]_2$. If $f : M_1 \to M_2$ is a homomorphism of hyper MV-algebras such that $f(F) \subseteq J$, then f induces a homomorphism $\overline{f}$ such that the following diagram is commutative

$$\begin{array}{ccc} M_1 & \to & M_2 \\ \downarrow & & \downarrow \\ \frac{M_1}{\sim_1} & \to & \frac{M_2}{\sim_2} \end{array} \qquad (1)$$

Moreover, if f is surjective and $f(F) = J$, then $\overline{f}$ is an isomorphism.

Proof: Define $g : M_1 \to \frac{M_2}{\sim_2}$ by $g(x) = [f(x)]_2$. It is clear that g is a homomorphism. Let $x \in \ker g$. Then $g(x) = [1]_2$. So we have $[f(x)]_2 = [1]_2$ which implies $f(x) \sim_2 1$, i.e., $f(x) \in J$. Hence $x \in f^{-1}(J)$. Therefore $\ker g \subseteq f^{-1}(J)$. Similarly, we can show that $\ker g \supseteq f^{-1}(J)$ and then $F \subseteq f^{-1}(J) = \ker g$.
Now, by Theorem 4.18 there exists a unique homomorphism $\overline{f} : \frac{M}{\sim_1} \to \frac{M}{\sim_2}$ such that $\overline{f}([x]_1) := g(x) = [f(x)]_2$ which means that diagram (1) is commutative. The last part of Corollary follows from Theorem 4.19.

Corollary 4.23. (Third isomorphism theorem) let $\sim_1$ and $\sim_2$ be two good H-congruence relations on M, $F = [1]_1$ and $J = [1]_2$ be such that $F \subseteq J$. Then $\frac{(\frac{M}{\sim_1})}{\sim_3} \cong \frac{M}{\sim_2}$ for some good H-congruence relation on $\frac{M}{\sim_1}$.

Proof: We define a relation$\sim_3$ on $\frac{M}{\sim_1}$ as follows:

$$[x]_1 \sim_3 [y]_1 \text{ if and only if } x \sim_2 y.$$

We see that $\sim_3$ is a good H-congruence relation on $\frac{M}{\sim_1}$ and $[[1]_1]_3 = \frac{J}{\sim_1}$. Thus $\frac{(\frac{M}{\sim_1})}{\sim_3}$ is a hyper MV-algebra by Theorem 4.11.
Define $f : \frac{M}{\sim_1} \to \frac{M}{\sim_2}$ by $f([x]_1) := [x]_2$. First we show that f is well defined. Let $[x]_1 = [y]_1$. Then $x \sim_1 y$ which implies that $x \oplus y^* \approx_1 y \oplus y^*$. Now by Lemma 4.5, we have $x \oplus y^* \sim_1 \{1\}$. Then there exists $t \in x \oplus y^*$ such that $t \sim_1 1$, i.e., $t \in F \subseteq J$ and hence $x \oplus y^* \sim_2 \{1\}$. Similarly, we can show that $y \oplus x^* \sim_2 \{1\}$. Therefore $[x]_2 = [y]_2$ because $\sim_2$ is good H-congruence relations on M. So f is well defined.
Since f is an epimorphism and $\ker f = \frac{J}{\sim_1}$. By Theorem 4.19, we have $\frac{(\frac{M}{\sim_1})}{\sim_3} \cong \frac{M}{\sim_2}$

Definition 4.24. Let J be a positive implicative subset of M which is a hyper MV-filter of M. Then J is called a positive implicative hyper MV-filter of M.

Notation: (i) The set of all positive implicative hyper MV- filters of M is denoted by *PIHF*(M),
(ii) The set of all positive implicative hyper MV- filters of M containing F is denoted by *PIHF*(M, F).

Theorem 4.25. Let $\sim$ be a good H-congruence relation on M and $F = [1]$. If $J \in$ *PIHF*(M, F), then $\frac{J}{\sim} \in$ *PIHF*$(\frac{M}{\sim})$.

Proof: Suppose

$$\frac{J}{\sim} << ([x]^{\overline{*}} \overline{\oplus} [y]) \overline{\oplus} [z] \text{ and } \frac{J}{\sim} << [y]^{\overline{*}} \overline{\oplus} [z].$$

Then there exist $[t] \in ([x]^{*}\overline{\oplus}[y])\overline{\oplus}[z]$ and $[r] \in \frac{J}{\sim}$ such that $[r] << [t]$. Thus there exist $s \in (x^{*} \oplus y) \oplus z$ and $w \in J$ such that $s \sim t$ and $w \sim r$.
Since $[r] << [t]$ and $[r] = [w]$, we have $[1] \in [w]^{*}\overline{\oplus}[t]$. So there exists $u \in r^{*} \oplus t$ such that $u \sim 1$, i.e., $u \in F \subseteq J$. Thus $J << w^{*} \oplus t$. Consequently $t \in J$. Since $s \sim t$ and $\sim$ is a good H-congruence relation on M, then $s \oplus t^{*} \approx t \oplus t^{*}$. By Lemma 4.5, we have $1 \sim s \oplus t^{*}$. Then there exists $a \in s \oplus t^{*}$ such that $a \sim 1$, i.e., $a \in F \subseteq J$. Thus $J << s \oplus t^{*}$. Consequently, $s \in J$. Hence $J << (x^{*} \oplus y) \oplus z$. Similarly, we can show that $J << y^{*} \oplus z$. Since J is a positive implicative hyper MV-filter of M we have $J << x^{*} \oplus z$ and hence $\frac{J}{\sim} << [x]^{*}\overline{\oplus}[z]$. Then $\frac{J}{\sim} \in IHF(\frac{M}{\sim})$.

Theorem 4.26. Let $\sim$ be a good H-congruence relation on M, $F = [1]$ and $K \in PIHF(\frac{M}{\sim})$. Then there exists $J \in PIHF(M, F)$ such that $K = \frac{J}{\sim}$.

Proof: Define $J = \{x \in M : [x] \in K\}$. Since $[1] \in K$, then $1 \in J$. Using Theorem 2.10(3), then J is a hyper MV-filter of M , $F \subseteq J$ and $K = \frac{J}{\sim}$.

Assume that $J << (x^{*} \oplus y) \oplus z$ and $J << y^{*} \oplus z$. Then we can show that $K << ([x]^{*}\overline{\oplus}[y])\overline{\oplus}[z]$ and $K << [y]^{*}\overline{\oplus}[z]$ Since K is a positive implicative hyper MV-filter of $\frac{M}{\sim}$, then $K << [x]^{*}\overline{\oplus}[z]$. So there exist $[t] \in [x]^{*}\overline{\oplus}[z]$ and $[r] \in K$ such that$[r] << [t]$. Thus there exists $s \in x^{*} \oplus z$ such that $s \sim t$. Since $[r] << [t]$, we have $[1] \in [r]^{*}\overline{\oplus}[t]$. So there exists $u \in r^{*} \oplus t$ such that $u \sim 1$, i.e., $u \in F \subseteq J$. Thus $J << r^{*} \oplus t$ and then $t \in J$. Since $s \sim t$ and $\sim$ is a good H-congruence relation on M, then $s \oplus t^{*} \approx t \oplus t^{*}$. By Lemma 4.5, we have $\{1\} \sim s \oplus t^{*}$. Thus $J << s \oplus t^{*}$. Consequently, $s \in J$. Hence $J << x^{*} \oplus z$ and J is a positive implicative hyper MV-filter of M.

Corollary 4.27. Let $\sim$ be a good H-congruence relation on M and $F = [1]$. Then there exists a one to one correspondence between $PIHF(M, F)$ and $PIHF(\frac{M}{\sim})$.

Proof: It is proved by Theorem 4.25 and Theorem 4.26.

Definition 4.28. The hyper MV-algebra M is called simple, if it has only two hyper MV-filters $\{1\}$ and M.

Example 4.29. Let $M = \{0, a, b, 1\}$. Consider the following tables:

$\oplus$	0	a	b	1
0	$\{0\}$	$\{0, a\}$	$\{0, b\}$	$\{0, a, b, 1\}$
a	$\{0, a\}$	$\{0, a, b, 1\}$	$\{0, a, b\}$	$\{0, a, b, 1\}$
b	$\{0, b\}$	$\{0, a, b\}$	$\{0, a, b, 1\}$	$\{0, a, b, 1\}$
1	$\{0, a, b, 1\}$	$\{0, a, b, 1\}$	$\{0, a, b, 1\}$	$\{0, a, b, 1\}$

$*$	0	a	b	1
	1	a	b	0

Then $M = \{0, a, b, 1\}$ is a simple hyper MV-algebra.

Remark 4.30. The hyper MV-algebra M is simple if and only if $\{1\}$ is a maximal hyper MV-filter of M .

Theorem 4.31. Let $\sim$ be a good H-congruence relation on M and $F = [1]$. Then $\frac{M}{\sim}$ is simple if and only if F is a maximal hyper MV-filter of M.

Proof: Let $\frac{M}{\sim}$ be simple. By Remark 4.30 $\{[1]\}$ is a maximal hyper MV-filter of $\frac{M}{\sim}$. Then

$\{[1]\} \neq \frac{M}{\sim}$. Thus there exists some $[x] \in \frac{M}{\sim}$ such that $[x] \neq [1]$. So we have $x \notin F$. Hence $F \neq M$. Assume that F is not a maximal hyper MV-filter of M. By Theorem 3.14, there exists a maximal hyper MV-filter J of M Such that $F \subset J$. Consider canonical epimorphism $\pi : M \to \frac{M}{\sim}$ in which $\ker \pi = F$. Using Theorem 3.16 (1) $\frac{J}{\sim}$ is a maximal hyper MV-filter of $\frac{M}{\sim}$. On the other hand, we have $\{[1]\} = \{[x] : x \in F\} \subseteq \{[x] : x \in J\} = \frac{J}{\sim}$. So $\{[1]\} = \frac{J}{\sim}$ .Then for all $x \in J$, we have $x \sim 1$, i.e., $x \in F$. Hence $J \subset F$ which is a contradiction.

Conversely, let F be a maximal hyper MV-filter of M. Assume that $\frac{M}{\sim}$ is not simple. Then by Remark 4.30 $\{[1]\}$ is not a maximal hyper MV-filter of $\frac{M}{\sim}$. By Theorem 3.14, there exists a maximal hyper MV-filter K of $\frac{M}{\sim}$ Such that $\{[1]\} \subset K$. By Theorem 3.16 (2) $\pi^{-1}(K)$ is a maximal hyper MV-filter of M such that $F \subseteq \pi^{-1}(K)$. So $F = \pi^{-1}(K)$. Thus $\{[1]\} = \pi(F) = K$ which is a contradiction.

5 The fundamental relation on hyper MV-algebras

Definition 5.1. [1] Let $\sim$ be an equivalence relation on M and A, B be non-empty subsets of M. Then $A \equiv B$ if for all $a \in A$ and for all $b \in B$, we have $a \sim b$.

Definition5.2. Let $\sim$ be an equivalence relation on M such that for all $x, y, z \in M$
(i) if $x \sim y$, then $x^* \sim y^*$,
(ii) if $x \sim y$ and $z \sim w$, then $x \oplus z \equiv y \oplus w$.
Then $\sim$ is called a strongly H- congruence relation on M.

Lemma 5.3. Let $\sim$ be an equivalence relation on M and A, B be non-empty subsets of M. If $A \equiv B$ and $B \equiv C$, then $A \equiv C$.

Proof: It is clear.

Lemma 5.4. Let $\sim$ be an equivalence relation on M. Then $\sim$ is a strongly H-congruence relation on M if and only if for all $x, y, z \in M$
(1) if $x \sim y$, then $x^* \sim y^*$,
(2) if $x \sim y$, then $x \oplus z \equiv y \oplus z$.

Proof: The proof follows from Lemma 5.3.

Proposition 5.5. If $\sim$ is a strongly H-congruence relation on M, then $\sim$ is a H-congruence relation on M.

Proof: The proof is straightforward.

Remark 5.6: If $< M, \oplus, *, 0 >$ is a hyper MV-algebra such that for all $x, y \in M$, $x \oplus y$ is a singleton as $\{a_{xy}\}$, then $< M, +, *, 0 >$ is an MV-algebra where $x + y = a_{xy}$.

Theorem 5.7. Let $\sim$ be a good and strongly H-congruence relation on M. Then $\frac{M}{\sim}$ is an MV-algebra.

Proof: Since $\sim$ is a good and strongly H-congruence relation on M, we have $[x]\overline{\oplus}[y] = \{[t] : t \in x\oplus y \equiv x\oplus y\}$ for all $x, y \in M$. Therefore $Card([x]\overline{\oplus}[x]) = 1$. Hence $\frac{M}{\sim}$ is an MV-algebra.

Example 5.8. Let $M = \{0, a, b, 1\}$. Consider the following tables:

$\oplus$	0	a	b	1
0	$\{0\}$	$\{0, a\}$	$\{b\}$	$\{b, 1\}$
a	$\{0, a\}$	$\{0, a\}$	$\{b, 1\}$	$\{b, 1\}$
b	$\{b\}$	$\{b, 1\}$	$\{b, 1\}$	$\{b, 1\}$
1	$\{b, 1\}$	$\{b, 1\}$	$\{b, 1\}$	$\{b, 1\}$

$*$	0	a	b	1
	1	b	a	0

Then $< M, \oplus, *, 0 >$ is a hyper MV-algebra. Define

$$\sim = \{\ (0, 0), (a, a),\ (b, b),\ (1, 1),\ (1, b),\ (b, 1),\ (0, a),\ (a, 0)\ \}.$$

Then $\sim$ is a good and strongly H-congruence relation on M. We can show that $F = [1] = [b] = \{1, b\}$, $[0] = [a] = \{0, a\}$ and $\frac{M}{\sim} = \{[0], [1]\}$.
Consider the following tables:

$\overline{\oplus}$	$[0]$	$[1]$
$[0]$	$\{[0]\}$	$\{[1]\}$
$[1]$	$\{[1]\}$	$\{[1]\}$

$\overline{*}$	$[0]$	$[1]$
	$[\ 1]$	$[0]$

$< \frac{M}{\sim}, \overline{\oplus}, \overline{*}, [0] >$ is a an MV-algebra.

Remark 5.9. (i) The hyper operation " $\oplus$ " is called the sum of x, y .
(ii) The hyper operation $x \otimes y = (x^* \oplus y^*)^*$ is called the product of x, y .

Lemma 5.10. Let $< M, \oplus, *, 0 >$ be a hyper MV-algebra. Then
(1) $a \in a \otimes 1$,
(2) $(A \oplus B)^* = A^* \otimes B^*$,
(3) $(\oplus_{i=1}^{m}(\otimes_{j=1}^{n} x_{j_i}))^* = \otimes_{i=1}^{m}(\otimes_{j=1}^{n} x_{j_i})^*$

Proof: (1) easily is proved by Proposition 2.2 (11).
For (2) we have

$$\begin{aligned} A^* \otimes B^* &= \bigcup\{a^* \otimes b^* : a \in A, b \in B\} \\ &= \bigcup\{(a \oplus b)^* : a \in A, b \in B\} \\ &= \bigcup(\{a \oplus b : a \in A, b \in B\})^* \\ &= (A \oplus B)^* \end{aligned}$$

(3) is proved (2) and using induction on m .

Definition 5.11. Let M be a hyper MV-algebra and U be the set of all the finite sum of finite products of elements of M. Then

(i) We define the relation γ on M as follows:

$$a\gamma b \text{ if and only if } \{a, b\} \subseteq u \text{ where } u \in U.$$

(ii) We define the relation γ^* on M as follows:
$a\gamma^* b$ if and only if there exist $z_1, \dots, z_{m+1} \in M$ with $z_1 = a$, $z_{m+1} = b$ and $u_i \in U$ such that$\{z_i, z_{i+1}\} \subseteq u_i$ for $i = 1, \dots, m$ (In fact γ^* is the transitive cluser of the relation γ).

γ^* is called the fundamental relation on$M$.

Theorem 5.12. The relation γ^* is an equivalence relation on M.

Proof: Since $a \in a \otimes 1$ by Lemma 5.10(1), then

$$\{a\} \subseteq a \otimes 1 \subseteq 0 \oplus (a \otimes 1) \subseteq (0 \otimes 1) \oplus (a \otimes 1).$$

Hence $a\gamma^* a$.
Let $a\gamma^* b$. Thus there exist $z_1, \dots, z_{m+1} \in M$ with $z_1 = a$, $z_{m+1} = b$ and $u_i \in U$ such that $\{z_i, z_{i+1}\} \subseteq u_i$ for $i = 1, \dots, m$. We define $y_i := z_{m-i+2}$ for $1 \leq i \leq m+1$. Then we have $\{y_i, y_{i+1}\} \subseteq v_i$ for some $v_i \in U$ for $i = 1, \dots, m$.
Let $a\gamma^* b$ and $b\gamma^* c$. Thus there exist $x_1, \dots, x_{n+1} \in M$ with $x_1 = a$, $x_{n+1} = b$ and $u_i \in U$ such that $\{x_i, x_{i+1}\} \subseteq u_i$ for $1 \leq i \leq n$ and there exist $y_1, \dots, y_{m+1} \in M$ with $y_1 = a$, $y_{m+1} = b$ and $v_j \in U$ such that $\{y_j, y_{j+1}\} \subseteq v_j$ for $1 \leq j \leq m$. We define $z_i := x_i$ for all $1 \leq i \leq n$ and $z_i := y_i$ for all $n+1 \leq i \leq m+1$. Then we can show that $\{z_i, z_{i+1}] \subseteq u_i$ for all $1 \leq i \leq n$ and $\{z_i, z_{i+1}\} \subseteq v_i$ for all $n+1 \leq i \leq m$. So $a\gamma^* c$. Hence γ^* is an equivalence relation on M.

Remark 5.13. We denote the equivalence class of the element a by $\gamma^*(a)$ and we call it the fundamental class of a.

Theorem 5.14. Let M be a hyper MV-algebra. Then the equivalence relation γ^* is the smallest good H-congruence relation defined on M such that $\frac{M}{\gamma^*}$ is an MV-algebra.

Proof: First we prove that γ^* is a strongly H-congruence relation on M. Let $a\gamma^* b$ and $c \in M$. Then there exist $x_1, \dots, x_{n+1} \in M$ with $x_1 = a$, $x_{n+1} = b$ and $u_i \in U$ such that$\{x_i, x_{i+1}\} \subseteq u_i$ for $1 \leq i \leq n$. Let $z_i \in x_i \oplus c$ for all $1 \leq i \leq n+1$. Therefore by Lemma 5.10 we have

$$\{z_i, z_{i+1}\} \subseteq (x_i \oplus c) \cup (x_{i+1} \oplus c) \subseteq u_i \oplus c \subseteq u_i \oplus (c \otimes 1) := v_i \text{ for all } 1 \leq i \leq n.$$

Thus $z_1 \gamma^* z_{n+1}$ for all $z_1 \in a \oplus c$ and for all $z_{n+1} \in b \oplus c$ and hence the condition (2) of Lemma 5.4 holds.
Also, we define $y_i = x_i^*$ for all $1 \leq i \leq n+1$. By Lemma 5.10 (3), we know u_i^* is finite product of elements of M and then $\{x_i, x_{i+1}\} \subseteq u_i^* \subseteq u_i^* \oplus (0 \otimes 1) \in U$ for all $1 \leq i \leq n$ thus $a^* \gamma^* b^*$. Hence the condition (1) of Lemma 5.4 holds and γ^* is a strongly H-congruence relation on M.
We claim that γ^* is a good H-congruence relation. Let $(a^* \oplus b) \sim \{1\}$ and $(b^* \oplus a) \sim \{1\}$. Since γ^* is strongly H-congruence relation on M, we have $(a^* \oplus b)^* \sim \{0\}$, $(b^* \oplus a)^* \sim \{0\}$ and then$(a^* \oplus b)^* \oplus b \equiv 0 \oplus b$, $(b^* \oplus a)^* \oplus a \equiv a \oplus 0$. Using Lemma 5.3 and (hMV4), we have $0 \oplus a \equiv 0 \oplus b$. Since $a \in 0 \oplus a$ and $b \in 0 \oplus b$ by Proposition 2.2 (11)$a\gamma^* b$.
Since γ^* is good and strongly H-congruence relation on M, then we get that $\frac{M}{\gamma^*}$ is an MV-algebra by Theorem 5.7.
Let σ be a good H-congruence relation on M such that $\frac{M}{\sigma}$ is an MV-algebra, then we can write for all $a, b \in M$

$$\sigma(a) \overline{\oplus} \sigma(b) = \sigma(c) \qquad \text{for all } c \in a \oplus b \ (1) \ ,$$

$$\sigma(a) \overline{\otimes} \sigma(b) = \sigma(d) \qquad \text{for all } d \in a \otimes b \ (2).$$

let $u = \oplus_{i=1}^{m}(\otimes_{j=1}^{n} x_{j_i}) \in U$. Then there exist $a_i \in M$ such that $\otimes_{j=1}^{n} x_{j_i} \subseteq \sigma(a_i)$ for all $1 \leq i \leq n$ by (2). Using relation (1), we get

$$u \subseteq \oplus_{i=1}^{m} \sigma(a_i) = \sigma(a) \text{ for all } a \in \oplus_{i=1}^{m} a_i \ .$$

And hence $\gamma \subseteq \sigma$. Since σ is transitive we obtain$\gamma^* \subseteq \sigma$. This means that it is the smallest good H-congruence relation defined on M such that $\frac{M}{\gamma^*}$ is an MV-algebra.

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Shokoofeh Ghorbani
Department of Mathematics
Shahid Bahonar University of Kerman
Kerman, Iran
sh.ghorbani@graduate.uk.ac.ir

Esfandiar Eslami
Department of Mathematics
Shahid Bahonar University of Kerman
Kerman, Iran
eeslami@mail.uk.ac.ir

Abbas Hasankhani
Department of Mathematics
Shahid Bahonar University of Kerman
Kerman, Iran
abhasan@mail.uk.ac.ir