POSITIVE IMPLICATIVE HYPER K-IDEALS II

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Abstract. In this note first we define the notions of positive implicative hyper K-ideals of types 1, 2,..., 26, 27 . Also we define the notions of commutative hyper K-ideals of types 1, 2, ...,8, 9. Then we give many examples to show that these notions are different together. Finally we prove some theorems and obtain some related results . In particular we determine the relationships between the implicative hyper K-ideals and commutative hyper K-ideals of a hyper K-algebra of order 3, which satisfies the simple condition.

1 Introduction

The hyperalgebraic structure theory was introduced by F. Marty [7] in 1934. Imai and Iseki [5] in 1966 introduced the notion of BCK-algebra. Borzooei, Jun and Zahedi et.al. [3,9] applied the hyperstructure to BCK-algebra and introduced the concept of hyper K-algebra which is a generalization of BCK-algebra. Boorzooei and Zahedi [4] introduced 8 different types of positive implicative hyper$K$-ideals also Torkzadeh and Zahedi [8] introduced 4 different types of commutative hyper K-ideals . In this note we define 27 different types of positive implicative hyper K-ideals and 9 different types of commutative hyper K-ideals. Then we obtain some results as mentioned in the abstract.

2 Preliminaries

Definition 2.1. [3] Let H be a nonempty set and " $\circ$ " be a *hyperoperation* on H, that is " $\circ$ " is a function from $H \times H$ to $P^*(H) = P(H)\backslash\{\emptyset\}$. Then H is called a hyper k- algebra if it contains a constant "0" and satisfies the following axioms:
(HK1) $(x \circ z) \circ (y \circ z) < x \circ y$,
(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,
(HK3) $x < x$,
(HK4) $x < y,\ y < x \Rightarrow x = y$,
(HK5) $0 < x$.

for all $x, y, z \in H$, where $x < y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A < B$ is defined by $\exists a \in A$, $\exists b \in B$ such that $a < b$. Note that if $A, B \subseteq H$, then by $A \circ B$ we mean the subset $\bigcup_{\substack{a \in A \\ b \in B}} a \circ b$ of H.

From now $(H, \circ, 0)$ is a hyper K-ideal, unless otherwise is stated.

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Theorem 2.2. [3] For all $x, y, z \in H$ and for all non-empty subsets A, B and C of H the following statements hold:
(i) $x \circ y < z \Leftrightarrow x \circ z < y$,
(ii) $(x \circ z) \circ (x \circ y) < y \circ z$,
(iii) $x \circ (x \circ y) < y$,
(iv) $x \circ y < x$,
(v) $A \subseteq B$ implies $A < B$,
(vi) $x \in x \circ 0$,
(vii) $(A \circ C) \circ (A \circ B) < B \circ C$,
(viii) $(A \circ C) \circ (B \circ C) < (A \circ B)$,
(ix) $A \circ B < C \Leftrightarrow A \circ C < B$,
(x) $A \circ B < A$,
(xi) $(A \circ C) \circ B = (A \circ B) \circ C$.

Definition 2.3. [3] Let I be a nonempty subset of H and $0 \in I$. Then,
(i) I is called a weak hyper K-ideal of H if $x \circ y \subseteq I$ and $y \in I$ imply that $x \in I$, for all $x, y \in H$.
(ii) I is called a hyper K-ideal of H if $x \circ y < I$ and $y \in I$ imply that $x \in I$, for all $x, y \in H$.

Definition 2.4. [3] Let I be a nonempty subset of H. Then we say that I satisfies the additive condition if for all $x, y \in H$, $x < y$ and $y \in I$ imply that $x \in I$.

Definition 2.5. [3] Let H be a hyper K-algebra. An element $a \in H$ is called to be a left(resp. right) scalar if $|a \circ x| = 1$(resp. $|x \circ a| = 1$) for all $x \in H$.

Definition 2.6. [4] Let $H = \{0, 1, 2\}$ be a hyper K-algebra. We say that H satisfies the simple condition if the conditions $1 \not< 2$ and $2 \not< 1$ holds.

Definition 2.7. [1] A nonempty subset I of H is called an implicative hyper K -ideal if it satisfies: (i) $0 \in I$,
(ii) $(x \circ z) \circ (y \circ x) < I$ and $z \in I$ imply that $x \in I$, for all $x, y, z \in I$.

Theorem 2.8. Let I be a nonempty subset of H and $0 \in I$. Then the following statements are equivalent:
(i) $(x \circ y) \bigcap I \neq \emptyset$ and $y \in I$ imply that $x \in I$,
(ii) $(x \circ y) < I$ and $y \in I$ imply that $x \in I$.

3 Positive implicative hyper K-ideals

Definition 3.1. Let I be a nonempty subset of H such that $0 \in I$. Then I is called a positive implicative hyper K-ideal of
(i) type 1, if for all $x, y, z \in H$, $(x \circ y) \circ z) \subseteq I$ and $(y \circ z) \subseteq I$ imply that $(x \circ z) \subseteq I$,

(ii) type 2, if for all $x, y, z \in H$, $((x \circ y) \circ z) \subseteq I$ and $(y \circ z) \subseteq I$ imply that $(x \circ z) \bigcap I \neq \emptyset$,

(iii) type 3, if for all $x, y, z \in H$, $((x \circ y) \circ z) \subseteq I$ and $(y \circ z) \subseteq I$ imply that $x \circ z < I$,

(iv) type 4, if for all $x, y, z \in H$, $((x \circ y) \circ z) \subseteq I$ and $(y \circ z) \bigcap I \neq \emptyset$ imply that $(x \circ z) \subseteq I$,

(v) type 5, if for all $x, y, z \in H$, $((x \circ y) \circ z) \subseteq I$ and $(y \circ z) \bigcap I \neq \emptyset$ imply that $(x \circ z) \bigcap I \neq \emptyset$

(vi) type 6, if for all $x, y, z \in H$, $((x \circ y) \circ z) \subseteq I$ and $(y \circ z) \bigcap I \neq \emptyset$ imply that $x \circ z < I$,

(vii) type 7, if for all $x, y, z \in H$, $((x \circ y) \circ z) \subseteq I$ and $y \circ z < I$ imply that $x \circ z < I$,

(viii) type 8, if for all $x, y, z \in H$, $((x \circ y) \circ z) \subseteq I$ and $y \circ z < I$ imply that $(x \circ z) \bigcap I \neq \emptyset$,

(ix) type 9 , if for all $x, y, z \in H$, $((x \circ y) \circ z) \subseteq I$ and $y \circ z < I$ imply that $(x \circ z) \subseteq I$,

(x) type 10, if for all $x, y, z \in H$, $((x \circ y) \circ z) \bigcap I \neq \emptyset$ and $(y \circ z) \subseteq I$ imply that $(x \circ z) \bigcap I \neq \emptyset$,

(xi) type 11, if for all$x, y, z \in H$, $((x \circ y) \circ z) \bigcap I \neq \emptyset$ and $(y \circ z) \subseteq I$ imply that$(x \circ z) \subseteq I$,

(xii) type 12, if for all$x, y, z \in H$, $((x \circ y) \circ z) \bigcap I \neq \emptyset$ and $(y \circ z) \subseteq I$ imply that $x \circ z < I$,

(xiii) type 13, if for all$x, y, z \in H$, $((x \circ y) \circ z) \bigcap I \neq \emptyset$ and $(y \circ z) \bigcap I \neq \emptyset$ imply that $(x \circ z) \subseteq I$,

(xiv) type 14, if for all$x, y, z \in H$, $((x \circ y) \circ z) \bigcap I \neq \emptyset$ and $(y \circ z) \bigcap I \neq \emptyset$ imply that $(x \circ z) \bigcap I \neq \emptyset$,

(xv) type 15, if for all$x, y, z \in H$, $((x \circ y) \circ z) \bigcap I \neq \emptyset$ and $(y \circ z) \bigcap I \neq \emptyset$ imply that$x \circ z < I$,

(xvi) type 16, if for all$x, y, z \in H$, $((x \circ y) \circ z) \bigcap I \neq \emptyset$ and $y \circ z < I$ imply that $x \circ z < I$,

(xvii) type 17 , if for all $x, y, z \in H$, $((x \circ y) \circ z) \bigcap I \neq \emptyset$ and $y \circ z < I$ imply that $(x \circ z) \bigcap I \neq \emptyset$,

(xviii) type 18, if for all $x, y, z \in H$, $((x \circ y) \circ z) \bigcap I \neq \emptyset$ and $y \circ z < I$ imply that $(x \circ z) \subseteq I$,

(xix) type 19 , if for all $x, y, z \in H$, $((x \circ y) \circ z) < I$ and $(y \circ z) \bigcap I \neq \emptyset$ imply that $x \circ z < I$,

(xx) type 20, if for all $x, y, z \in H$, $((x \circ y) \circ z) < I$ and $(y \circ z) \bigcap I \neq \emptyset$ imply that $(x \circ z) \subseteq I$,

(xxi) type 21, if for all $x, y, z \in H$, $(x \circ y) \circ z) < I$ and $(y \circ z) \bigcap I \neq \emptyset$ imply that $(x \circ z) \bigcap I \neq \emptyset$,

(xxii) type 22, if for all $x, y, z \in H$, $((x \circ y) \circ z) < I$ and $(y \circ z) \subseteq I$ imply that $(x \circ z) \subseteq I$,

(xxiii) type 23 , if for all $x, y, z \in H$, $((x \circ y) \circ z) < I$ and $(y \circ z) \subseteq I$ imply that $x \circ z < I$,

(xxiv) type 24 , if for all $x, y, z \in H$, $((x \circ y) \circ z) < I$ and $(y \circ z) \subseteq I$ imply that $(x \circ z) \bigcap I \neq \emptyset$,

(xxv) type 25 , if for all $x, y, z \in H$, $((x \circ y) \circ z) < I$ and $y \circ z < I$ imply that $x \circ z < I$,

(xxvi) type 26 , if for all $x, y, z \in H$, $((x \circ y) \circ z) < I$ and $y \circ z < I$ imply that $(x \circ z) \bigcap I \neq \emptyset$,

(xxvii) type 27 , if for all $x, y, z \in H$, $((x \circ y) \circ z) < I$ and $y \circ z < I$ imply that $(x \circ z) \subseteq I$.

For simplicity of notation we use "PIHKI " instead of " Positive Implicative Hyper K-ideal" .

Theorem 3.2. Let I be a hyper K-ideal of H , $A \subseteq H$ and $b \in I$. Then $(A \circ b) \bigcap I \neq \emptyset$ implies that $(A \bigcap I) \neq \emptyset$.

Proof. Since $(A \circ b) \bigcap I \neq \emptyset$ and $b \in I$, then there exists $t \in (A \circ b) \bigcap I$. So $t \in (a \circ b)$ and $t \in I$ for some $a \in A$. Then $(a \circ b) \bigcap I \neq \emptyset$. Now I is a hyper K-ideal ,thus $a \in I$ and hence $A \bigcap I \neq \emptyset$.

Theorem 3.3 Let I be a hyper K-ideal of H . Then the following statements are equivalent:
(i) $(x \circ y) < I$,
(ii) $(x \circ y) \bigcap I \neq \emptyset$.

Proof. (i)$\Rightarrow$ (ii) Let $x \circ y < I$, then there exists $a \in I$ and $t \in x \circ y$ such that $t < a$. Thus $0 \in t \circ a$ and hence $(x \circ y) \circ a \bigcap I \neq \emptyset$. Therefore by Theorem 3.2 $(x \circ y) \bigcap I \neq \emptyset$.

(ii)$\Rightarrow$ (i) It is obvious.

Theorem 3.4. If I is a hyper K-ideal of H and A and B are nonempty subsets of H, then the following statements are equivalent:
(i) $(A \circ B) < I$,
(ii) $(A \circ B) \bigcap I \neq \emptyset$.

Proof.. The proof follows from Theorem 3.3.

Example 3.5. (i) The following table shows that a hyper K-algebra structure on $H = \{0, 1, 2\}$.

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1,2\}$	$\{0,1\}$	$\{0,2\}$
2	$\{2\}$	$\{1,2\}$	$\{0,1,2\}$

It is easy to check that $I = \{0, 2\}$ is a PIHKI hyper$K$-ideal of types 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 ,12 , 14, 15 ,16, 17, 19, 22, 23, 25, 26.

(ii) Let (X,$\star$,0) be a BCK-algebra and define a hyper opration $\circ$ on X by $x \circ y = \{x \star y\}$ for all $x, y \in I$. If I is a positive implicative ideal of the BCK-algebra of X , then (I,$\star$,0) is a PIHKI of types 1, 2, 3,..., 27 .

Theorem 3.6. Let $0 \in H$ be a right scalar element and I be a PIHKI of type 11,13,14,21,22 or 24. Then I is a hyper K-ideal .

Proof. Let $x, y \in H$, I be a PIHKI of type 11 , $(x \circ y) \bigcap I \neq \emptyset$ and $y \in I$. Since $0 \in H$ is a right scalar element , we have $((x \circ y) \circ 0) \bigcap I \neq \emptyset$ and $\{y\} = y \circ 0 \subseteq I$. Thus $\{x\} = x \circ 0 \subseteq I$, then $x \in I$. Therefore I is a hyper K-ideal. The proof of each of the PIHKI of types 13,14,20,21,22,24 is the same as above.

Example 3.7.(i) The following table shows a hyper K-algebra structure on $H = \{0, 1, 2\}$.

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1,2\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,1\}$	$\{1,2\}$
2	$\{1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$

Then $I = \{0, 1\}$ is a PIHKI of type 21, while I is not a hyper K-ideal, because $(2\circ1) \bigcap I \neq \emptyset$ and $1 \in I$,but $2 \notin I$. Also we see that $0 \in H$ is not a right scalar element.

(ii) Consider the following hyper K-algebra

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1,2\}$	$\{0,1\}$	$\{0,2\}$
2	$\{2\}$	$\{1,2\}$	$\{0,1,2\}$

We see that $0 \in H$ is not a right scalar element and $I = \{0, 2\}$ is PIHKI of types 11, 14, 22, 24 , while I is not a hyper K-ideal, because $(1 \circ 2) \bigcap I \neq \emptyset$ and $2 \in I$, but $1 \notin I$. $I = \{0, 1\}$ is a PIHKI of type 13 , while I is not a hyper K-ideal, because $2 \circ 1 \bigcap I \neq \emptyset$ and $1 \in I$,but $2 \notin I$.
Note that Example 3.7 shows that the condition "$0 \in H$ is a right scalar" is necessary in Theorem 3.6.

Theorem 3.8. Let $0 \in H$ be a right scalar element and I is additive. If I is a PIHKI of type 12, 15, 16, 19 or 23 , then I is a weak hyper K-ideal .

Proof. Let I be a PIHKI of type 12, $x, y \in H$, $(x \circ y) \subseteq I$ and $y \in I$. Since $0 \in H$ is a right scalar element $((x \circ y) \circ 0) \bigcap I \neq \emptyset$ and $(y \circ 0) \subseteq I$ imply that $x \circ 0 < I$. So there exists $t \in x \circ 0$ and $i \in I$ such that $t < i$, i.e. $x \circ o < i$. Therefore $x \circ i < 0$, hence there exists $k \in x \circ i$ such that $k < 0$. Since $0 < k$, thus k=0. Thus $0 \in x \circ i$ and hence $x < i$. Now since I is an additive , then we get that $x \in I$. Therefore I is weak hyper K-ideal. The proofs of PIHKI of types 15, 16, 19, 23 are the same as above.

Theorem 3.9. Let $0 \in H$ be a right scalar element of H and I is PIHKI of type 18, 20, 26 or 27 . Then I is a weak hyper K-ideal .

Proof. The proof is similar to the proof of Theorem 3.8.

Example 3.10. (i) Consider the following hyper K-algebra

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1,2\}$	$\{0,1\}$	$\{0,2\}$
2	$\{2\}$	$\{1,2\}$	$\{0,1,2\}$

In this example $I = \{0, 2\}$ is a PIHKI of types 12, 15, 16, 19, 23, while I is not a hyper K-ideal. Also we see that I is not additive and $0 \in H$ is not a right scalar.

(ii) The following table shows a hyper K-algebra structure on $H = \{0, 1, 2\}$.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0, 1\}$	$\{1, 2\}$
2	$\{2\}$	$\{0, 2\}$	$\{0, \}$

We see that $I = \{0, 1\}$ is a PIHKI of type 25 , but it is not of type 14. Since $((2\circ1)\circ0)\bigcap I \neq \emptyset$ and $(1\circ0)\bigcap I \neq \emptyset$, but $(2\circ0)\bigcap I = \emptyset$. Note that here the simple condition dose not satisfy.

Theorem 3.11. Let H be a hyper K-algebra of order 3 such that H satisfies the simple condition and $I \neq \{0\}$. Then I is a PIHKI of type 14 if and only if I is a PIHKI of type 25.

Proof. With out loss of generality assume that $I = \{0, 1\}$.
($\Rightarrow$) On the contrary, let$I = \{0, 1\}$ does not be a PIHKI of type 14. Then there exists $x, y, z \in H$ such that $((x \circ y) \circ z)\bigcap I \neq \emptyset$ and $(y \circ z)\bigcap I \neq \emptyset$, but $(x \circ z)\bigcap I = \emptyset$. Then $((x\circ y)\circ z) < I$ and $(y\circ z) < I$. So by hypothesis $2 = x\circ z < I = \{0, 1\}$ which is impossible, because H satisfies the simple condition.

($\Leftarrow$) On the contrary, let $I = \{0, 1\}$ does not be a PIHKI of type 25. Then there exists $x, y, z \in H$ such that $(x \circ y) \circ z) < I$, $y \circ z < I$ and $(x \circ z) \not< I$, then $(x \circ z)$=2. Since $((x \circ y) \circ z) < I = \{0, 1\}$, $(y \circ z) < I$ and H satisfies the simple condition then $((x \circ y) \circ z)\bigcap I \neq \emptyset$ and $(y \circ z)\bigcap I \neq \emptyset$. By hypothesis $(x \circ z)\bigcap I \neq \emptyset$, i.e. $2 \in I$, which is impossible.

Open problem 3.12. Is there an example of PIHKI of type 14 that it is not of PIHKI of type 25?

Notation : Let $I \subseteq H$ and $a \in I$. Then we let $I_a = \{x \in H|(x \circ a)\bigcap I \neq \emptyset\}$.

Theorem 3.13. Let H be a hyper K-algebra . Then I is a PIHKI of type 14 if and only if for all $a \in H$, I_a is a hyper K-ideal.

Proof. ($\Rightarrow$) Let for all $x, y, a \in H$, $(x \circ y)\bigcap I_a \neq \emptyset$ and $y \in I_a$. Then $(x \circ y) \circ a)\bigcap I \neq \emptyset$ and $(y \circ a)\bigcap I \neq \emptyset$. Since I is a PIHKI of type 14 , then $(x \circ a)\bigcap I \neq \emptyset$. Hence $x \in I_a$.

($\Leftarrow$) Let for all $x, y, a \in H$, $((x \circ y) \circ a)\bigcap I \neq \emptyset$ and $(y \circ a)\bigcap I \neq \emptyset$. Since I_a is a hyper K-ideal, we get that $x \in I$.

Theorem 3.14 . Let I be a PIHKI of type 14 and I be a hyper K-ideal. Then for all $a \in H$, I_a is the least hyper K-ideal containing $I\bigcup\{a\}$.

Proof. Let $x \in I$. Since $0 \in ((x \circ a) \circ x)$, we get that $((x \circ a) \circ x)\bigcap I \neq \emptyset$. Then $x \in I_a$ and hence $I \subseteq I_a$. Now $0 \in (a \circ a)\bigcap I \neq \emptyset$, then Hence $a \in I_a$.
Let B be a hyper K-ideal of H containing $I\bigcup\{a\}$. We show that $I_a \subseteq B$. If $x \in I_a$, then $(x\circ a)\bigcap I \neq \emptyset$. Since $I \subseteq B$ we get that $(x\circ a)\bigcap B \neq \emptyset$ and $a \in B$, then $x \in B$. Therefore $I_a \subseteq B$.

Theorem 3.15. Let I be a nonempty subsets of H. Then the following statements hold:
(i) If I is a PIHKI of type 4, then I is PIHKI of types 1,6,

(ii) If I is a PIHKI of type 5, then I is PIHKI of types 2,6 ,

(iii) If I is a PIHKI of type 6, then I is a PIHKI of type 3,

(iv) If I is a PIHKI of type 8, then I is a PIHKI of type 7,

(v) If I is a PIHKI of type 9, then I is PIHKI of types 7,8,

(vi) If I is a PIHKI of type 11, then I is PIHKI of types 10,12 ,

(vii) If I is a PIHKI of type 10, then I is a PIHKI of type 12,

(viii) If I is a PIHKI of type 13, then I is PIHKI of types 14,15,

(ix) If I is a PIHKI of type 14, then I is a PIHKI of 15,

(x) If I is a PIHKI of type 18, then I is a PIHKI of 16, 17,

(xi) If I is a PIHKI of type 17, then I is a PIHKI of type 16,

(xii) If I is a PIHKI of type 20, then I is a PIHKI of type 3,

(xiii) If I is a PIHKI of type 21, then I is a PIHKI of type 19,

(xiv) If I is a PIHKI of type 24, then I is a PIHKI of type 23,

(xv) If I is a PIHKI of type 22, then I a PIHKI of type 24,

(xvi) If I is a PIHKI of type 27 , then I is is a PIHKI of type 26.

Proof. The proof is straightforward.

Example 3.16. (i) The following table shows a hyper K-algebra structure on H.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1,2\}$

We can see that $I = \{0,1\}$ is a PIHKI of type 6 , while I is not PIHKI of type 4, because $((2 \circ 1) \circ 0) \subseteq I$ and $(1 \circ 0) \bigcap I \neq \emptyset$, but $(2 \circ 0) \not\subseteq I$.

(ii) The following table shows a hyper K-algebra structure on H.

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,2\}$	$\{1,2\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1,2\}$

We can see that $I=\{0,1\}$ is a PIHKI of type 7 , while I is not PIHKI of type 8, because $((2\circ 1)\circ o)\subseteq I$ and $(1\circ 0)<I$,but $(2\circ 0)\bigcap I=\emptyset$. Also $I=\{0,1\}$ is not a PIHKI of type 9. Since $((2\circ 1)\circ 0)\subseteq I$ and $(1\circ 0)<I$, but $(2\circ 0)\not\subseteq I$.

(iii) The following table shows a hyper K-algebra structure on H.

$\circ$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1,2\}$	$\{0,1\}$	$\{0,2\}$
2	$\{2\}$	$\{1,2\}$	$\{0,1,2\}$

$I=\{0,2\}$ is a PIHKI of type 10 , but I is not PIHKI of type 11, because $((0\circ 1)\circ 2)\bigcap I\neq\emptyset$ and $(1\circ 2)\subseteq I$, but $(0\circ 2)\not\subseteq I$.

(iv) The following table shows a hyper K-algebra structure on H.

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,2\}$	$\{1,2\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1,2\}$

$I=\{0,1\}$ is a PIHKI of type 12 , but I is not PIHKI of type 11. Since $((2\circ 1)\circ 0)\bigcap I\neq\emptyset$ and $(1\circ 0)\subseteq I$, but $(2\circ 0)\not\subseteq I$.

(v) the following table shows a hyper K-algebra structure on H.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,1\}$	$\{0,1,2\}$

$I=\{0,1\}$ is a PIHKI of type 15 , but I is not PIHKI of type 13. Since $((2\circ 1)\circ 2)\bigcap I\neq\emptyset$ and $(1\circ 2)\bigcap I\neq\emptyset$, but $(2\circ 2)\not\subseteq I$.

(vi) Consider Example 3.16(iii) . $I=\{0,2\}$ is a PIHKI of type 14, but I is not PIHKI of type 13, because $((0\circ 1)\circ 2)\bigcap I\neq\emptyset$ and $(1\circ 2)\bigcap I\neq\emptyset$,but $(0\circ 2)\not\subseteq I$.

(vii) The table in Example 3.16(ii) shows that $I=\{0,2\}$ is a PIHKI of type 15, but I is not PIHKI of type 14. Since $(2\circ 1)\circ 0)\bigcap I\neq\emptyset$ and $(1\circ 0)\bigcap I\neq\emptyset$, but $(2\circ 0)\bigcap I=\emptyset$.

(viii) The following table shows that a hyper K-algebra structure on H.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,2\}$

$I = \{0, 1\}$ is a PIHKI of type 16 and I is not PIHKI of type 18 , because $((2 \circ 1) \circ 2) \bigcap I \neq \emptyset$ and $(1 \circ 2) \bigcap I \neq \emptyset$,but $(2 \circ 2) \nsubseteq I$.

(ix) Consider Example 3.16(iii). We see that $I = \{0, 2\}$ is a PIHKI of type 17, while I is not a PIHKI of type 18. Since $((0 \circ 1) \circ 2) \bigcap I \neq \emptyset$ and $1 \circ 2 < I$, but $(0 \circ 2) \nsubseteq I$.

(x) Example 3.16(ii) shows that $I = \{0, 1\}$ is a PIHKI of type 16, while I is not a PIHKI of type 17 . Since $((2 \circ 1) \circ 0) \bigcap I \neq \emptyset$ and $(1 \circ 0) < I$, but $(2 \circ 0) \bigcap I = \emptyset$.

(xi) Consider Example 3.16(ii). We see that $I = \{0, 1\}$ is a of type 19, while I is not PIHKI of type 20 . Since $((2 \circ 1) \circ 2) \bigcap I \neq \emptyset$ and $1 \circ 2 \bigcap I \neq \emptyset$, but $(1 \circ 2) \nsubseteq I$.

(xii) Example 3.16(iii) shows that $I = \{0, 2\}$ is a PIHKI of type 24, while I is not a PIHKI of type 22 , because $((0 \circ 1) \circ 2) < I$ and $(1 \circ 2) \subseteq I$, but $(0 \circ 2) \nsubseteq I$.

(xiii) Consider Example 3.16(ii). We see that $I = \{0, 1\}$ is a PIHKI of type 25, but I is not a PIHKI of type 26 . Since $(2 \circ 1) \circ 0 < I$, $(1 \circ 0) < I$ and $(2 \circ 0) \bigcap I = \emptyset$.

(xiv) Example 3.16(iii) shows that $I = \{0, 2\}$ is a PIHKI of type 26, while I is not PIHKI of type 27 , because $(0 \circ 1) \circ 2 < I$ and $(1 \circ 2) < I$, but $(0 \circ 2) \nsubseteq I$.

(xv) Consider Example 3.16(iii) .We see that $I = \{0, 1\}$ is of types 2, 3 , while I is not a PIHKI type 1 . Since $((2 \circ 1) \circ 0) \subseteq I$ and $(1 \circ 0) \subseteq I$, but $(2 \circ 0) \nsubseteq I$.

Theorem 3.17. Let I is a hyper K-ideal of H. Then the following statements are equivalent:

(i) I is a PIHKI of type 14,

(ii) I is a PIHKI of type 15,

(iii) I is a PIHKI of type 16,

(iv) I is a PIHKI of type 17,

(v) I is a PIHKI of type 19,

(vi) I is a PIHKI of type 21,

(vii) I is a PIHKI of type 25,

(viii) I is a PIHKI of type 26.

Proof. By considering Theorem 3.4 , the proof is easy.

Theorem 3.18. Let I be a hyper K-ideal. Then the following statements are equivalent:

(i) I is a PIHKI of type 13,

(ii) I is a PIHKI of type 18,

(iii) I is a PIHKI of type 20,

(iv) I is a PIHKI of type 27.

Proof. By considering Theorem 3.4 , the proof is easy.

Theorem 3.19. Let I be a hyper K-ideal. Then the following statements are equivalent:

(i) I is a PIHKI of type 10,

(ii) I is a PIHKI of type 23,

(iii) I is a PIHKI of type 12,

(iv) I is a PIHKI of type 24.

Proof. By considering Theorem 3.4, the proof is easy.

4 Commutative hyper K-ideals

Definition 4.1. Let I be a nonempty subset of H such that $0 \in I$.
Then I is called a commutative hyper k-ideal of

(i) type 1 , if for all $x, y, z \in H$, $((x \circ y) \circ z) \bigcap I \neq \emptyset$ and $z \in I$ imply that $(x \circ (y \circ (y \circ x)) \subseteq I$,

(ii) type 2 , if for all $x, y, z \in H$, $((x \circ y) \circ z) \bigcap I \neq \emptyset$ and $z \in I$ imply that $(x \circ (y \circ (y \circ x)) \bigcap I \neq \emptyset$,

(iii) type 3 , if for all $x, y, z \in H$, $((x \circ y) \circ z) \bigcap I \neq \emptyset$ and $z \in I$ imply that $(x \circ (y \circ (y \circ x)) < I$,

(iv) type 4 , if for all $x, y, z \in H$, $((x \circ y) \circ z) \subseteq I$, $z \in I$ imply that $(x \circ (y \circ (y \circ x)) \subseteq I$,

(v) type 5 , if for all $x, y, z \in H$, $((x \circ y) \circ z) \subseteq I$ and $z \in I$ imply that $(x \circ (y \circ (y \circ x)) \bigcap I \neq \emptyset$,

(vi) type 6 , if for all $x, y, z \in H$, $((x \circ y) \circ z) \subseteq I$ and $z \in I$ imply that $(x \circ (y \circ (y \circ x)) < I$,

(vii) type 7 , if for all $x, y, z \in H$, $((x \circ y) \circ z) < I$, $z \in I$ imply that $(x \circ (y \circ (y \circ x)) \subseteq I$,

(viii) type 8 , if for all $x, y, z \in H$, $((x \circ y) \circ z) < I$ and $z \in I$ imply that $(x \circ (y \circ (y \circ x)) \bigcap I \neq \emptyset$,

(ix) type 9 , if for all $x, y, z \in H$, $((x \circ y) \circ z) < I$ and $z \in I$ imply that $(x \circ (y \circ (y \circ x)) < I$.

For simplicity of notation we use "CHKI " instead of "Commutative hyper K-ideal ".

Example 4.2.(i) The following table shows a hyper K-algebra structure on H.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,2\}$	$\{0,2\}$
2	$\{2\}$	$\{0,2\}$	$\{0,2\}$

We see that $I = \{0, 2\}$ is a CHKI of types 3, 6, 7, 9, while it is not CHKI of types 1, 2, 4, 5 or 8, beacause $((1 \circ 0) \circ 2) = \{0, 2\}$ and $2 \in I$, but $(1 \circ (0 \circ (0 \circ (0 \circ 1)) = 1 \notin I$. Also $I = \{0, 2\}$ is an implicative hyper K-ideal.
(ii) The following table shows a hyper K-algebra structure on H.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0,1\}$
2	$\{2\}$	$\{2\}$	$\{0\}$

Then $I = \{0, 1\}$ is CHKI of types 1, 2, 3, 4, 5, 6, 7, 8, 9. Also it is $I = \{0, 1\}$ is an implicative hyper K-ideal .

(iii) The following table shows a hyper K-algebra structure on H.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{0,1\}$
2	$\{2\}$	$\{2\}$	$\{0,1\}$

In this example $I = \{0, 2\}$ is CHKI of type 9, while it is not of type 2 , because $((1 \circ 0) \circ 2) \bigcap I \neq \emptyset$ and $2 \in I$, but $(1 \circ (0 \circ (0 \circ 1)) \bigcap I = \emptyset$. Also $I = \{0, 2\}$ is not an implicative hyper$K$-ideal, because $(1 \circ 0) \circ (0 \circ 1) < I$ and $0 \in I$, but $1 \notin I$.

Theorem 4.3. Let I be a CHKI of types 1 or 7 . Then I is a hyper K-ideal.

Proof. Let $(x \circ y) \bigcap I \neq \emptyset$ and $y \in I$. Then there exists $z \in (x \circ y) \bigcap I$. We have $z \in (x \circ 0) \circ y)$, so $((x \circ 0) \circ y) \bigcap I \neq \emptyset$. Since I is a CHKI of type 1, thus $(x \circ (0 \circ (0 \circ x)) \subseteq I$. Therefore $x \in (x \circ (0 \circ (0 \circ x))$ implies that $x \in I$. Hence I is a hyper K-ideal.

Theorem 4.4. Let $0 \in H$ be a left scalar and I be an additive CHKI of types 2 or 3. Then I is a hyper K-ideal.

Proof. Let $(x \circ y) \bigcap I \neq \emptyset$, $y \in I$. Then $((x \circ 0) \circ y) \bigcap I \neq \emptyset$. Since I is a CHKI of type 2, thus $(x \circ (0 \circ (0 \circ x)) \bigcap I \neq \emptyset$, by hypothesis $(x \circ 0) \bigcap I \neq \emptyset$. Therefore $x \circ 0 < I$. So there exists $k \in x \circ 0$ and $t \in I$ such that $k < t$, i.e $(x \circ t) < 0$, hence there exists $z \in (x \circ t)$ such that $z < t$ .We have z=0 , thus $0 \in x \circ t$. Now since I is an additive, we get that $x \in I$. Hence I is a hyper K-ideal. The proof of the other types is similar to above.

Theorem 4.5. Let $0 \in H$ be a scalar , I is an additive and I be a CHKI of types 5 or 6. Then I is a weak hyper K-ideal.

Proof. The proof is similar to the proof of Theorem 4.4.

Theorem 4.6. Let $H = \{0, 1, 2\}$ be a hyper K-algebra of order 3 that satisfies the simple condition and $\{0\} \neq I \subseteq H$. If I is an implicative hyper K-ideal , then I is commutative hyper K-ideal of types 1, 2, 3, 4, 5, 6, 7, 8, 9.

Proof. Let I be an implicative hyper K-ideal. Without loss of generality assume that $I = \{0, 1\}$. By the proof of Theorem 4.15[1], H has the following hyper structure .

$\circ$	0	1	2
0	$\{0\} or \{0,1\}$	$\{0\} or \{0,1\}$	$\{0\} or \{0,1\}$
1	$\{1\}$	$\{0\} or \{0,1\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0\} or \{0,1\}$

$I \neq \{0\}$ is an implicative hyper K-ideal. Then we show that I is a CHKI of type 1.

Consider the following different cases:

Case (i) : if x=0 or x=1, then for any $y \in H$, we have $(0\circ(y\circ(y\circ 0)) \subseteq I$ or $(1\circ(y\circ(y\circ 1)) \subseteq I$, and hence in these cases we are done,

Case (ii) : if x=2, then for any $y, z \in H$, we can see that $((2 \circ y) \circ z) \bigcap I = \emptyset$. For example let y=0 , z=1. Then $((2 \circ 0) \circ 1) = \{2\} \circ 1 = \{2\}$ and $\{2\} \bigcap I = \emptyset$. Thus there are nothing to prove, and hence theorem is proved.

The proof of the other types are similar to above .

Example 4.7. The following table shows a hyper K-algebra structure on H.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{0,1\}$
2	$\{2\}$	$\{2\}$	$\{0,1\}$

In this example $I = \{0, 2\}$ is a CHKI of type 9, while it is not implicative hyper K-ideal, because $(1 \circ 0) \circ (0 \circ 1) < I$ and $0 \in I$, but $0 \notin I$.
$I = \{0\}$ is a CHKI of types 2 and 9, but it is not an implicative hyper K-ideal, because $(1 \circ 0) \circ (2 \circ 1) < I$ and $0 \in I$, but $1 \notin I$.

Theorem 4.8. Let H be a hyper K-algebra of order 3 such that H satisfies the simple condition . Then $I = \{0, 1\}$ is a CHKI of type 2 if and only if $I = \{0, 1\}$ is a CHKI of type 9.

Proof. The proof is similar to the proof of Theorem 3.11.

Open problem 4.9. Is there an example of CHKI of type 2 that it is not of type 9?

Theorem 4.10. Let H be a hyper K-algebra of order 3 such that H satisfies the simple condition , $I = \{0, 1\}$ and I is CHKI of types 1,2, ..., 8 or 9. Then I is an implicative hyper K-ideal.

Proof. Let I be a CHKI of type 1. We show that $(x \circ z) \circ (y \circ x) < I$ and $z \in I$ imply that $x \in I$. By Theorems 17.3 and 19.3 of [9] , we have $2 \circ 0 = \{2\}$, $2 \circ 1 = \{2\}$, $1 \circ 2 = \{1\}$, $1 \circ 0 = \{1\}$, $x \circ y \neq \{0, 2\}$ and $x \circ y \neq \{0, 1, 2\}$ for all $x, y \in H$.
Now, let $x = 2$. In the folloiwng statements we show that , this case is imposssible.

(i) Let z=0. We consider the following subcases :
(a) If y=0, then by above we have $(2 \circ 0) \circ (0 \circ 2) = 2 \circ \{0, 1\} = \{2\} \bigcup \{2\}$. So by hypothesis $(2 \circ 0) \circ (0 \circ 2) < \{0, 1\}$, therefore $\{2\} < \{0, 1\}$, which implies that $2 < 1$. Thus we obtain a contradiction, because H satisfies the simple condition.
(b) If y=1, then $(2 \circ 0) \circ (1 \circ 2) = \{2\}$. By hypthesis $\{2\} < \{0, 1\}$. Therefore $2 < 1$, which is a contradiction.
(c) If y=2, then $(2 \circ 2) \subseteq \{0, 1\}$. So $(2 \circ 0) \circ (2 \circ 2) \subseteq \{2\}$. By hypothesis $\{2\} < \{0, 1\}$, hence $2 < 1$, which is a contradiction .
(ii) Let z=1. Then a similar argument as the case of (i), gives a contradiction.
Note that by hypothesis $z \in I$, so $z \neq 2$. Hence x=2 is impossible . Thus $x \in I$. Therefore I is an implicative hyper K-ideal.

Remark 4.11.. Note that Theorems 4.6 , 4.8 and 4.10 satisfy for $I = \{0, 2\}$ too.

Corollary 4.12. Let H be a hyper K-algebra of order 3 such that H satisfies the simple condition. Then I is CHKI of types 1,2, ..., 8 or 9 if and only if I is an implicative hyper K-ideal.

Proof. The proof follows from Theorems 4.6 and 4.10.

Example 4.13 (i) In Example 4.2 (i) we see that $I = \{0, 2\}$ is an implicative hyper K-ideal, while it is not CHKI of types 1, 2, 4, 5 and 8. Here H does not satisfy in the simple condition.

(ii) In Example 4.2(i) $I = \{0, 2\}$ is CHKI of types 3, 6, 7 and 9, while it is not a hyper K-ideal. Since $0 \in H$ is not left scalar and it is not additive.

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