A CHARACTERIZATION OF ω_1-STRONGLY COUNTABLE-DIMENSIONAL METRIZABLE SPACES

Masahiro Matsumoto

Received March 26, 2007; revised June 14, 2007

Abstract. In this paper, we characterize the class of ω_1-strongly countable-dimensional metrizable spaces by a special metric. A characterization of locally finite-dimensional metrizable spaces is also obtained.

1 Introduction If every finite open cover of a metrizable space X has a finite open refinement of order $\leq n+1$, then X has *covering dimension* $\leq n$, $\dim X \leq n$. For $\varepsilon > 0$, we let $S_\varepsilon(x)$ denote the ε-ball $\{y \in X \mid \rho(x,y) < \varepsilon\}$ about x.

In [5], [6] and [7], J. Nagata gave a characterization of metrizable spaces of dim $\leq n$ by a special metric.

Theorem 1.1 (J. Nagata [5], [6], [7]) *The following conditions are equivarent for a metrizable space X:*

(1) $\dim X \leq n$.

(2) *There is an admissible metric ρ satisfying the following condition: for every $\varepsilon > 0$, every point x of X and every $n+2$ many points $y_1, ..., y_{n+2}$ of X with $\rho(S_{\varepsilon/2}(x), y_i) < \varepsilon$ for each $i = 1, ..., n+2$, there are distinct natural numbers i and j such that $\rho(y_i,\ y_j) < \varepsilon$.*

(3) *There is an admissible metric ρ satisfying the following condition: for every point x of X and every $n+2$ many points $y_1, ..., y_{n+2}$ of X, there are ditinct natural numbers i and j such that $\rho(y_i,\ y_j) \leq \rho(x,\ y_j)$.*

For the case of the separable metrizable spaces, J. de Groot [2] gave the following characterization.

Theorem 1.2 (J. de Groot [2]) *A separable metrizable space X has $\dim X \leq n$ if and only if X can introduce an admissible totally bounded metric satisfying the following condition:*

For every point x of X and every $n+2$ many points $y_1, ..., y_{n+2}$ of X, there are natural numbers i, j and k such that $i \neq j$ and $\rho(y_i,\ y_j) \leq \rho(x,\ y_k)$.

A metrizable space X is *strongly countable-dimensional* if X can be represented as a countable union of closed finite-dimensional subspaces. Let $\mathbb{N}$ denote the set of all natural numbers.

In [8], J. Nagata extended Theorems 1.1 and 1.2 to strongly countable-dimensional metrizable spaces.

Theorem 1.3 (J. Nagata [8]) *The following conditions are equivarent for a metrizable space X:*

(1) *X is strongly countable-dimensional.*

2000 *Mathematics Subject Classification.* 54A05.

Key words and phrases. ω_1-strongly countable-dimensional, locally finite-dimensional.

(2) *There is an admissible metric ρ satisfying the following condition: for every point x of X, there is an $n(x) \in \mathbb{N}$ such that for every $n(x)+2$ many points $y_1, ..., y_{n(x)+2}$ of X, there are ditinct natural numbers i and j such that $\rho(y_i,\ y_j) \leq \rho(x,\ y_j)$.*

(3) *There is an admissible metric ρ satisfying the following condition: for every point x of X, there is an $n(x) \in \mathbb{N}$ such that for every $n(x)+2$ many points $y_1, ..., y_{n(x)+2}$ of X, there are natural numbers i, j and k such that $i \neq j$ and $\rho(y_i,\ y_j) \leq \rho(x,\ y_k)$.*

In [3], Y. Hattori characterized the class of strongly countable-dimensional spaces by extending the condition (2) of Theorem 1.1.

Theorem 1.4 (Y. Hattori [3]) *A metrizable space X is strongly countable-dimensional if and only if X can introduce an admissible metric ρ satisfying the following condition:*

For every point x of X, there is an $n(x) \in \mathbb{N}$ such that for every $\varepsilon > 0$, and every $n+2$ many points $y_1, ..., y_{n(x)+2}$ of X with $\rho(S_{\varepsilon/2}(x), y_i) < \varepsilon$ for each $i = 1, ..., n(x)+2$, there are distinct natural numbers i and j such that $\rho(y_i,\ y_j) < \varepsilon$.

2 ω_1-strongly countable-dimensional spaces In this section, we characterize the class of ω_1-strongly countable-dimensional metrizable spaces by a special metric. A characterization of locally finite-dimensional metrizable spaces is also obtained.

Definition 2.1 A metrizable space X is *locally finite-dimensional* if for every point $x \in X$ there exists an open subspace U of X such that $x \in U$ and $\dim U < \infty$.

The first infinite ordinal number is denoted by ω and ω_1 is the first uncountable ordinal number.

Definition 2.2 A metrizable space X is called an *ω_1-strongly countable-dimensional space* if $X = \bigcup\{P_\xi \mid 0 \leq \xi < \xi_0\}$, $\xi_0 < \omega_1$, where P_ξ is an open subset of $X - \bigcup\{P_\eta \mid 0 \leq \eta < \xi\}$ and $\dim P_\xi < \infty$.

For a metrizable space X and a non-negative integer n, we put

$$P_n(X) = \bigcup\{U \mid U \text{ is an open subspace of } X \text{ and } \dim U \leq n\}.$$

We notice that for each ordinal number α, we can put $\alpha = \lambda(\alpha) + n(\alpha)$, where $\lambda(\alpha)$ is a limit ordinal number or 0 and $n(\alpha)$ is a non-negative integer.

Definition 2.3 Let X be a metrizable space and α either an ordinal number ≥ 0 or the integer -1. Then *strong small transfinite dimension sind* of X is defined as follows:

(1) $\operatorname{sind} X = -1$ if and only if $X = \emptyset$.

(2) $\operatorname{sind} X \leq \alpha$ if X is expressed in the form $X = \bigcup\{P_\xi \mid \xi < \alpha\}$, where $P_\xi = P_{n(\xi)}(X - \bigcup\{P_\eta \mid \eta < \lambda(\xi)\})$.

Furthermore, if $\operatorname{sind} X$ is defined, we say that X has *strong small transfinite dimension*.

Clearly, a metrizable space X is locally finite-dimensional if and only if $\operatorname{sind} X \leq \omega$ (R. Engelking [1]). And X is ω_1-strongly countable-dimensional if and only if there is a $\xi_0 < \omega_1$ such that $\operatorname{sind} X \leq \xi_0$.

Theorem 2.9 is a main theorem. Thus we characterize the class of ω_1-strongly countable-dimensional metrizable spaces by a special metric. To prove this theorem, we need Theorem 2.4.

Theorem 2.4 *Let α be an ordinal number with $\alpha < \omega_1$ and let n be a non-negative integer. The following conditions are equivarent for a metrizable space X:*

(a) $\operatorname{sind} X \leq \omega\alpha + n$.

(b) *There are an admissible metric ρ for X and a family $\{X_\beta \mid 0 \leq \beta \leq \alpha\}$ of closed sets of X satisfing the following conditions:* (*b*-1) $X_0 = X$, $X_\beta \supset X_{\beta'}$ *for* $\beta \leq \beta' \leq \alpha$, $X_\beta = \bigcap\{X_{\beta'} \mid \beta' < \beta\}$ *if β is a limit, and $X_\alpha = \emptyset$ if $n = 0$.* (*b*-2) *For every point x of X there are an open neighborhood $U(x)$ of x in $X_{\beta(x)}$, where $\beta(x) = \max\{\beta \mid x \in X_\beta\}$, and an $n(x) \in N_{\beta(x)}$ such that for every $\varepsilon > 0$, every point x' of $U(x)$ and every $n(x)+2$ many points $y_1, ..., y_{n(x)+2}$ of X with $\rho(S_{\varepsilon/2}(x'), y_i) < \varepsilon$ for each $i = 1, ..., n(x)+2$, there are distinct natural numbers i and j such that $\rho(y_i,\ y_j) < \varepsilon$, where*

$$N_{\beta(x)} = \begin{cases} \mathbb{N}, & \text{if } \beta(x) < \alpha, \\ \{n-1\}, & \text{if } \beta(x) = \alpha. \end{cases}$$

(c) *There are an admissible metric ρ for X and a family $\{X_\beta \mid 0 \leq \beta \leq \alpha\}$ of closed sets of X satisfing the following conditions:* (*c*-1) $X_0 = X$, $X_\beta \supset X_{\beta'}$ *for* $\beta \leq \beta' \leq \alpha$, $X_\beta = \bigcap\{X_{\beta'} \mid \beta' < \beta\}$ *if β is a limit, and $X_\alpha = \emptyset$ if $n = 0$.* (*c*-2) *For every point x of X there are an open neighborhood $U(x)$ of x in $X_{\beta(x)}$, where $\beta(x) = \max\{\beta \mid x \in X_\beta\}$, and an $n(x) \in N_{\beta(x)}$ such that for every point x' of $U(x)$ and every $n(x)+2$ many points $y_1, ..., y_{n(x)+2}$ of X, there are distinct natural numbers i and j such that $\rho(y_i,\ y_j) \leq \rho(x',\ y_j)$, where*

$$N_{\beta(x)} = \begin{cases} \mathbb{N}, & \text{if } \beta(x) < \alpha, \\ \{n-1\}, & \text{if } \beta(x) = \alpha. \end{cases}$$

Remark 2.5 Let $\{X_\beta \mid 0 \leq \beta \leq \alpha\}$ be a family of closed sets of X satisfing the condition (b-1). Then we shall show that for every point x of X, there is a maximum element $\beta(x)$ of $\{\beta \mid x \in X_\beta\}$. Indeed, if $x \in X_{\lambda(\alpha)}$, then $\beta(x) = \max\{\beta \mid x \in X_\beta,\ \lambda(\alpha) \leq \beta \leq \alpha\}$. Now, we suppose that $x \in X_{\lambda(\alpha)}$, there is a minimum element $\beta_0 > 0$ of $\{\beta \mid x \notin X_\beta\}$. Assume that β_0 is limit. By the condition (b-1), $x \in \bigcap\{X_\beta \mid \beta < \beta_0\} = X_{\beta_0}$. This contradicts the definition of β_0. Therefore β_0 is not limit and hence $\beta(x) = \beta_0 - 1$.

To prove this theorem, we need the following lemmas. Essentially, the following lemma is the same as [3; Lemma 1.5]. By a minor modification in the proof of [3; Lemma 1.5], we obtain the following lemma.

Lemma 2.6 ([3; Lemma 2.5], [8; Lemma 1]) *Let n be a non-negative integer and let $\{F_m \mid m = 0,\ 1, ...\}$ be a closed cover of a metrizable space X such that $\dim F_m \leq (n-1)+m$, $F_m \subset F_{m+1}$ for $m = 0,\ 1, ...$ Then for every open cover $\mathcal{U}$ of X, there are a sequence $\mathcal{V}_1$, $\mathcal{V}_2$, ... of discrete families of open sets of X and an open cover $\mathcal{W}$ of X which satisfy the following conditions:*

(1) $\bigcup\{\mathcal{V}_k \mid k \in \mathbb{N}\}$ *is a cover of X.*

(2) $\bigcup\{\mathcal{V}_k \mid k \in \mathbb{N}\}$ *refines $\mathcal{U}$.*

(3) *If $W \in \mathcal{W}$ satisfies $W \cap F_m \neq \emptyset$, then W meets at most one member of $\mathcal{V}_k$ for $k \leq (n+0)+(n+1)+...+(n+m)$ and meets no member of $\mathcal{V}_k$ for $k > (n+0)+(n+1)+...+(n+m)$.*

Let Q^* denote the set of all rational numbers of the form $2^{-m_1} + ... + 2^{-m_t}$, where m_1,...,m_t are natural numbers satisfying $1 \leq m_1 < ... < m_t$.

Essentially, the following lemma is the same as [3; Lemma 1.6]. By a minor modification in the proof of [3; Lemma 1.6], we obtain the following lemma.

Lemma 2.7 ([3; Lemma 2.6], [8; Lemma 3]) *Let n be a non-negative integer and let $\{F_m \mid m = 0,\ 1, ...\}$ be a closed cover of a metrizable space X such that $\dim F_m \leq (n-1)+m$,*

$F_m \subset F_{m+1}$ for $m = 0,\ 1, \ldots.$ Then for every $q \in Q^*$, there is an open cover $\mathcal{S}(q)$ which satisfies the following conditions:

(1) $\mathcal{S}(q) = \bigcup_{i=1}^{\infty} \mathcal{S}^i(q)$, where each $\mathcal{S}^i(q)$ is discrete in X.

(2) $\{St(x, \mathcal{S}(q)) \mid q \in Q^*\}$ is a neighborhood base at $x \in X$.

(3) Let $p,\ q \in Q^*$ and $p < q$. Then $\mathcal{S}(p)$ refines $\mathcal{S}(q)$.

(4) Let $p,\ q \in Q^*$ and $p < q$. If $S_1 \in \mathcal{S}^i(p)$ and $S_2 \in \mathcal{S}^i(q)$, then $S_1 \cap S_2 = \emptyset$ or $S_1 \subset S_2$.

(5) Let $p,\ q \in Q^*$ and $p + q < 1$. Let $S_1 \in \mathcal{S}(p)$, $S_2 \in \mathcal{S}(q)$ and $S_1 \cap S_2 \neq \emptyset$. Then there is an $S_3 \in \mathcal{S}(p+q)$ such that $S_1 \cup S_2 \subset S_3$.

(6) For every $q \in Q^*$ and every $S \in \bigcup\{\mathcal{S}^i(q) \mid i > (n+0) + (n+1) + \ldots + (n+m)\}$, $S \cap F_m = \emptyset$.

For a cover $\mathcal{U}$ of a set X, we denote $\mathcal{U}^* = \{\mathrm{St}(U,\mathcal{U}) \mid U \in \mathcal{U}\}$ and $\mathcal{U}^{**} = (\mathcal{U}^*)^*$, where $\mathrm{St}(U,\mathcal{U}) = \bigcup\{V \in \mathcal{U} \mid U \cap V \neq \emptyset\}$.

Proof of Theorem 2.4. We prove the implication (a) $\Rightarrow$ (c). Let $\operatorname{sind} X \leq \omega\alpha + n$. We put

$$Y_\gamma = X - \bigcup\{P_\xi \mid \xi < \gamma\} \quad \text{for} \quad \gamma \leq \omega\alpha + n$$

and

$$X_\beta = Y_{\omega\beta} \quad \text{for} \quad \beta \leq \alpha.$$

Clearly, the family of closed sets $\{X_\beta \mid 0 \leq \beta \leq \alpha\}$ satisfies the condition (c-1). Note that X_β is a closed set of X and $P_{\omega\beta+m}$ is an open set of X_β such that $P_{\omega\beta+m} \subset P_{\omega\beta+(m+1)}$ for $m = 0, 1, \ldots.$ Also $P_{\omega\alpha+(n-1)}$ is a closed set of X. Hence for each $\beta \leq \alpha$ there is a family $\{W_{\omega\beta+m} \mid m = 0,\ 1, \ldots\}$ of open sets of X_β such that

(1) $\overline{W_{\omega\beta+m}} \subset P_{\omega\beta+m}$,

(2) $\overline{W_{\omega\beta+m}} \subset W_{\omega\beta+(m+1)}$,

(3) $\bigcup_{m=0}^{\infty} W_{\omega\beta+m} = \bigcup_{m=0}^{\infty} P_{\omega\beta+m}$.

Since $\{\beta \mid 0 \leq \beta < \alpha\}$ is countable, there is a mapping f from $\mathbb{N}$ onto $\{\beta \mid 0 \leq \beta < \alpha\}$. For each $m = 0, 1, \ldots$, we put

$$V_0 = P_{\omega\alpha+(n-1)},$$

$$V_1 = P_{\omega\alpha+(n-1)} \cup W_{\omega f(1)+(n-1)+1},$$

$$V_2 = P_{\omega\alpha+(n-1)} \cup W_{\omega f(1)+(n-1)+2} \cup W_{\omega f(2)+(n-1)+2},$$

$$\ldots$$

$$V_m = P_{\omega\alpha+(n-1)} \cup W_{\omega f(1)+(n-1)+m} \cup W_{\omega f(2)+(n-1)+m} \cup \ldots \cup W_{\omega f(m)+(n-1)+m},$$

$$\ldots$$

Then $V_0, V_1, \ldots$ are subsets of X satisfing the following conditions:

(4) $\overline{V_m} \subset V_{m+1}$.

(5) $\dim \overline{V_m} \leq (n-1) + m$.

(6) $X = \bigcup_{m=0}^{\infty} V_m$.

The latter half of the proof is similar to the proof of [3; Theorem 1.4]. By Lemma 2.7, for every $q \in Q^*$, there is an open cover $\mathcal{S}(q)$ which satisfies the following conditions:

(7) $\mathcal{S}(q) = \bigcup_{i=1}^{\infty} \mathcal{S}^i(q)$, where each $\mathcal{S}^i(q)$ is discrete in X.

(8) $\{St(x, \mathcal{S}(q)) \mid q \in Q^*\}$ is a neighborhood base at $x \in X$.

(9) Let $p,\ q \in Q^*$ and $p < q$. Then $\mathcal{S}(p)$ refines $\mathcal{S}(q)$.

(10) Let $p,\ q \in Q^*$ and $p < q$. If $S_1 \in \mathcal{S}^i(p)$ and $S_2 \in \mathcal{S}^i(q)$, then $S_1 \cap S_2 = \emptyset$ or $S_1 \subset S_2$.

(11) Let $p,\ q \in Q^*$ and $p + q < 1$. Let $S_1 \in \mathcal{S}(p)$, $S_2 \in \mathcal{S}(q)$ and $S_1 \cap S_2 \neq \emptyset$. Then there is an $S_3 \in \mathcal{S}(p + q)$ such that $S_1 \cup S_2 \subset S_3$.

(12) For every $q \in Q^*$ and every $S \in \bigcup\{\mathcal{S}^i(q) \mid i > (n + 0) + (n + 1) + ... + (n + m)\}$, $S \cap \overline{V_m} = \emptyset$.

We define a function $\rho : X \times X \to [0,\ 1]$ as follows: For $x,\ y \in X$,

$$\rho(x,\ y) = \begin{cases} 1, & \text{if } y \notin St(x,\ \mathcal{S}(q)) \text{ for every } q \in Q^*, \\ \inf\{q \in Q^* \mid y \in St(x,\ \mathcal{S}(q))\}, & \text{otherwise.} \end{cases}$$

It follows that ρ is an admissible metric for X by the proof of [9; Ch. 5.3, (D)]. To prove that the metric ρ and the family of closed sets $\{X_\beta \mid 0 \leq \beta \leq \alpha\}$ satisfy the condition (c-2), let x be a point of X. Put $n_0 = \min\{m \mid x \in V_m\}$ and $n(x) = (n+0)+(n+1)+...+(n+n_0)-1$. Clearly, if $n_0 = 0$ then $x \in V_0 = P_{\omega\alpha+(n-1)} \subset X_\alpha$.

Now we shall show that if $n_0 > 0$, then $x \in W_{\omega\beta(x)+(n-1)+n_0} \subset X_{\beta(x)}$. By the definition of n_0, $x \in V_{n_0} = P_{\omega\alpha+(n-1)} \cup W_{\omega f(1)+(n-1)+n_0} \cup W_{\omega f(2)+(n-1)+n_0} \cup ... \cup W_{\omega f(n_0)+(n-1)+n_0}$. Since $x \notin P_{\omega\alpha+(n-1)}$ by $n_0 > 0$, there is a natural number i such that $x \in W_{\omega f(i)+(n-1)+n_0}$. Hence $x \in W_{\omega f(i)+(n-1)+n_0} \subset P_{\omega f(i)+(n-1)+n_0} \subset X_{f(i)} - X_{f(i)+1}$. Also since $\beta(x) = \max\{\beta \mid x \in X_\beta\} < \alpha$, $x \in X_{\beta(x)} - X_{\beta(x)+1}$. Hence $f(i) = \beta(x)$ and hence $x \in W_{\omega\beta(x)+(n-1)+n_0} \subset X_{\beta(x)}$.

We put

$$U(x) = \begin{cases} P_{\omega\alpha+(n-1)}, & \text{if } n_0 = 0, \\ W_{\omega\beta(x)+(n-1)+n_0}, & \text{if } n_0 > 0. \end{cases}$$

Then $U(x)$ is an open neighborhood of x in $X_{\beta(x)}$. Let x' be a point of $U(x)$ and $y_1, ..., y_{n(x)+2}$ be points of X. We can assume that $\rho(x',\ y_j) < 1$ for each $j = 1, ..., n(x) + 2$. Let $\varepsilon > 0$ be given. For each $j = 1, ..., n(x) + 2$, there is a $q(j) \in Q^*$ such that $\rho(x',\ y_j) \leq q(j) < \rho(x',\ y_j) + \varepsilon$. Then, by the definition of ρ, there is an $S_j \in \mathcal{S}(q(j))$ such that $x',\ y_j \in S_j$. Let $S_j \in \mathcal{S}^{i(j)}(q(j))$. Then, by (12), there are distinct natural numbers j and k such that $i(j) = i(k)$. Assume that $q(j) \leq q(k)$. By (10), we obtain $S_j \subset S_k$. Hence $y_j,\ y_k \in S_k$ and hence $\rho(y_j,\ y_k) \leq q(k) < \rho(x',\ y_k) + \varepsilon$. Therefore, there are distinct natural numbers j and k such that $\rho(y_j,\ y_k) < \rho(x',\ y_k) + 1/m$ for each $m \in \mathbb{N}$ and hence $\rho(y_j,\ y_k) \leq \rho(x',\ y_k)$. This completes the proof of the implication (a) $\Rightarrow$ (c).

Next, we prove the implication (a) $\Rightarrow$ (b). The proof is similar to the proof of [3; Theorem 1.1]. We use the above notations. By Lemma 2.6, there is a sequence $\mathcal{U}_1, \mathcal{U}_2, ...$ of open covers of X which satisfies the following conditions.

(13) $\mathcal{U}^{**}_{k+1}$ refines $\mathcal{U}_k$ for each $k \in \mathbb{N}$.

(14) $\{\mathrm{St}(x, \mathcal{U}_k) \mid k \in \mathbb{N}\}$ is a neighborhood base at $x \in X$.

(15) For each $x \in \overline{V_m}$ and each $k \in \mathbb{N}$, $\mathrm{St}^2(x, \mathcal{U}^*_{k+1})$ meets at most $(n + 0) + (n + 1) + ... + (n + m)$ members of $\mathcal{U}_k$.

For each $q = 2^{-m_1} + ... + 2^{-m_t} \in Q^*$ and each $U \in \mathcal{U}_{m_1}$, we put

$S(U; m_1) = U$,
$S(U; m_1, ..., m_k) = \mathrm{St}^2(S(U; m_1, ..., m_{k-1}),\ \mathcal{U}_{m_k})$ for $2 \leq k \leq t$,
$S(U; q) = S(U; m_1, ..., m_t)$
and

$$\mathcal{S}(q) = \{S(U; q) \mid U \in \mathcal{U}_{m_1}\}.$$

We define a function $\rho : X \times X \to [0,\ 1]$ as follows: For $x,\ y \in X$,

$$\rho(x,\ y) = \begin{cases} 1, & \text{if } y \notin St(x,\ \mathcal{S}(q)) \text{ for every } q \in Q^*, \\ \inf\{q \in Q^* \mid y \in St(x,\ \mathcal{S}(q))\}, & \text{otherwise.} \end{cases}$$

It follows that ρ is an admissible metric for X (see the proof of [9; Ch. 5.3,(D)]). To prove that the metric ρ and the family of closed sets $\{X_\beta \mid 0 \le \beta \le \alpha\}$ satisfy the condition (b-2), let x be a point of X. Put $n_0 = \min\{m \mid x \in V_m\}$ and $n(x) = (n+0)+(n+1)+...+(n+n_0)-1$. We put

$$U(x) = \begin{cases} P_{\omega\alpha+(n-1)}, & \text{if } n_0 = 0, \\ W_{\omega\beta(x)+(n-1)+n_0}, & \text{if } n_0 > 0. \end{cases}$$

Then $U(x)$ is an open neighborhood of x in $X_{\beta(x)}$. Let $x' \in U(x)$, $\varepsilon > 0$ and $y_1, ..., y_{n(x)+2} \in X$ with $\rho(S_{\varepsilon/2}(x'), y_i) < \varepsilon$ for each $i = 1, ..., n(x)+2$. For each $i = 1, ..., n(x)+2$, let x_i be a point of X such that $\rho(x', x_i) < \varepsilon/2$ and $\rho(x_i, y_i) < \varepsilon$. Put

$$\delta_i = \max\{2\rho(x', x_i), \rho(x_i, y_i)\}$$

and $\delta = \max\{\delta \mid i = 1, ..., n(x)+2\}$. Then there is a $q = 2^{-m_1} + ... + 2^{-m_t} \in Q^*$ such that $\delta < q < \varepsilon$. Since $\rho(x', x_i) < 2^{-(m_1+1)} + ... + 2^{-(m_t+1)}$, there is a $U_i \in \mathcal{U}_{m_1+1}$ such that $x', x_i \in S(U_i; m_1+1, ..., m_t+1)$. Hence $x_i \in \mathrm{St}(x', \mathcal{U}^*_{m_1+1})$. On the other hand, there is a $U'_i \in \mathcal{U}_{m_1}$ such that $x_i, y_i \in S(U'_i; m_1, ..., m_t)$. Therefore $\mathrm{St}(x_i, \mathcal{U}_{m_1+1}) \cap U'_i \neq \emptyset$ and hence $\mathrm{St}^2(x, \mathcal{U}^*_{m_1+1}) \cap U'_i \neq \emptyset$. By (15), there are distinct natural numbers i and j such that $U'_i = U'_j$. Then $y_i, y_j \in S(U'_i; m_1, ..., m_t) = S(U'_j; m_1, ..., m_t)$. Therefore, $\rho(y_i, y_j) \le q < \varepsilon$. This completes the proof of the implication (a) $\Rightarrow$ (b).

Next, we prove the implication (c) $\Rightarrow$ (a). Let ρ be an admissible metric for X and $\{X_\beta \mid 0 \le \beta \le \alpha\}$ be a family of closed sets of X which satisfy the conditions (c-1) and (c-2).

We shall show that for every $\beta \le \alpha$

$$(16)\quad X - \bigcup\{P_\xi \mid \xi < \omega\beta\} \subset X_\beta.$$

The validity of (16) is clear for $\beta = 0$. To prove (16) by transfinite induction we assume (16) for $\gamma < \beta$.

Let $x \notin X_\beta$. Then there are an open neighborhood $U(x)$ of x in $X_{\beta(x)}$ and an $n(x) \in N_{\beta(x)}$ such that for every point x' of $U(x)$ and every $n(x)+2$ many points $y_1, ..., y_{n(x)+2}$ of X, there are distinct natural numbers i and j such that $\rho(y_i,\ y_j) \le \rho(x',\ y_j)$. Since $x \notin X_\beta$, $\beta(x) < \beta$.

If $x \in \bigcup\{P_\xi \mid \xi < \omega\beta(x)\}$, then $x \in \bigcup\{P_\xi \mid \xi < \omega\beta\}$ by $\beta(x) < \beta$.

We shall also show that if $x \in X - \bigcup\{P_\xi \mid \xi < \omega\beta(x)\}$, then $x \in \bigcup\{P_\xi \mid \xi < \omega\beta\}$. Since $U(x)$ is an open neighborhood of x in $X_{\beta(x)}$, by the induction hypothesis

$$V(x) = U(x) \cap (X - \bigcup\{P_\xi \mid \xi < \omega\beta(x)\})$$

is an open neighborhood of x in $X - \bigcup\{P_\xi \mid \xi < \omega\beta(x)\}$. Also $\dim V(x) \le \dim U(x) \le n(x)$ by Theorem 1.1. Hence,

$$x \in V(x) \subset P_{n(x)}(X - \bigcup\{P_\xi \mid \xi < \omega\beta(x)\})$$

$$= P_{\omega\beta(x)+n(x)} \subset \bigcup\{P_\xi \mid \xi < \omega(\beta(x)+1)\}$$

$$\subset \bigcup\{P_\xi \mid \xi < \omega\beta\}.$$

Therefore for every $\beta \le \alpha$, (16) holds. In particular,

$$(17) \quad X - \bigcup\{P_\xi \mid \xi < \omega\alpha\} \subset X_\alpha.$$

We shall show that

$$X - \bigcup\{P_\xi \mid \xi < \omega\alpha\} \subset \bigcup\{P_\xi \mid \omega\alpha \le \xi < \omega\alpha + n\}.$$

If $n = 0$ then $X_\alpha = \emptyset$, and hence

$$X - \bigcup\{P_\xi \mid \xi < \omega\alpha\} = \emptyset$$

by (17).

Assume that $n > 0$. Let $x \in X - \bigcup\{P_\xi \mid \xi < \omega\alpha\}$. Then there is an open neighborhood $U(x)$ of x in $X_{\beta(x)}$ such that there is an $n(x) \in N_{\beta(x)}$ such that for every point x' of $U(x)$ and every $n(x)+2$ many points $y_1, ..., y_{n(x)+2}$ of X, there are distinct natural numbers i and j such that $\rho(y_i,\ y_j) \le \rho(x',\ y_j)$. Since $x \in X_\alpha$ by (17), $\beta(x) = \alpha$. Hence $U(x)$ is an open neighborhood of x in X_α. By (17)

$$V(x) = U(x) \cap (X - \bigcup\{P_\xi \mid \xi < \omega\alpha\})$$

is an open neighborhood of x in $X - \bigcup\{P_\xi \mid \xi < \omega\alpha\}$. Also $\dim V(x) \le \dim U(x) \le n(x)$ by Theorem 1.1. Furthermore $n(x) < n$ by $n(x) \in N_\alpha = \{n-1\}$. Hence,

$$x \in V(x) \subset P_{n(x)}(X - \bigcup\{P_\xi \mid \xi < \omega\alpha\})$$

$$= P_{\omega\alpha+n(x)} \subset \bigcup\{P_\xi \mid \omega\alpha \le \xi \le \omega\alpha + n(x)\}$$

$$\subset \bigcup\{P_\xi \mid \omega\alpha \le \xi < \omega\alpha + n\}.$$

Therefore $X = \bigcup\{P_\xi \mid 0 \le \xi < \omega\alpha + n\}$ and hence $\mathrm{sind} X \le \omega\alpha + n$. This completes the proof of the implication (c) $\Rightarrow$ (a).

Fainally, the proof of the implication (b) $\Rightarrow$ (a) is the same as the proof of the implication (c) $\Rightarrow$ (a). $\square$

By Theorems 1.2 and 2.4, we obtain the following theorem.

Theorem 2.8 *Let α be an ordinal number with $\alpha < \omega_1$ and let n be a non-negative integer. The following conditions are equivarent for a compact metrizable space X:*

(*a*) $\mathrm{sind}\, X \le \omega\alpha + n$.

(*d*) *There are an admissible totally bounded metric ρ for X and a family $\{X_\beta \mid 0 \le \beta \le \alpha\}$ of closed sets of X satisfing the following conditions:* (d-1) $X_0 = X$, $X_\beta \supset X_{\beta'}$ *for* $\beta \le \beta' \le \alpha$, $X_\beta = \bigcap\{X_{\beta'} \mid \beta' < \beta\}$ *if β is a limit, and $X_\alpha = \emptyset$ if $n = 0$.* (d-2) *For every point x of X there are an open neighborhood $U(x)$ of x in $X_{\beta(x)}$, where $\beta(x) = \max\{\beta \mid x \in X_\beta\}$, and an $n(x) \in N_{\beta(x)}$ such that for every point x' of $U(x)$ and every $n(x)+2$ many points $y_1, ..., y_{n(x)+2}$ of X, there are natural numbers i, j and k such that $i \ne j$ and $\rho(y_i,\ y_j) \le \rho(x',\ y_k)$, where*

$$N_{\beta(x)} = \begin{cases} \mathbb{N}, & \text{if } \beta(x) < \alpha, \\ \{n-1\}, & \text{if } \beta(x) = \alpha. \end{cases}$$

Proof. The implication (a) $\Rightarrow$ (d) is obvious by the implication (a) $\Rightarrow$ (c) of Theorem 2.4.

We prove the implication (d) $\Rightarrow$ (a). Let ρ be an admissible metric for X and $\{X_\beta \mid 0 \leq \beta \leq \alpha\}$ be a family of closed sets of X which satisfy the conditions (d-1) and (d-2). Let x be a point of X. There are an open neighborhood $U(x)$ of x in $X_{\beta(x)}$ and an $n(x) \in N_{\beta(x)}$ such that for every point x' of $U(x)$ and every $n(x)+2$ many points $y_1, ..., y_{n(x)+2}$ of X, there are natural numbers i, j and k such that $i \neq j$ and $\rho(y_i,\ y_j) \leq \rho(x',\ y_k)$. Then $\dim U(x) \leq n(x)$ by Theorem 1.2. Hence the implication (d) $\Rightarrow$ (a) follows from the proof of the implication (c) $\Rightarrow$ (a) of Theorem 2.4. $\square$

By Theorem 2.4, we obtain the Main Theorem 2.9 and Theorem 2.10.

Theorem 2.9 *The following conditions are equivalent for a metrizable space X:*

(a) *X is an ω_1-strongly countable-dimensional space.*

(b) *There are an admissible metric ρ for X, an ordinal number $\alpha < \omega_1$ and a family $\{X_\beta \mid 0 \leq \beta \leq \alpha\}$ of closed sets of X satisfing the following conditions:* (b-1) *$X_0 = X$, $X_\beta \supset X_{\beta'}$ for $\beta \leq \beta' \leq \alpha$ and $X_\beta = \bigcap\{X_{\beta'} \mid \beta' < \beta\}$ if β is a limit.* (b-2) *For every point x of X there are an open neighborhood $U(x)$ of x in $X_{\beta(x)}$, where $\beta(x) = \max\{\beta \mid x \in X_\beta\}$, and an $n(x) \in \mathbb{N}$ such that for every $\varepsilon > 0$, every point x' of $U(x)$ and every $n(x)+2$ many points $y_1, ..., y_{n(x)+2}$ of X with $\rho(S_{\varepsilon/2}(x'), y_i) < \varepsilon$ for each $i = 1, ..., n(x)+2$, there are distinct natural numbers i and j such that $\rho(y_i,\ y_j) < \varepsilon$.*

(c) *There are an admissible metric ρ for X, an ordinal number $\alpha < \omega_1$ and a family $\{X_\beta \mid 0 \leq \beta \leq \alpha\}$ of closed sets of X satisfing the following conditions:* (c-1) *$X_0 = X$, $X_\beta \supset X_{\beta'}$ for $\beta \leq \beta' \leq \alpha$ and $X_\beta = \bigcap\{X_{\beta'} \mid \beta' < \beta\}$ if β is a limit.* (c-2) *For every point x of X there are an open neighborhood $U(x)$ of x in $X_{\beta(x)}$, where $\beta(x) = \max\{\beta \mid x \in X_\beta\}$, and an $n(x) \in \mathbb{N}$ such that for every point x' of $U(x)$ and every $n(x)+2$ many points $y_1, ..., y_{n(x)+2}$ of X, there are distinct natural numbers i and j such that $\rho(y_i,\ y_j) \leq \rho(x',\ y_j)$.*

(d) *There are an admissible metric ρ for X, an ordinal number $\alpha < \omega_1$ and a family $\{X_\beta \mid 0 \leq \beta \leq \alpha\}$ of closed sets of X satisfing the following conditions:* (d-1) *$X_0 = X$, $X_\beta \supset X_{\beta'}$ for $\beta \leq \beta' \leq \alpha$ and $X_\beta = \bigcap\{X_{\beta'} \mid \beta' < \beta\}$ if β is a limit.* (d-2) *For every point x of X there are an open neighborhood $U(x)$ of x in $X_{\beta(x)}$, where $\beta(x) = \max\{\beta \mid x \in X_\beta\}$, and an $n(x) \in \mathbb{N}$ such that for every point x' of $U(x)$ and every $n(x)+2$ many points $y_1, ..., y_{n(x)+2}$ of X, there are natural numbers i, j and k such that $i \neq j$ and $\rho(y_i,\ y_j) \leq \rho(x',\ y_k)$.*

Proof. The conditions (a), (b) and (c) are equivalent by Theorem 2.4. The implication (c) $\Rightarrow$ (d) is obvious.

We prove the implication (d) $\Rightarrow$ (a). Let ρ be an admissible metric for X and $\{X_\beta \mid 0 \leq \beta \leq \alpha\}$ be a family of closed sets of X which satisfy the conditions (d-1) and (d-2). Let x be a point of X. There are an open neighborhood $U(x)$ of x in $X_{\beta(x)}$ and an $n(x) \in \mathbb{N}$ such that for every point x' of $U(x)$ and every $n(x)+2$ many points $y_1, ..., y_{n(x)+2}$ of X, there are natural numbers i, j and k such that $i \neq j$ and $\rho(y_i,\ y_j) \leq \rho(x',\ y_k)$. Then $\mu \dim U(x) \leq n(x)$, where $\mu \dim$ denotes the metric dimension, by [8; p. 500]. By M. Katětov's theorem [4] and J. Nagata [9; p. 93],

$$\dim U(x) \leq 2\mu \dim U(x) \leq 2n(x).$$

Therefore $\operatorname{sind} X \leq \omega\alpha + \omega$ by the proof of the implication (c) $\Rightarrow$ (a) of Theorem 2.4, and hence X is an ω_1-strongly countable-dimensional space. $\square$

Theorem 2.10 *The following conditions are equivalent for a metrizable space X:*

(*a*) *X is a locally finite-dimensional space.*

(*b*) *There is an admissible metric ρ for X satisfing the following conditions: For every point x of X, there are an $n(x) \in \mathbb{N}$ and an open neighborhood $U(x)$ of x in X such that for every $\varepsilon > 0$, every point x' of $U(x)$ and every $n(x)+2$ many points $y_1, ..., y_{n(x)+2}$ of X with $\rho(S_{\varepsilon/2}(x'), y_i) < \varepsilon$ for each $i = 1, ..., n(x)+2$, there are distinct natural numbers i and j such that $\rho(y_i,\ y_j) < \varepsilon$.*

(*c*) *There is an admissible metric ρ for X satisfing the following conditions: For every point x of X, there are an $n(x) \in \mathbb{N}$ and an open neighborhood $U(x)$ of x in X such that for every point x' of $U(x)$ and every $n(x)+2$ many points $y_1, ..., y_{n(x)+2}$ of X, there are distinct natural numbers i and j such that $\rho(y_i,\ y_j) \leq \rho(x',\ y_j)$.*

(*d*) *There is an admissible metric ρ for X satisfing the following conditions: For every point x of X, there are an $n(x) \in \mathbb{N}$ and an open neighborhood $U(x)$ of x in X such that for every point x' of $U(x)$ and every $n(x)+2$ many points $y_1, ..., y_{n(x)+2}$ of X, there are natural numbers i, j and k such that $i \neq j$ and $\rho(y_i,\ y_j) \leq \rho(x',\ y_k)$.*

References

[1] R. Engelking, *Theory of Dimensions Finite and Infinite*, Heldermann Verlag (1995).

[2] J. de Groot, *On a metric that characterizes dimension*, Canad. J. Math. 9 (1957), 511-514.

[3] Y. Hattori, *On special metrics characterizing topological properties*, Fund. Math. 126 (1986), 133-145.

[4] M. Katětov, *On the relations between the metric and topological dimensions*, Czech. Math. J. 8 (1958) 163-166.

[5] J. Nagata, *On a relation between dimension and metrization*, Proc. Jap. Ac. 32 (1956), 237-240.

[6] J. Nagata, *On a special metric characterizing a metric space of* $\dim \leq n$, Proc. Japan Acad. 39 (1963), 278-282.

[7] J. Nagata, *On a special metric and dimension*, Fund. Math. 55 (1964), 181-194.

[8] J. Nagata, *Topics in dimension theory*, General Topology and its Relations to Modern Analysis and Algebra (Proc. Fifth Prague Topology Symposium 1981), Heldermann Verlag, Berlin (1982), 497-507.

[9] J. Nagata, *Modern Dimension Theory, revised and extended edn.*, Heldermann, Berlin, 1983.

[10] J. Nagata, *Open problems left in my wake of research*, Topology and its Appl. 146-147 (2005), 5-13.

[11] Z. Shmuely, *On strongly countable-dimensional sets*, Duke Math. J. 38 (1971), 169-173.

Masahiro Matsumoto
Institute of Mathematics
University of Tsukuba
Ibaraki, 305-8571 Japan
e-mail: masahiro@math.tsukuba.ac.jp