CORRIGENDUM: IMPLICATIVE BCS-ALGEBRA SUBREDUCTS OF SKEW BOOLEAN ALGEBRAS

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1. Introduction

An implicative BCS-algebra $\langle A; \backslash, 0\rangle$ is a non-commutative analogue of an implicative BCK- or Tarski algebra. The variety iBCS of all implicative BCS-algebras has been investigated extensively in [1]. In [2] the authors studied implicative BCS-algebras and their connections with the skew Boolean algebras of Cornish [3] and Leech [4]. This note corrects some of the results claimed in [2], which are too strong to hold in the general case.

2. Implicative BCS-Algebra Subreducts

In [2, Section 3, p. 633] it is observed that any member $\langle A; \wedge, ^*, 0\rangle$ of the variety PCSL of pseudo-complemented semilattices has a canonical implicative BCS-algebra term reduct $\langle A; \backslash, 0\rangle$, where for all $a, b \in A$, $a\backslash b := a \wedge b^*$. Theorem 3.6 of [2] asserts that iBCS coincides with the class of all such term reducts of members of PCSL. This claim is too strong, since there exists a four-element implicative BCS-algebra that is not the $\langle\backslash, 0\rangle$-term reduct of any four-element pseudo-complemented semilattice. Instead, the following slightly weaker result holds.

Theorem 2.1. [2, Theorem 3.6] iBCS *is the class of all* $\langle\backslash, 0\rangle$*-term subreducts of* PCSL.

Proof. It suffices to show that for any implicative BCS-algebra $\mathbf{A}$ there exists a pseudo-complemented semilattice $\mathbf{A}^*$ such that $\mathbf{A}$ is isomorphic to a subalgebra of the $\langle\backslash, 0\rangle$-term reduct of $\mathbf{A}^*$. So assume without loss of generality that $\mathbf{A}$ is a subalgebra of a product $\prod_{\gamma\in\Gamma} \mathbf{A}_\gamma$ of a family $(\mathbf{A}_\gamma)_{\gamma\in\Gamma}$ of subdirectly irreducible implicative BCS-algebras. By [1, Theorem 5.13] the algebra $\mathbf{A}_\gamma$ is, for each $\gamma \in \Gamma$, the $\langle\backslash, 0\rangle$-term reduct of a subdirectly irreducible pseudo-complemented semilattice $\mathbf{A}^*_\gamma$. Hence $\mathbf{A}$ is a subalgebra of the $\langle\backslash, 0\rangle$-term reduct of the pseudo-complemented semilattice $\mathbf{A}^* := \prod_{\gamma\in\Gamma} \mathbf{A}^*_\gamma$. ∎

A similar proof can be employed to show iBCS is the class of all $\langle\backslash, 0\rangle$-term subreducts of the variety of pseudo-complemented distributive lattices, rather than the class of all $\langle\backslash, 0\rangle$-term reducts, as indicated in [2].

3. The Quasivariety $\mathbf{Q}(\mathbf{B}_2)$

In [2, Section 4] the quasivariety $\mathbf{Q}(\mathbf{B}_2)$ generated by the three-element subdirectly irreducible implicative BCS-algebra $\mathbf{B}_2$ is investigated. (The quasivariety $\mathbf{Q}(\mathbf{B}_1)$ generated by the two-element implicative BCS-algebra $\mathbf{B}_1$ is just the variety of all implicative BCK- or Tarski algebras.) Lemma 4.1 of [2] introduces a certain map $\varphi_a : A \to (A \wedge a) \times \mathrm{ann}(a)$,

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which plays an important role in the study of $\mathbf{Q}(\mathbf{B}_2)$. It is erroneously claimed in part (3) of this lemma that the map φ_a is an epimorphism. The correct statement of the lemma is:

Lemma 3.1. [2, Lemma 4.1] *Let $\mathbf{A}$ be an implicative BCS-algebra and let $a \in A$ be fixed. The following assertions hold:*

1. *The maps $c \mapsto c\wedge a$ and $c \mapsto c\backslash a$ are epimorphisms from $\mathbf{A}$ onto $A\wedge a := \{b\wedge a : b \in A\}$ and $\operatorname{ann}(a) := \{b \in A : a \wedge b = 0\}$ respectively.*
2. *The relations Φ_a and Ψ_a, defined respectively for all $b, c \in A$ by:*
$$b \equiv c \pmod{\Phi_a} \quad \textit{if and only if} \quad b \wedge a = c \wedge a$$
$$b \equiv c \pmod{\Psi_a} \quad \textit{if and only if} \quad b\backslash a = c\backslash a$$
are congruences on $\mathbf{A}$. Moreover, when $\mathbf{A} \in \mathbf{Q}(\mathbf{B}_2)$, both Φ_a and Ψ_a are $\mathbf{Q}(\mathbf{B}_2)$-congruences on $\mathbf{A}$.
3. *The sets $A \wedge a$ and $\operatorname{ann}(a)$ are (the base sets of) retracts of $\mathbf{A}$. Thus the map $\varphi_a : A \to (A \wedge a) \times \operatorname{ann}(a)$ defined for all $c \in A$ by:*
$$\varphi_a(c) := \langle c \wedge a, c\backslash a\rangle$$
is a homomorphism.

Proof. (3) This follows immediately from the proofs of [2, Lemma 4.1(1), Lemma 4.1(2)]. ∎

In view of this correction, the statement and proof of [2, Theorem 4.3] and the proof of [2, Theorem 4.7], which rely on the lemma, also need to be corrected. These two theorems characterise and describe the quasivariety $\mathbf{Q}(\mathbf{B}_2)$.

Theorem 3.2. [2, Theorem 4.3] *The following are equivalent for $\mathbf{A} \in \mathsf{iBCS}$.*

1. $\mathbf{A} \in \mathbf{Q}(\mathbf{B}_2)$.
2. $\mathbf{A} \models x \wedge z \approx y \wedge z \;\&\; x\backslash z \approx y\backslash z \supset x \approx y$.
3. *For any $a \in A$, the map φ_a of Lemma 3.1(3) is one-to-one.*
4. *$\mathbf{A}$ is the $\langle\backslash, 0\rangle$-subreduct of a skew Boolean algebra.*

Proof. (2) $\Leftrightarrow$ (3) For any $b, c \in A$, we have $\varphi_a(b) = \varphi_a(c)$ if and only if $b\backslash a = c\backslash a$ and $b \wedge a = c \wedge a$. Hence $\mathbf{A} \models x \wedge z \approx y \wedge z \;\&\; x\backslash z \approx y\backslash z \supset x \approx y$ if and only if φ_a is one-to-one. ∎

Theorem 3.3. [2, Theorem 4.7] *To within isomorphism, the only $\mathbf{Q}(\mathbf{B}_2)$-subdirectly irreducible members of $\mathbf{Q}(\mathbf{B}_2)$ are the three-element and two-element flat implicative BCS-algebras $\mathbf{B}_2$ and $\mathbf{B}_1$.*

Proof. Suppose $\mathbf{A} \in \mathbf{Q}(\mathbf{B}_2)$ is $\mathbf{Q}(\mathbf{B}_2)$-subdirectly irreducible. Let $a \in A$ be such that $a \neq 0$ and let Φ_a and Ψ_a be the $\mathbf{Q}(\mathbf{B}_2)$-congruences of Lemma 3.1(2). Since $a\backslash a = 0 = 0\backslash a$ we have $a \equiv 0 \pmod{\Psi_a}$. Hence $\Psi_a \neq \omega$. However, $\Psi_a \cap \Phi_a = \omega$, which implies that $\Phi_a = \omega$, since $\mathbf{A}$ is $\mathbf{Q}(\mathbf{B}_2)$-subdirectly irreducible. This in turn implies $\mathbf{A}$ is flat and the result now follows, since (i) the flatness of $\mathbf{A}$ forces $\operatorname{Con}\mathbf{A} = \operatorname{Con}_{\mathbf{Q}(\mathbf{B}_2)}\mathbf{A}$ (by [2, Lemma 4.5(4)]) and (ii) to within isomorphism, $\mathbf{B}_2$ and $\mathbf{B}_1$ are the only flat subdirectly irreducible implicative BCS-algebras. ∎

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