DYNAMICAL SYSTEM FOR FOREST KINEMATIC MODEL UNDER DIRICHLET CONDITIONS

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Abstract. We are concerned with a forest kinetic model equipped with the Dirichlet boundary conditions which has been presented by Kuzunetsov et al. [4]. We construct global solutions and construct a dynamical system determined from the Cauchy problem of the model equations. It is also shown that the dynamical system possesses a bounded absorbing set and every trajectory has a nonempty ω-limit set in a suitable weak topology. These results are then a modification of those obtained in our previous paper [1] from the Neumann condition case to the Dirichlet condition case.

1 Introduction This paper together with the forth coming two papers are going to be devoted to rewriting our previous results [1, 2, 3] to the case of the Dirichlet boundary conditions. In the papers [1, 2, 3], we have studied a prototype forest kinematic model which was presented by Kuznetsov, Antonovsky, Biktashev and Aponina [4]:

$$
(1.1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} = \beta\delta w - \gamma(v)u - fu & \text{in} \quad \Omega \times (0,\infty), \\ \dfrac{\partial v}{\partial t} = fu - hv & \text{in} \quad \Omega \times (0,\infty), \\ \dfrac{\partial w}{\partial t} = d\Delta w - \beta w + \alpha v & \text{in} \quad \Omega \times (0,\infty), \\ \dfrac{\partial w}{\partial n} = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x),\ w(x,0) = w_0(x) & \text{in} \quad \Omega. \end{cases}
$$

The system consists of a mono-species and contains only two age classes, the young age class and the old age class. The unknown functions $u(x,t)$ and $v(x,t)$ denote the tree densities of young and old age classes, respectively, at a position $x \in \Omega$ and at time $t \in [0,\infty)$, and the third unknown function $w(x,t)$ denotes the density of seeds in the air at $x \in \Omega$ and $t \in [0,\infty)$ satisfying the Neumann boundary conditions, Ω being a two-dimensional domain in which the forest exists and $\partial\Omega$ being its boundary.

In [1] we have constructed not only global solutions but also a dynamical system determined from the initial-boundary value problem in the underlying space X given by (3.1). In [2] we have shown that the dynamical system has a somewhat simple structure, namely, the dynamical system enjoys a Lyapunov function and every trajectory has nonempty ω-limit set, although the ω-limit set is defined for the moment by a weak topology of X, which

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consists of stationary solutions. In the third paper [3], we have investigated stability and instability for homogeneous stationary solutions and have also shown in some case that there exist infinite number of discontinuous stationary solutions. Through the papers (cf. also [5]), we find out that, when $ab^2 > 3(c+f)$ (see (1.3)), (1.1) possesses stationary solutions which have a clear curve in Ω on which the graphs of u and v have discontinuous gaps. In the view point of forestry, this curve may be considered as a forest boundary which is highly interesting (see [4]) created by the system in a canonical way. At the same time, we find out that the Neumann boundary conditions we imposed on w imply that, in some cases (i.e., $0 < h < \frac{f\alpha\delta}{ab^2+c+f}$), the curve coincides with the whole boundary $\partial\Omega$ and, in other cases (i.e., $\frac{f\alpha\delta}{ab^2+c+f} < h < \frac{f\alpha\delta}{c+f}$), the curve is not uniquely determined by the parameters appearing in the equations of (1.1).

It is then very natural to ask how the forest boundary is determined from the various parameters appearing in (1.1) which characterize a forest ecosystem. To study this problem we are led to introduce the initial-boundary value problem in which we impose on w the Dirichlet boundary conditions, i.e., (1.2) below, for this model can be expected to have a unique forest boundary if the condition $ab^2 > 3(c+f)$ is satisfied and if the parameter h is suitably small.

In fact, let these conditions be satisfied and let $(\overline{u}, \overline{v}, \overline{w})$ be a stable stationary solution to (1.2). We can prove that the zero solution $(0,0,0)$ is unstable if h is sufficiently small; therefore, $(\overline{u}, \overline{v}, \overline{w})$ must be non zero. Consider a point $x \in \Omega$ which is far from the boundary $\partial\Omega$ in such a way that the value $(\overline{u}(x), \overline{v}(x), \overline{w}(x))$ is free from the Dirichlet boundary conditions. Then, Theorem 2.8 of [3] suggests that $(\overline{u}(x), \overline{v}(x), \overline{w}(x))$ may be close to $\big(\frac{h}{f}(b+\sqrt{D}), b+\sqrt{D}, \frac{\alpha}{\beta}(b+\sqrt{D})\big)$, where $D = \frac{f\alpha\delta-(c+f)h}{ah}$. On the other hand, at the boundary point $x \in \partial\Omega$, $(\overline{u}(x), \overline{v}(x), \overline{w}(x))$ must be zero. So, these observations deduce that $\overline{w}(x)$ changes its values from 0 to $\frac{\alpha}{\beta}(b+\sqrt{D})$. Since $f\overline{u}(x) - h\overline{v}(x) = 0$ and $\beta\delta\overline{w}(x) - \gamma(\overline{v}(x))\overline{u}(x) - f\overline{u}(x) = 0$ for all $x \in \Omega$, we have $\overline{w}(x) = Q(\overline{v}(x))$, where $Q(v)$ is a cubic function given by $Q(v) = \frac{h}{f\beta\delta}[\gamma(v) + f]v$ due to (1.3). The condition $ab^2 > 3(c+f)$ implies that $Q(v)$ possesses two extremal values (cf. Figure 2 of [3]), and it is impossible that the value $\overline{v}(x)$ changes continuously if $\overline{w}(x)$ changes monotonically increasingly and continuously from 0 to $\frac{\alpha}{\beta}(b+\sqrt{D})$, namely, $\overline{v}(x)$ must have a discontinuous gap. Note that we can show that $\overline{w}(x)$ must be a concave function, i.e., $\Delta\overline{w} \le 0$.

We are therefore concerned with the initial-boundary value problem

$$
(1.2) \qquad \begin{cases}
\dfrac{\partial u}{\partial t} = \beta\delta w - \gamma(v)u - fu & \text{in} \quad \Omega \times (0,\infty), \\
\dfrac{\partial v}{\partial t} = fu - hv & \text{in} \quad \Omega \times (0,\infty), \\
\dfrac{\partial w}{\partial t} = d\Delta w - \beta w + \alpha v & \text{in} \quad \Omega \times (0,\infty), \\
w = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\
u(x,0) = u_0(x),\ v(x,0) = v_0(x),\ w(x,0) = w_0(x) & \text{in} \quad \Omega
\end{cases}
$$

in a two-dimensional domain $\Omega \subset \mathbb{R}^2$. The first equation of (1.2) denotes the growth of young trees in Ω, and the second the growth of old trees. The third equation describes the kinetics of seeds; $d > 0$ is a diffusion constant of seeds, and $\alpha > 0$ and $\beta > 0$ are seed production and seed deposition rates respectively. While, $0 < \delta \leqslant 1$ is a seed establishment rate, $\gamma(v) > 0$ is a mortality of young trees which is allowed to depend on the old-tree density v, $f > 0$ is an aging rate, and $h > 0$ is a mortality of old trees. On w, the Dirichlet boundary conditions are imposed on the boundary $\partial\Omega$. Nonnegative initial functions $u_0(x) \geqslant 0$,

$v_0(x) \geqslant 0$ and $w_0(x) \geqslant 0$ are given in Ω.

Throughout the paper, Ω is a bounded, convex or $\mathcal{C}^2$ domain in $\mathbb{R}^2$. According to [10], the Poisson problem $-d\Delta w + \beta w = v$ in Ω under the Dirichlet boundary conditions $w = 0$ on $\partial\Omega$ enjoys the optimal shift property that $v \in L^2(\Omega)$ always implies that $w \in H^2(\Omega)$. We assume as in [4, p. 220] that the mortality of young trees is given by a square function of the form

$$\gamma(v) = a(v-b)^2 + c, \tag{1.3}$$

where a, b, $c > 0$ are positive constants. This means that the mortality takes its minimum when the old-age tree density is a specific value b. As mentioned, d, f, h, α and $\beta > 0$ are all positive constants (> 0) and $0 < \delta \leqslant 1$.

Under these assumptions, we will prove global existence of solution to (1.2) and will construct a dynamical system determined from Problem (1.2). But the proof can be carried out in a quite analogous way as in [1].

2 Preliminary

We shall first recall the known results for abstract semilinear evolution equations studied in [6]. Consider the Cauchy problem

$$\begin{cases} \dfrac{dU}{dt} + AU = F(U), \quad 0 < t \leqslant T, \\ U(0) = U_0, \end{cases} \tag{2.1}$$

in a Banach space X. Here, A is a densely defined, closed linear operator of X, the spectral set of which is contained in a sectorial domain $\Sigma = \{\lambda \in \mathbb{C}; |\arg \lambda| < \omega\}$ with some angle $0 < \omega < \frac{\pi}{2}$, and the resolvent satisfies the estimate

$$\|(\lambda - A)^{-1}\|_{\mathcal{L}(X)} \leqslant \frac{M}{|\lambda| + 1}, \quad \lambda \notin \Sigma \tag{2.2}$$

with some constant $M \geq 1$. Therefore, $-A$ generates an analytic semigroup e^{-tA} on X. U_0 is an initial value in $\mathcal{D}(A^\mu)$ with the estimate

$$\|A^\mu U_0\| \leqslant R, \tag{2.3}$$

here μ is some exponent such that $0 \leqslant \mu < 1$ and $R > 0$ is a constant. $F(U)$ is a nonlinear mapping from $\mathcal{D}(A^\eta)$ to X with $\mu \leqslant \eta < 1$ and is assumed to satisfy a Lipschitz condition of the form

$$\begin{aligned} \|F(U) - F(V)\| &\leqslant \varphi(\|A^\mu U\| + \|A^\mu V\|) \\ &\times [\|A^\eta(U - V)\| + (\|A^\eta U\| + \|A^\eta V\|)\|A^\mu(U - V)\|], \qquad U, V \in \mathcal{D}(A^\eta), \end{aligned} \tag{2.4}$$

where $\varphi(\cdot)$ is some increasing continuous function. Then the following theorem is known.

Theorem 2.1 ([6, Theorem 3.1])**.** *Let $0 \leqslant \mu \leqslant \eta < 1$ and let (2.2), (2.3) and (2.4) be satisfied. Then (2.1) possesses a unique local solution in the function space*

$$\begin{cases} U \in \mathcal{C}([0, T_R]; \mathcal{D}(A^\mu)) \cap \mathcal{C}^1((0, T_R]; X) \cap \mathcal{C}((0, T_R]; \mathcal{D}(A)), \\ t^{1-\mu} U \in \mathcal{B}((0, T_R]; \mathcal{D}(A)), \end{cases}$$

where $T_R > 0$ being determined by R. Moreover, the estimate

$$t^{1-\mu}\|AU(t)\| + \|A^\mu U(t)\| \leqslant C_R, \qquad 0 < t \leqslant T_R$$

holds with some constant C_R determined by R alone.

We shall next list some well-known results in the theory of function spaces. Let Ω denote a bounded, convex or $\mathcal{C}^2$ domain in $\mathbb{R}^2$. For $0 \leqslant s \leqslant 2$, $H^s(\Omega)$ denotes the Sobolev space, its norm being denoted by $\|\cdot\|_{H^s}$ (see [10, Chap. 1] and [12]). For $0 \leqslant s_0 \leqslant s \leqslant s_1 \leqslant 2$, $H^s(\Omega)$ coincides with the complex interpolation space $[H^{s_0}(\Omega), H^{s_1}(\Omega)]_\theta$, where $s = (1-\theta)s_0 + \theta s_1$, and the estimate

$$\|\cdot\|_{H^s} \leqslant C\|\cdot\|_{H^{s_0}}^{1-\theta}\|\cdot\|_{H^{s_1}}^{\theta} \tag{2.5}$$

holds. When $0 \leqslant s < 1$, $H^s(\Omega) \subset L^p(\Omega)$, where $\frac{1}{p} = \frac{1-s}{2}$, with continuous embedding

$$\|\cdot\|_{L^p} \leqslant C_s\|\cdot\|_{H^s}. \tag{2.6}$$

When $s = 1$, $H^1(\Omega) \subset L^q(\Omega)$ for any finite $2 \leqslant q < \infty$ with the estimate

$$\|\cdot\|_{L^q} \leqslant C_{pq}\|\cdot\|_{H^1}^{1-\frac{p}{q}}\|\cdot\|_{L^p}^{\frac{p}{q}}, \tag{2.7}$$

where $1 \leqslant p < q < \infty$. When $s > 1$, $H^s(\Omega) \subset \mathcal{C}(\overline{\Omega})$ with continuous embedding

$$\|\cdot\|_{\mathcal{C}} \leqslant C_s\|\cdot\|_{H^s}. \tag{2.8}$$

Consider a sesquilinear form given by

$$a(u,v) = d\int_\Omega \nabla u \cdot \nabla\overline{v}dx + \beta\int_\Omega u\overline{v}dx, \qquad u,\ v \in \mathring{H}^1(\Omega)$$

on the space $\mathring{H}^1(\Omega)$, where $\mathring{H}^1(\Omega)$ is the closure of $\mathcal{C}_0^\infty(\Omega)$ in the space $H^1(\Omega)$. It is known that $\mathring{H}^1(\Omega)$ is characterized by

$$\mathring{H}^1(\Omega) = H_D^1(\Omega) = \{u \in H^1(\Omega);\ u = 0 \text{ on } \partial\Omega\}.$$

From this form, we can define a realization Λ of the Laplace operator $-d\Delta + \beta$ in $L^2(\Omega)$ under the Dirichlet boundary conditions on the boundary $\partial\Omega$ (see [9, Chap. VI]). The realization $\Lambda \geqslant \beta$ is a positive definite self-adjoint operator of $L^2(\Omega)$ and, according to [10], its domain is characterized by

$$\mathcal{D}(\Lambda) = H_D^2(\Omega) = \{u \in H^2(\Omega);\ u = 0 \text{ on } \partial\Omega\}. \tag{2.9}$$

For $0 \leq \theta \leq 1$, the fractional powers Λ^θ of Λ are defined and are also positive definite self-adjoint operators of $L^2(\Omega)$. As shown in [7], we can characterize for $0 \leqslant \theta \leqslant 1$, $\theta \neq \frac{1}{4}, \frac{3}{4}$, their domains in the form

$$\mathcal{D}(\Lambda^\theta) = \begin{cases} H^{2\theta}(\Omega), & \text{when } 0 \leqslant \theta < \frac{1}{4}, \\ H_D^{2\theta}(\Omega) = \{u \in H^{2\theta}(\Omega);\ u = 0 \text{ on } \partial\Omega\}, & \text{when } \frac{1}{4} < \theta \leqslant 1,\ \theta \neq \frac{3}{4}, \end{cases} \tag{2.10}$$

and the following estimates

$$C_1\|\Lambda^\theta\cdot\|_{L^2} \leqslant \|\cdot\|_{H^{2\theta}} \leqslant C_2\|\Lambda^\theta\cdot\|_{L^2}, \qquad 0 \leq \theta \leq 1,\ \theta \neq \tfrac{1}{4}, \tfrac{3}{4} \tag{2.11}$$

hold with some constants $0 < C_1 < C_2$.

Finally, consider an initial value problem

$$\begin{cases} \dfrac{du}{dt} = p(t)u + q(t), & 0 < t \leqslant T, \\ u(0) = u_0 \end{cases}$$

in the Banach space $L^\infty(\Omega)$. Here, p, $q \in \mathcal{C}([0,T]; L^\infty(\Omega))$ are given $L^\infty(\Omega)$ valued continuous functions and $u_0 \in L^\infty(\Omega)$ is an initial value. For each $0 < t \leqslant T$, put a function

$$\varphi(s) = e^{\int_s^t p(r)dr} u(s), \quad 0 \leqslant s \leqslant t.$$

Since the operator: $f \mapsto e^f$ is a Fréchet differentiable mapping from $L^\infty(\Omega)$ into itself and its derivative is given by a multiplication operator: $g \mapsto e^f g$ on $L^\infty(\Omega)$, it follows that $\varphi \in \mathcal{C}^1((0,t]; L^\infty(\Omega))$ and

$$\frac{\partial}{\partial s}[e^{\int_s^t p(r)dr} u(s)] = e^{\int_s^t p(r)dr}[-p(s)]u(s) + e^{\int_s^t p(r)dr}\frac{du}{ds}(s) = e^{\int_s^t p(r)dr} q(s).$$

Integrating this equality in $s \in [0,t]$, we obtain the formula

$$u(t) = e^{\int_0^t p(r)dr} u_0 + \int_0^t e^{\int_s^t p(r)dr} q(s) ds, \qquad 0 \leqslant t \leqslant T. \tag{2.12}$$

3 Local solutions We shall construct local solution to Problem (1.2) by handling it as an abstract equation of the form (2.1).

The underlying space X is set as a product function space of the form

$$X = \left\{ \begin{pmatrix} u \\ v \\ w \end{pmatrix}; \ u \in L^\infty(\Omega), \ v \in L^\infty(\Omega) \ \text{ and } \ w \in L^2(\Omega) \right\}. \tag{3.1}$$

The linear operator A is defined by

$$A = \begin{pmatrix} f & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & \Lambda \end{pmatrix} \quad \text{with} \quad \mathcal{D}(A) = \left\{ \begin{pmatrix} u \\ v \\ w \end{pmatrix}; \ u, v \in L^\infty(\Omega) \text{ and } w \in H_D^2(\Omega) \right\}, \tag{3.2}$$

where Λ is the operator defined in Section 2. It is clear that A satisfies (2.2). Moreover, for $0 \leqslant \theta \leqslant 1$,

$$A^\theta = \begin{pmatrix} f^\theta & 0 & 0 \\ 0 & h^\theta & 0 \\ 0 & 0 & \Lambda^\theta \end{pmatrix} \quad \text{with} \quad \mathcal{D}(A^\theta) = \left\{ \begin{pmatrix} u \\ v \\ w \end{pmatrix}; \ u, v \in L^\infty(\Omega) \ \text{ and } \ w \in \mathcal{D}(\Lambda^\theta) \right\}.$$

The nonlinear operator F is given by

$$F(U) = \begin{pmatrix} \beta\delta w - \gamma(v)u \\ fu \\ \alpha v \end{pmatrix}, \quad U = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in \mathcal{D}(A^\eta), \tag{3.3}$$

where η is an arbitrarily fixed exponent in such a way that $\frac{1}{2} < \eta < 1$. Initial value U_0 is taken from X. Then, the problem (1.2) is formulated in the form (2.1).

Let us now verify that the condition (2.4) is fulfilled with $\mu = 0$ and the η fixed above. In fact, for $U, V \in \mathcal{D}(A^\eta)$, we have

$$F(U) - F(V) = \begin{pmatrix} \beta\delta(w_1 - w_2) - \gamma(v_1)(u_1 - u_2) - [\gamma(v_1) - \gamma(v_2)]u_2 \\ f(u_1 - u_2) \\ \alpha(v_1 - v_2) \end{pmatrix},$$

where

$$U = \begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix} \qquad \text{and} \qquad V = \begin{pmatrix} u_2 \\ v_2 \\ w_2 \end{pmatrix}.$$

In view of (1.3), (2.8) and (2.11), it follows that the estimate

$$\begin{aligned} \|F(U) - F(V)\| &\leqslant C[\|w_1 - w_2\|_{L^\infty} + \|v_1 - v_2\|_{L^2} \\ &+ (\|u_1\|_{L^\infty}^2 + \|v_1\|_{L^\infty}^2 + \|u_2\|_{L^\infty}^2 + \|v_2\|_{L^\infty}^2 + 1)(\|u_1 - u_2\|_{L^\infty} + \|v_1 - v_2\|_{L^\infty})] \\ &\leqslant C[\|A^\eta(U - V)\| + (\|U\|^2 + \|V\|^2 + 1)\|U - V\|] \end{aligned} \tag{3.4}$$

holds with some constant C independent of U and V. This then shows that (2.4) is certainly fulfilled with $\mu = 0$ and $\frac{1}{2} < \eta < 1$. By virtue of Theorem 2.1, we now conclude the following result.

Theorem 3.1. *For any initial function u_0, $v_0 \in L^\infty(\Omega)$ and $w_0 \in L^2(\Omega)$, (1.2) possesses a unique local solution in the function space*

$$\begin{cases} u,\ v \in \mathcal{C}([0, T_0]; L^\infty(\Omega)) \cap \mathcal{C}^1((0, T_0]; L^\infty(\Omega)), \\ w \in \mathcal{C}([0, T_0]; L^2(\Omega)) \cap \mathcal{C}^1((0, T_0]; L^2(\Omega)) \cap \mathcal{C}((0, T_0]; H_D^2(\Omega)), \\ tw \in \mathcal{B}((0, T_0]; H_D^2(\Omega)). \end{cases} \tag{3.5}$$

Here, $T_0 > 0$ is determined by the norm $\|u_0\|_{L^\infty} + \|v_0\|_{L^\infty} + \|w_0\|_{L^2}$ alone. Moreover, the estimate

$$t\|\Lambda w(t)\| + \|w(t)\| \leqslant C_0, \quad 0 < t \leqslant T_0$$

holds with some constant C_0 determined by $\|u_0\|_{L^\infty} + \|v_0\|_{L^\infty} + \|w_0\|_{L^2}$ alone.

4 Nonnegativity of solutions We shall next verify that nonnegativity of initial functions implies that of the local solution obtained in Theorem 3.1.

Theorem 4.1. *Let $u_0, v_0 \in L^\infty(\Omega)$ and $w_0 \in L^2(\Omega)$ with $u_0 \geqslant 0$, $v_0 \geqslant 0$ and $w_0 \geqslant 0$. Then* (1.2) *possesses a unique local solution such that*

$$\begin{cases} 0 \leqslant u, v \in \mathcal{C}([0, T_0]; L^\infty(\Omega)) \cap \mathcal{C}^1((0, T_0]; L^\infty(\Omega)), \\ 0 \leqslant w \in \mathcal{C}([0, T_0]; L^2(\Omega)) \cap \mathcal{C}^1((0, T_0]; L^2(\Omega)) \cap \mathcal{C}((0, T_0]; H_D^2(\Omega)), \\ tw \in \mathcal{B}((0, T_0]; H_D^2(\Omega)). \end{cases}$$

Here, $T_0 > 0$ is determined by the norm $\|u_0\|_{L^\infty} + \|v_0\|_{L^\infty} + \|w_0\|_{L^2}$ alone. Moreover, the estimate

$$t\|\Lambda w(t)\| + \|w(t)\| \leqslant C_0, \qquad 0 < t \leqslant T_0$$

holds with some constant C_0 determined by $\|u_0\|_{L^\infty} + \|v_0\|_{L^\infty} + \|w_0\|_{L^2}$ alone.

Proof. By Theorem 3.1, (1.2) possesses a unique local solution (u, v, w) in function space (3.5) with $T_0 = T_{01}$ determined by the norm $\|u_0\|_{L^\infty} + \|v_0\|_{L^\infty} + \|w_0\|_{L^2}$.

Let us now consider an auxiliary problem

$$(4.1) \qquad \begin{cases} \dfrac{\partial \widetilde{u}}{\partial t} = \beta\delta\widetilde{w} - \gamma(\widetilde{v})\widetilde{u} - f\widetilde{u} & \text{in } \ \Omega \times (0,\infty), \\ \dfrac{\partial \widetilde{v}}{\partial t} = f\widetilde{u} - h\widetilde{v} & \text{in } \ \Omega \times (0,\infty), \\ \dfrac{\partial \widetilde{w}}{\partial t} = d\Delta\widetilde{w} - \beta\widetilde{w} + \alpha\chi(\operatorname{Re}\widetilde{v}) & \text{in } \ \Omega \times (0,\infty), \\ \widetilde{w} = 0 & \text{on } \ \partial\Omega \times (0,\infty), \\ \widetilde{u}(x,0) = u_0(x), \ \widetilde{v}(x,0) = v_0(x), \ \widetilde{w}(x,0) = w_0(x) & \text{in } \ \Omega. \end{cases}$$

Here, $\chi(\widetilde{v})$ is a cutoff function given by

$$\chi(\widetilde{v}) = \begin{cases} \widetilde{v} & \text{if } \ \widetilde{v} \geqslant 0, \\ 0 & \text{if } \ \widetilde{v} < 0. \end{cases}$$

It is clear that

$$|\chi(\widetilde{v}_1) - \chi(\widetilde{v}_2)| \leqslant |\widetilde{v}_1 - \widetilde{v}_2|, \quad \widetilde{v}_1, \widetilde{v}_2 \in \mathbb{R}.$$

By repeating the same arguments as in Section 3, we can deduce that (4.1) possesses a unique local solution $(\widetilde{u}, \widetilde{v}, \widetilde{w})$ in the function space (3.5) with $T_0 = T_{02}$ determined by the norm $\|u_0\|_{L^\infty} + \|v_0\|_{L^\infty} + \|w_0\|_{L^2}$. Our goal is then to show the nonnegativity of $\widetilde{u}, \widetilde{v}$ and $\widetilde{w}$. In this case, $\chi(\widetilde{v}) = \widetilde{v}$ and therefore $(\widetilde{u}, \widetilde{v}, \widetilde{w})$ is also a local solution of (1.2) in $[0, T_{02}]$. Then, by the uniqueness of solutions, we conclude that $(u, v, w) = (\widetilde{u}, \widetilde{v}, \widetilde{w})$ in $[0, T_0]$ with $T_0 = \min\{T_{01}, T_{02}\}$. This means that (1.2) possesses a unique nonnegative local solution in the function space (3.5) with T_0 determined by the norm $\|u_0\|_{L^\infty} + \|v_0\|_{L^\infty} + \|w_0\|_{L^2}$.

Let us first notice that $(\widetilde{u}, \widetilde{v}, \widetilde{w})$ are real valued. Indeed, it is clear that if $(\widetilde{u}, \widetilde{v}, \widetilde{w})$ is a local solution of (4.1), then its complex conjugate is also a local solution; so they must coincide; which shows that $(\widetilde{u}, \widetilde{v}, \widetilde{w})$ is real valued.

Let us now verify the nonnegativity. For this purpose, we shall use another cutoff function. Let $H(\tau)$ be the $\mathcal{C}^{1,1}$ function defined by

$$H(\tau) = \begin{cases} \dfrac{\tau^2}{2} & \text{if } \ \tau < 0, \\ 0 & \text{if } \ \tau \geqslant 0. \end{cases}$$

We consider the function

$$\psi_1(t) = \int_\Omega H(\widetilde{w}(x,t))dx, \quad 0 \leqslant t \leqslant T_{02}.$$

Clearly, $\psi_1(t)$ is a nonnegative $\mathcal{C}^1$ function with the derivative

$$\psi_1'(t) = -d\int_\Omega H''(\widetilde{w})|\nabla\widetilde{w}|^2dx - \beta\int_\Omega H'(\widetilde{w})\widetilde{w}dx + \alpha\int_\Omega H'(\widetilde{w})\chi(\widetilde{v})dx \leqslant 0, \quad 0 \leqslant t \leqslant T_{02}.$$

Since $\psi_1(0) = 0$, it follows that $\psi_1(t) = 0$ for every $t \in [0, T_{02}]$, that is, $\widetilde{w}(t) \geqslant 0$ in $[0, T_{02}]$.

By the same argument, putting

$$\psi_2(t) = \int_\Omega H(\widetilde{u}(x,t))dx, \quad 0 \leqslant t \leqslant T_{02},$$

we observe that $\psi_2(t)$ is a nonnegative $\mathcal{C}^1$ function with the derivative

$$\psi_2'(t) = \beta\delta \int_\Omega H'(\widetilde{u})\widetilde{w}dx - \int_\Omega [\gamma(\widetilde{v}) + f]H'(\widetilde{u})\widetilde{u}dx, \quad 0 \leqslant t \leqslant T_{02}.$$

Since $\widetilde{w} \geqslant 0$, it is seen that $\psi_2'(t) \leq 0$. Since $\psi_2(0) = 0$, it follows that $\psi_2(t) \equiv 0$ and hence $\widetilde{u}(t) \geqslant 0$ on $[0, T_{02}]$. It is the same for $\widetilde{v}$. Hence, theorem has been proved. □

5 Global solutions

We shall establish a priori estimates of local solutions.

Proposition 5.1. *Let $u_0, v_0 \in L^\infty(\Omega)$ and $w_0 \in L^2(\Omega)$ with $u_0 \geqslant 0$, $v_0 \geqslant 0$ and $w_0 \geqslant 0$. Let (u, v, w) be any local solution of (1.2) on an interval $[0, T_{u,v,w})$ such that*

$$\begin{cases} 0 \leqslant u, v \in \mathcal{C}([0, T_{u,v,w}); L^\infty(\Omega)) \cap \mathcal{C}^1((0, T_{u,v,w}); L^\infty(\Omega)), \\ 0 \leqslant w \in \mathcal{C}([0, T_{u,v,w}); L^2(\Omega)) \cap \mathcal{C}^1((0, T_{u,v,w}); L^2(\Omega)) \cap \mathcal{C}((0, T_{u,v,w}); H_D^2(\Omega)). \end{cases}$$

Then, the estimate

$$\begin{aligned} (5.1) \quad & \|u(t)\|_{L^\infty} + \|v(t)\|_{L^\infty} + \|w(t)\|_{L^2} \\ & \leqslant C[e^{-\rho t}(\|u_0\|_{L^\infty} + \|v_0\|_{L^\infty} + \|w_0\|_{L^2}) + 1], \qquad 0 \leqslant t < T_{u,v,w} \end{aligned}$$

holds with some constant $C > 0$ and some exponent $\rho > 0$ independent of (u, v, w).

Proof. Throughout the proof, we shall use notation $C_1, C_2, \dots$ and universal notation C, ρ, ρ' to denote positive constants and positive exponents which are determined by the constants a, b, c, d, f, h, α, β and δ and by Ω. In these, C, ρ and ρ' may be change from occurrence to occurrence.

Step 1. Estimate for $\|u\|_{L^2}$, $\|v\|_{L^2}$ and $\|w\|_{L^2}$. Multiply the first equation of (1.2) by u and integrate the product in Ω. Then we have

$$\begin{aligned} (5.2) \quad \frac{1}{2}\frac{d}{dt}\int_\Omega u^2dx + f\int_\Omega u^2dx &= \beta\delta\int_\Omega wudx - \int_\Omega \gamma(v)u^2dx \\ &\leqslant \frac{f}{2}\int_\Omega u^2dx + C_1\int_\Omega w^2dx - \int_\Omega \gamma(v)u^2dx. \end{aligned}$$

Multiply the third equation of (1.2) by w and integrate the product in Ω. Then,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega w^2dx + \beta\int_\Omega w^2dx = -d\int_\Omega |\nabla w|^2dx + \alpha\int_\Omega vwdx \leqslant \frac{\beta}{2}\int_\Omega w^2dx + C_2\int_\Omega v^2dx.$$

Let $C_3 > 0$ be constant such that $C_1C_3 \leqslant \frac{\beta}{4}$. Multiply (5.2) by C_3 and add the product to the above inequality. Then we obtain that

$$\begin{aligned} (5.3) \quad & \frac{C_3}{2}\frac{d}{dt}\int_\Omega u^2dx + \frac{1}{2}\frac{d}{dt}\int_\Omega w^2dx + \frac{C_3 f}{2}\int_\Omega u^2dx + \frac{\beta}{4}\int_\Omega w^2dx \\ & \leqslant C_2\int_\Omega v^2dx - C_3\int_\Omega \gamma(v)u^2dx. \end{aligned}$$

Next, multiply the second equation of (1.2) by v and integrate the product in Ω. Then,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v^2dx + h\int_\Omega v^2dx = f\int_\Omega uvdx.$$

Let $C_4 > 0$ be constant such that $C_4h \geqslant 2C_2$. Multiply the above equation by C_4 and add the product to the inequality (5.3) to obtain

$$\frac{C_3}{2}\frac{d}{dt}\int_\Omega u^2dx + \frac{C_4}{2}\frac{d}{dt}\int_\Omega v^2dx + \frac{1}{2}\frac{d}{dt}\int_\Omega w^2dx + \frac{C_3f}{2}\int_\Omega u^2dx + C_2\int_\Omega v^2dx \\ + \frac{\beta}{4}\int_\Omega w^2dx \leqslant C_4f\int_\Omega uvdx - C_3\int_\Omega \gamma(v)u^2dx.$$

We notice that

$$C_4fuv - C_3\gamma(v)u^2 = -\left[C_3a(v-b)^2u^2 - C_4f(v-b)u + \frac{C_4^2f^2}{4C_3a}\right] \\ -\left[C_3cu^2 - C_4fbu + \frac{C_4^2f^2b^2}{4C_3c}\right] + \frac{C_4^2f^2}{4C_3}\left(\frac{1}{a}+\frac{b^2}{c}\right) \leqslant \frac{C_4^2f^2}{4C_3}\left(\frac{1}{a}+\frac{b^2}{c}\right).$$

Therefore,

$$\frac{d}{dt}\int_\Omega (C_3u^2 + C_4v^2 + w^2)dx + \rho\int_\Omega (C_3u^2 + C_4v^2 + w^2)dx \leqslant C.$$

Solving this, we conclude that

$$C_3\|u(t)\|_{L^2}^2 + C_4\|v(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2 \leqslant Ce^{-\rho t}(C_3\|u_0\|_{L^2}^2 + C_4\|v_0\|_{L^2}^2 + \|w_0\|_{L^2}^2) + C.$$

Hence, it follows that

$$\|u(t)\|_{L^2} + \|v(t)\|_{L^2} + \|w(t)\|_{L^2} \\ \leqslant C[e^{-\rho t}(\|u_0\|_{L^\infty} + \|v_0\|_{L^\infty} + \|w_0\|_{L^2}) + 1], \qquad 0 \leqslant t < T_{u,v,w}. \tag{5.4}$$

Step 2. Estimate for $\|w(t)\|_{H^{2\eta}}$. Using the representation by the semigroup, we can write $w(t)$ in the form

$$\Lambda^\eta w(t) = \Lambda^\eta e^{-t\Lambda}w_0 + \int_0^t [\Lambda^\eta e^{-\frac{t-\tau}{2}\Lambda}]e^{-\frac{t-\tau}{2}\Lambda}\alpha v(\tau)d\tau.$$

In view of (2.11),

$$\|w(t)\|_{H^{2\eta}} \leqslant Ct^{-\eta}e^{-(\beta/2)t}\|w_0\|_{L^2} + C\int_0^t (t-\tau)^{-\eta}e^{-\frac{\beta}{2}(t-\tau)}\|v(\tau)\|_{L^2}d\tau,$$

here we used the estimate $\|e^{-t\Lambda}\| \leqslant e^{-t\beta}$ for $t \geqslant 0$. Moreover, by (5.4),

$$\int_0^t (t-\tau)^{-\eta}e^{-\frac{\beta}{2}(t-\tau)}\|v(\tau)\|_{L^2}d\tau \leqslant C\int_0^t (t-\tau)^{-\eta}e^{-\frac{\beta}{2}(t-\tau)}d\tau \\ + Ce^{-\rho' t}\int_0^t (t-\tau)^{-\eta}e^{-(\frac{\beta}{2}-\rho')(t-\tau)}e^{-(\rho-\rho')\tau}d\tau(\|u_0\|_{L^\infty} + \|v_0\|_{L^\infty} + \|w_0\|_{L^2}) \\ \leqslant Ce^{-\rho' t}(\|u_0\|_{L^\infty} + \|v_0\|_{L^\infty} + \|w_0\|_{L^2}) + C,$$

where $0 < \rho' < \min\{\frac{\beta}{2}, \rho\}$. Thus, we have obtained that

$$\|w(t)\|_{H^{2\eta}} \leqslant C[e^{-\rho t}(\|u_0\|_{L^\infty} + \|v_0\|_{L^\infty} + t^{-\eta}\|w_0\|_{L^2}) + 1], \qquad 0 \leqslant t < T_{u,v,w}. \tag{5.5}$$

Step 3. Estimate for $\|u(t)\|_{L^\infty}$ and $\|v(t)\|_{L^\infty}$. By using the formula (2.12), from the first equation of (1.2), we have

$$u(t) = e^{-\int_0^t \gamma(v(s))+f ds}u_0 + \beta\delta \int_0^t e^{-\int_\tau^t \gamma(v(s))+f ds}w(\tau)d\tau, \qquad 0 \leqslant t < T_{u,v,w}.$$

Therefore,

$$\|u(t)\|_{L^\infty} \leqslant e^{-ft}\|u_0\|_{L^\infty} + C\int_0^t e^{-f(t-\tau)}\|w(\tau)\|_{L^\infty}d\tau.$$

In addition, by (2.8) and (5.5),

$$\begin{aligned}\int_0^t e^{-f(t-\tau)}\|w(\tau)\|_{L^\infty}d\tau \leqslant C\int_0^t e^{-f(t-\tau)}\|w(\tau)\|_{H^{2\eta}}d\tau \leqslant C\int_0^t e^{-f(t-\tau)}d\tau \\ + Ce^{-\rho' t}\int_0^t (1+\tau^{-\eta})e^{-(f-\rho')(t-\tau)}e^{-(\rho-\rho')\tau}d\tau(\|u_0\|_{L^\infty} + \|v_0\|_{L^\infty} + \|w_0\|_{L^2}) \\ \leqslant Ce^{-\rho' t}(\|u_0\|_{L^\infty} + \|v_0\|_{L^\infty} + \|w_0\|_{L^2}) + C,\end{aligned}$$

where $0 < \rho' < \min\{f, \rho\}$. Thus, we conclude that

$$\|u(t)\|_{L^\infty} \leqslant C[e^{-\rho t}(\|u_0\|_{L^\infty} + \|v_0\|_{L^\infty} + \|w_0\|_{L^2}) + 1], \qquad 0 \leqslant t < T_{u,v,w}. \tag{5.6}$$

Similarly, by the second equation of (1.2) and (5.6) we also verify that

$$\|v(t)\|_{L^\infty} \leqslant C[e^{-\rho t}(\|u_0\|_{L^\infty} + \|v_0\|_{L^\infty} + \|w_0\|_{L^2}) + 1], \qquad 0 \leqslant t < T_{u,v,w}. \tag{5.7}$$

Combining (5.5), (5.6) and (5.7), we get the desired a priori estimates (5.1). □

As an immediate consequence of a priori estimates, we can prove the existence and uniqueness of global solutions.

Theorem 5.1. *Let u_0, $v_0 \in L^\infty(\Omega)$ and $w_0 \in L^2(\Omega)$ with $u_0 \geqslant 0$, $v_0 \geqslant 0$ and $w_0 \geqslant 0$. Then,* (1.2) *possesses a unique global solution such that*

$$\begin{cases} 0 \leqslant u, v \in \mathcal{C}([0,\infty); L^\infty(\Omega)) \cap \mathcal{C}^1((0,\infty); L^\infty(\Omega)), \\ 0 \leqslant w \in \mathcal{C}([0,\infty); L^2(\Omega)) \cap \mathcal{C}^1((0,\infty); L^2(\Omega)) \cap \mathcal{C}((0,\infty); H^2_D(\Omega)). \end{cases}$$

Proof. By Theorem 4.1, there exists a unique local solution (u, v, w) on an interval $[0, T_0]$. Moreover, by Proposition 5.1, $\|u(T_0)\|_{L^\infty} + \|v(T_0)\|_{L^\infty} + \|w(T_0)\|_{L^2}$ is estimated by $\|u_0\|_{L^\infty} + \|v_0\|_{L^\infty} + \|w_0\|_{L^2}$ alone. This then shows that the solution (u, v, w) can be extended as a local solution on an interval $[0, T_0 + \tau]$, where $\tau > 0$ is determined by $\|u(T_0)\|_{L^\infty} + \|v(T_0)\|_{L^\infty} + \|w(T_0)\|_{L^2}$, and hence depends only on $\|u_0\|_{L^\infty} + \|v_0\|_{L^\infty} + \|w_0\|_{L^2}$. Repeating this procedure, we obtain the result. □

We next verify the Lipschitz continuity of solution in initial data. Let B be a bounded set of initial values

$$B_R = \left\{ U_0 = \begin{pmatrix} u_0 \\ v_0 \\ w_0 \end{pmatrix} \in X; \|U_0\|_X \leqslant R \ \text{ with } \ u_0 \geqslant 0, v_0 \geqslant 0 \text{ and } w_0 \geqslant 0 \right\}$$

with any constant $R > 0$. By Theorem 5.1, there exists a unique global solution to (1.2) for each $U_0 \in B_R$.

Proposition 5.2. *Let U (resp. V) be the solution to* (1.2) *with initial value $U_0 \in B$ (resp. $V_0 \in B$). Then, for each $T > 0$ fixed, there exists some constants $C_{R,T} > 0$ depending on R and T alone such that*

$$t^\eta \|A^\eta [U(t) - V(t)]\| + \|U(t) - V(t)\| \leqslant C_{R,T} \|U_0 - V_0\|, \qquad 0 \leq t \leq T. \tag{5.8}$$

Proof. We have the formula

$$A^\eta [U(t) - V(t)] = A^\eta e^{-tA}(U_0 - V_0) + \int_0^t A^\eta e^{-(t-s)A}[F(U(s)) - F(V(s))]ds, \qquad 0 < t \leqslant T.$$

From (3.4) and Proposition 5.1, it follows that

$$t^\eta \|A^\eta [U(t) - V(t)]\| \leqslant A_\eta \|U_0 - V_0\| + C_R A_\eta t^\eta \int_0^t (t-s)^{-\eta} \|A^\eta [U(s) - V(s)]\| ds, \qquad 0 < t \leqslant T,$$

where $A_\eta = \sup_{0<t\leqslant T} t^\eta \|A^\eta e^{-tA}\|$. Then, by putting

$$p(t) = t^\eta \|A^\eta [U(t) - V(t)]\|, \qquad 0 < t \leqslant T,$$

we obtain the inequality

$$p(t) \leqslant A_\eta \|U_0 - V_0\| + C_R A_\eta \int_0^t t^\eta (t-s)^{-\eta} s^{-\eta} p(s) ds, \qquad 0 < t \leqslant T. \tag{5.9}$$

If $0 < t \leqslant \varepsilon$, we get

$$\begin{aligned} p(t) &\leqslant A_\eta \|U_0 - V_0\| + C_R A_\eta \int_0^t t^\eta (t-s)^{-\eta} s^{-\eta} ds \sup_{0<s\leqslant\varepsilon} p(s) \\ &\leqslant A_\eta \|U_0 - V_0\| + C_R A_\eta \varepsilon^{1-\eta} \sup_{0<s\leqslant\varepsilon} p(s). \end{aligned}$$

Therefore, taking $\varepsilon > 0$ sufficiently small, we conclude that

$$\sup_{0<t\leqslant\varepsilon} p(t) \leqslant C_R \|U_0 - V_0\|.$$

For $\varepsilon \leqslant t \leqslant T$, by (5.9) and the above estimate, we get

$$p(t) \leqslant \left(A_\eta + C_R \int_0^\varepsilon t^\eta (t-s)^{-\eta} s^{-\eta} ds \right) \|U_0 - V_0\| + C_R \varepsilon^{-\eta} \int_\varepsilon^t t^\eta (t-s)^{-\eta} p(s) ds.$$

Hence, it is deduced that

$$p(t) \leqslant C_{R,T}\|U_0 - V_0\|, \qquad \varepsilon \leqslant t \leqslant T.$$

Thus, we have obtained the first estimate of (5.8).

The second estimate of (5.8) then follows immediately from

$$\|U(t) - V(t)\| \leqslant A_0\|U_0 - V_0\| + C_R \int_0^t \|A^\eta[U(s) - V(s)]\| ds, \qquad 0 < t \leqslant T.$$

□

6 Dynamical system Let K be the space of initial values defined by

$$K = \left\{ \begin{pmatrix} u_0 \\ v_0 \\ w_0 \end{pmatrix} \in X;\ u_0 \geqslant 0, v_0 \geqslant 0 \text{ and } w_0 \geqslant 0 \right\}.$$

As shown in Section 5, for each $U_0 \in K$, there exists a unique global solution $U(t;U_0) = {}^t(u(t), v(t), w(t))$ to (1.2) and the solution is continuous with respect to the initial value. Therefore, we can define a semigroup $\{S(t)\}_{t\geqslant 0}$ acting on K by $S(t)U_0 = U(t;U_0)$. Such that the mapping $(t, U_0) \mapsto S(t)U_0$ is continuous from $[0, \infty) \times K$ into X, where K is equipped with the distance induced from the universal space X. Hence, we have constructed a dynamical system $(S(t), K, X)$ determined from (1.2).

We now verify that $(S(t), K, X)$ admits a bounded absorbing set. Indeed, let $R > 0$ be any radius and let U_0 be in K with $\|U_0\| \leqslant R$. Then, from (5.1) there exists a time t_R such that $\|U(t)\| \leqslant \widetilde{C} + 1$ for every $t \geqslant t_R$, where $\widetilde{C}$ is the constant appearing in (5.1). That is,

$$\sup_{\substack{U_0 \in K, \\ \|U_0\| \leqslant R}} \sup_{t \geqslant t_R} \|S(t)U_0\| \leqslant \widetilde{C} + 1.$$

This then shows that the set

$$\mathcal{B} = \{U \in K; \|U\| \leqslant \widetilde{C} + 1\}$$

is a bounded absorbing set of $(S(t), K, X)$.

Since $\mathcal{B}$ itself is absorbed by $\mathcal{B}$, there exists a time $t_{\mathcal{B}} > 0$ such that $S(t)\mathcal{B} \subset \mathcal{B}$ for every $t \geqslant t_{\mathcal{B}}$. We then consider the set

$$\widetilde{\mathcal{X}} = \bigcup_{0 \leqslant t < \infty} S(t)\mathcal{B} = \bigcup_{0 \leqslant t \leqslant t_{\mathcal{B}}} S(t)\mathcal{B}.$$

It is clean that $\widetilde{\mathcal{X}}$ is an absorbing and invariant bounded set of K. By Theorem 4.1 we then verify that

$$\|AS(t)U_0\| \leqslant C_{\widetilde{\mathcal{X}}} t^{-1}, \quad 0 < t \leqslant T_{\widetilde{\mathcal{X}}},\ U_0 \in \widetilde{\mathcal{X}}$$

with a sufficiently small time $T_{\widetilde{\mathcal{X}}} > 0$ and a constant $C_{\widetilde{\mathcal{X}}} > 0$. In view of such a smoothing effect, we introduce the subset

$$\mathcal{X} = S(T_{\widetilde{\mathcal{X}}})\widetilde{\mathcal{X}} \subset \widetilde{\mathcal{X}}.$$

It is easy to see that this subset is also an absorbing and invariant set. In addition, $\mathcal{X} \subset \mathcal{D}(A)$ with the estimate

$$\|AU\| = \|AS(T_{\widetilde{\mathcal{X}}})U_0\| \leqslant C_{\widetilde{\mathcal{X}}} T_{\widetilde{\mathcal{X}}}^{-1}, \quad U = S(T_{\widetilde{\mathcal{X}}})U_0 \in \mathcal{X},\ U_0 \in \widetilde{\mathcal{X}}.$$

We have thus verified the following result.

Theorem 6.1. *The dynamical system* $(S(t), K, X)$ *determined from the problem* (1.2) *can be reduced to a dynamical system* $(S(t), \mathcal{X}, X)$ *in which the phase space* $\mathcal{X}$ *is a bounded subset of* $\mathcal{D}(A)$.

Since $\mathcal{X}$ is a bounded set of $\mathcal{D}(A)$, it is meaningful to replace the universal space X by $X_\theta = \mathcal{D}(A^\theta)$ with any exponent $0 < \theta < 1$ and consider a dynamical system $(S(t), \mathcal{X}, X_\theta)$, where $\mathcal{X}$ is now a metric space with the distance $d_\theta(U, V) = \|A^\theta(U - V)\|$.

Corollary 1. *For each* $0 < \theta < 1$, $(S(t), \mathcal{X}, X_\theta)$ *is a dynamical system.*

Proof. By the moment inequality (cf. [12]) and the boundedness of $\mathcal{X}$ in $\mathcal{D}(A)$, it follows that

$$\|A^\theta(U - V)\| \leqslant C\|A(U - V)\|^\theta \|U - V\|^{1-\theta} \leqslant C_{\mathcal{X}} \|U - V\|^{1-\theta}, \quad U, V \in \mathcal{X}$$

with some constant $C_{\mathcal{X}}$. This shows that the mapping $(t, U) \mapsto S(t)U$ is continuous from $[0, \infty) \times \mathcal{X}$ into X_θ. □

7 ω-limit sets For each trajectory $S(t)U_0$, $U_0 \in \mathcal{X}$, the ω-limit set is defined by

$$\omega(U_0) = \bigcap_{t \geq 0} \overline{\{S(\tau)U_0;\ t \leq \tau < \infty\}} \qquad \text{(closure in the topology of } X).$$

We cannot expect, however, that $\omega(U_0)$ is a nonempty set for every initial value $U_0 \in \mathcal{X}$ (cf. [1, 2, 3]). So, we will introduce also some ω-limit set with a weak topology.

We will introduce the weak* topology of X: a sequence $\{{}^t(u_n, v_n, w_n)\}$ in X is said to be weak* convergent to ${}^t(u_0, v_0, w_0)$ as $n \to \infty$ if

$$\begin{cases} u_n \to u_0 & \text{weak* in } L^\infty(\Omega), \\ v_n \to v_0 & \text{weak* in } L^\infty(\Omega), \\ w_n \to w_0 & \text{strongly in } L^2(\Omega). \end{cases}$$

The weak* ω-limit set of $S(t)U_0$ is defined by

$$\text{(7.1)} \qquad \text{w*-}\omega(U_0) = \bigcap_{t \geqslant 0} \overline{\{S(\tau)U_0;\ t \leqslant \tau < \infty\}} \qquad \text{(closure in the weak* topology of } X).$$

Theorem 7.1. *For each* $U_0 \in \mathcal{X}$, w*-$\omega(U_0)$ *is a nonempty set.*

Proof. Let $U_0 = {}^t(u_0, v_0, w_0) \in \mathcal{X}$ and $U(t) = {}^t(u(t), v(t), w(t)) = S(t)U_0$. Since $\mathcal{B}$ is an absorbing set of $(S(t), \mathcal{X}, X)$, it follows that there exists a sequence of time $t_n \to \infty$ such that $S(t_n)U_0 \in \mathcal{B}$. Therefore, $\{u(t_n)\}$ is a bounded sequence in $L^\infty(\Omega)$. By Banach-Alaoglu's theorem, we can take a subsequence $\{u(t_{n'})\}$ of $\{u(t_n)\}$ such that $u(t_{n'}) \to \overline{u}$ weak* in $L^\infty(\Omega)$. Similarly, from the bounded sequence $\{v(t_{n'})\}$, we can extract a subsequence $\{v(t_{n''})\}$ such that $v(t_{n''}) \to \overline{v}$ weak* in $L^\infty(\Omega)$. Finally, by the boundedness of sequence $\{w(t_{n''})\}$ in $H^{2\eta}(\Omega)$, there exists a subsequence $\{w(t_{n'''})\}$ such that $w(t_{n'''}) \to \overline{w}$ strongly in $L^2(\Omega)$. Then, by the definition (7.1), we deduce that ${}^t(\overline{u}, \overline{v}, \overline{w})$ belongs to w*-$\omega(U_0)$. □

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