ASYMPTOTIC BEHAVIOR OF SOLUTIONS FOR FOREST KINEMATIC MODEL UNDER DIRICHLET CONDITIONS

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Abstract. We continue a study of the forest ecosystem model due to Kuzunetsov et al. [4] in which the Dirichlet conditions are imposed. In this paper, we introduce three kinds of ω-limit sets, namely, $\omega(U_0) \subset L^2\text{-}\omega(U_0) \subset \mathrm{w}^*\text{-}\omega(U_0)$, for each point U_0 of the dynamical system which has been constructed in our preceding paper [7]. Using a Lyapunov function, we will then investigate basic properties of the these ω-limit sets. Especially, it shall be shown that $L^2\text{-}\omega(U_0)$ consists of equilibria alone. These results are then a modification of those obtained in [2] from the Neumann condition case to the Dirichlet condition case.

1 Introduction We continue the study for a forest kinematic model

$$
(1.1)\qquad \begin{cases}
\dfrac{\partial u}{\partial t} = \beta\delta w - \gamma(v)u - fu & \text{in} \quad \Omega \times (0,\infty),\\
\dfrac{\partial v}{\partial t} = fu - hv & \text{in} \quad \Omega \times (0,\infty),\\
\dfrac{\partial w}{\partial t} = d\Delta w - \beta w + \alpha v & \text{in} \quad \Omega \times (0,\infty),\\
w = 0 & \text{on} \quad \partial\Omega \times (0,\infty),\\
u(x,0) = u_0(x),\ v(x,0) = v_0(x),\ w(x,0) = w_0(x) & \text{in} \quad \Omega
\end{cases}
$$

under the Dirichlet boundary conditions.

This system has been introduced by Kuzunetsov et al. [4] in order to describe the process of development of a forest ecosystem. They considered an age-structured continuous model which is of prototype in a two-dimensional domain Ω. The unknown functions $u(x,t)$ and $v(x,t)$ denote the tree densities of young and old age classes, respectively, at a position $x \in \Omega$ and time $t \in [0,\infty)$. The third unknown function $w(x,t)$ denotes the density of seeds in the air at $x \in \Omega$ and $t \in [0,\infty)$. The third equation describes the kinetics of seeds; $d > 0$ is a diffusion constant of seeds, and $\alpha > 0$ and $\beta > 0$ are seed production and seed deposition rates respectively. On w the Dirichlet boundary conditions are imposed. While the first and second equations describe the growth of young and old trees respectively; $0 < \delta \leqslant 1$ is a seed establishment rate, $\gamma(v) > 0$ is a mortality of young trees which is allowed to depend on the old-tree density v, $f > 0$ is an aging rate, and $h > 0$ is a mortality of old trees. It is assumed that $\gamma(v)$ is a square function which has a minimum at some value of v, namely,

$$
(1.2)\qquad \gamma(v) = a(v-b)^2 + c,
$$

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a, b, $c > 0$ being positive constants, see [4]. It is also assumed that Ω is a bounded, convex or $\mathcal{C}^2$ domain in $\mathbb{R}^2$.

In the previous paper [7], we have already formulated (1.1) as the Cauchy problem for an abstract parabolic evolution equation in the underlying function space

$$X = \left\{ \begin{pmatrix} u \\ v \\ w \end{pmatrix} ;\ u,\, v \in L^\infty(\Omega) \text{ and } w \in L^2(\Omega) \right\},$$

and have constructed not only global solutions for initial functions U_0 from

$$K = \left\{ U_0 = \begin{pmatrix} u_0 \\ v_0 \\ w_0 \end{pmatrix} ;\ 0 \leqslant u_0,\, v_0 \in L^\infty(\Omega) \text{ and } 0 \leqslant w_0 \in L^2(\Omega) \right\}$$

but also a dynamical system $(S(t), K, X)$ determined from (1.1) in the function space X with phase space K.

In this paper we will study asymptotic behavior of trajectories $S(t)U_0$, $U_0 \in K$. As pointed out in the series of papers [1, 2, 3] in which we handled the Neumann boundary conditions, the dynamical system $(S(t), K, X)$ does not enjoy any compact attractive set. So, as in [2], we shall introduce L^2 omega limit set L^2-$\omega(U_0)$ and weak* omega limit set w*-$\omega(U_0)$. Using the same Lyapunov function as in [2], we shall prove that L^2-$\omega(U_0)$ consists of stationary solutions alone.

2 Reviews In this section, we shall list the known results (1.1) which have been obtained in the previous paper [7], and shall also describe some consequences deduced from these which will be needed in the present paper.

The problem (1.1) is formulated as the Cauchy problem for a semilinear abstract evolution equation

$$\begin{cases} \frac{dU}{dt} + AU = F(U), \qquad 0 < t < \infty, \\ U(0) = U_0. \end{cases} \tag{2.1}$$

in the product space X. Here, A is a sectorial operator of X given by

$$A = \begin{pmatrix} f & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & \Lambda \end{pmatrix} \qquad \text{with} \qquad \mathcal{D}(A) = \left\{ \begin{pmatrix} u \\ v \\ w \end{pmatrix} ;\ u,\, v \in L^\infty(\Omega) \text{ and } w \in H^2_D(\Omega) \right\},$$

where Λ is a realization of the operator $-d\Delta + \beta$ in $L^2(\Omega)$ under the homogeneous Dirichlet boundary conditions $w = 0$ on the boundary $\partial\Omega$ and is a positive definite self-adjoint operator of $L^2(\Omega)$ and where $H^2_D(\Omega)$ is a closed subspace of $H^2(\Omega)$ consisting of functions w satisfying the homogeneous Dirichlet boundary conditions on $\partial\Omega$. Meanwhile, F is a nonlinear operator from $\mathcal{D}(A^\eta)$ into X given by

$$F(U) = \begin{pmatrix} \beta\delta w - \gamma(v)u \\ fu \\ \alpha v \end{pmatrix}, \qquad U = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in \mathcal{D}(A^\eta),$$

where η is some fixed exponent such that $\frac{1}{2} < \eta < 1$. Then, (1.1) is written in the form (2.1), see [7, Section 3].

According to [7, Theorem 5.2], for any U_0 from the space of initial values K, (2.1) possesses a unique global solution $U = {}^t(u, v, w)$ in the function space

$$0 \le u,\, v \in \mathcal{C}([0,\infty); L^\infty(\Omega)) \cap \mathcal{C}^1((0,\infty); L^\infty(\Omega)), \tag{2.2}$$

$$0 \le w \in \mathcal{C}([0,\infty); L^2(\Omega)) \cap \mathcal{C}((0,\infty); H^2_D(\Omega)) \cap \mathcal{C}^1((0,\infty); L^2(\Omega)). \tag{2.3}$$

Each u, v and w of the solution satisfies the following integral equation:

$$u(t) = e^{-\int_0^t \{\gamma(v(s)+f\}ds} u_0 + \beta\delta \int_0^t e^{-\int_s^t \{\gamma(v(\tau))+f\}d\tau} w(s)ds, \qquad 0 \le t < \infty, \tag{2.4}$$

$$v(t) = e^{-ht} v_0 + f \int_0^t e^{-(t-s)h} u(s)ds, \qquad 0 \le t < \infty, \tag{2.5}$$

$$w(t) = e^{-tA} w_0 + \alpha \int_0^t e^{-(t-s)A} v(s)ds, \qquad 0 \le t < \infty, \tag{2.6}$$

respectively. Here, e^{-tA} denotes the linear semigroup generated by A. Since $A \ge \beta$, it follows that $\|e^{-tA}\|_{L^2} \le e^{-\beta t}$.

We verify the following uniform estimates of solutions which were essentially established in [7, Proposition 5.1].

Proposition 2.1. *Let $U(t) = {}^t(u(t), v(t), w(t))$ be the global solution to* (2.1) *with $U_0 \in K$. Then,*

$$\|u(t)\|_{L^\infty} \le p(\|U_0\|_X), \qquad 0 \le t < \infty, \tag{2.7}$$

$$\|v(t)\|_{L^\infty} \le p(\|U_0\|_X), \qquad 0 \le t < \infty, \tag{2.8}$$

$$\|w(t)\|_{L^2} \le p(\|U_0\|_X), \qquad 0 \le t < \infty, \tag{2.9}$$

where $p(\cdot)$ denotes an appropriate continuous increasing function.

Proof. We already know that

$$\|U(t)\|_{L^2} \le p(\|U_0\|_{L^2}), \qquad 0 \le t < \infty.$$

We have from (2.6)

$$\begin{aligned} \|w(t)\|_{H^{2\eta}} &\le C\left\{\|A^\eta e^{-tA} w_0\|_{L^2} + \int_0^t \|A^\eta e^{-(t-s)A} v(s)\|_{L^2} ds\right\} \\ &\le C(1+t^{-\eta})e^{-\beta t}\|w_0\|_{L^2} + \int_0^t (1+(t-s)^{-\eta})e^{-\beta(t-s)} ds\, p(\|U_0\|_{L^2}) \\ &\le (1+t^{-\eta})p(\|U_0\|_{L^2}), \qquad 0 < t < \infty. \end{aligned}$$

As $\|w(t)\|_{L^\infty} \le C\|w(t)\|_{H^{2\eta}}$ (due to [7, (2.9)]), we obtain that

$$\|w(t)\|_{L^\infty} \le (1+t^{-\eta})p(\|U_0\|_{L^2}), \qquad 0 < t < \infty. \tag{2.10}$$

In view of (2.10), we use (2.4) to obtain that

$$\|u(t)\|_{L^\infty} \le \|u_0\|_{L^\infty} + \int_0^t e^{-f(t-s)}(1+s^{-\eta}) ds\, p(\|U_0\|_{L^2}) \le p(\|U_0\|_X), \qquad 0 \le t < \infty,$$

i.e., (2.7). Finally, (2.8) is easily observed by (2.5). $\square$

In addition, we verify the uniform estimates for the derivative of solutions.

Proposition 2.2. *For the derivative* $U'(t) = {}^t(u'(t), v'(t), w'(t))$,

$$\|u'(t)\|_{L^\infty} \le (1 + t^{-\eta}) p_1(\|U_0\|_X), \qquad 0 < t < \infty \tag{2.11}$$

$$\|v'(t)\|_{L^\infty} \le p_1(\|U_0\|_X), \qquad 0 < t < \infty \tag{2.12}$$

$$\|w'(t)\|_{L^2} + \|w(t)\|_{H^2} \le (1 + t^{-1}) p_1(\|U_0\|_X), \qquad 0 < t < \infty, \tag{2.13}$$

where $p_1(\cdot)$ *is an appropriate continuous increasing function.*

Proof. Using (2.7), (2.8) and (2.10) in the equation on u in (2.1), we immediately observe (2.11). Similarly, from the equation on v in (2.1) we observe (2.12). We know that $v \in \mathcal{C}([0,\infty); L^2(\Omega)) \cap \mathcal{C}^1([0,\infty); L^2(\Omega))$ with the estimate (2.12). Then, (2.13) is deduced by the standard arguments for the linear abstract equation applied to the equation for w in (2.1). Note that w is represented by (2.6). □

We next obtain uniform estimates for the second order derivative of solutions.

Proposition 2.3. *For the second order derivative* $U''(t) = {}^t(u''(t), v''(t), w''(t))$,

$$\|u''(t)\|_{L^\infty} \le (1 + t^{-1-\eta}) p_2(\|U_0\|_X), \qquad 0 < t < \infty, \tag{2.14}$$

$$\|v''(t)\|_{L^\infty} \le (1 + t^{-\eta}) p_2(\|U_0\|_X), \qquad 0 < t < \infty, \tag{2.15}$$

$$\|w''(t)\|_{L^2} + \|w'(t)\|_{H^2} \le (1 + t^{-2}) p_2(\|U_0\|_X), \qquad 0 < t < \infty, \tag{2.16}$$

where $p_2(\cdot)$ *is an appropriate continuous increasing function.*

Proof. From the second equation in (2.1),

$$v''(t) = f u'(t) - h v'(t), \qquad 0 < t < \infty.$$

Then, $v \in \mathcal{C}^2((0,\infty); L^\infty(\Omega))$ and the estimate (2.15) is seen by (2.11) and (2.12).

With any $\tau > 0$, we consider the Cauchy problem for a linear evolution equation

$$\begin{cases} \frac{dw^1}{dt} + \Lambda w^1 = \alpha v'(t), & \tau < t < \infty, \\ w^1(\tau) = w'(\tau) \end{cases}$$

in $L^2(\Omega)$, where $w^1 = w^1(t)$ is the unknown function. Since v' is in $\mathcal{C}^1([\tau,\infty); L^2(\Omega))$, this problem has a unique solution $w^1 \in \mathcal{C}^1((\tau,\infty); L^2(\Omega))$. By a direct calculation it is verified that $w^1(t) = w'(t)$ for any $t \in [\tau,\infty)$. Therefore,

$$w'(t) = e^{-(t-\tau)\Lambda} w'(\tau) + \alpha \int_\tau^t e^{-(t-s)\Lambda} v'(s) ds, \qquad \tau \le t < \infty.$$

Taking $\tau = \frac{t}{2}$, we repeat the same argument as for (2.13) to obtain that

$$\|w''(t)\|_{L^2} + \|w'(t)\|_{H^2} \le C(1 + t^{-1}) \|w'(\tfrac{t}{2})\|_{L^2} + C\{p_2(\|U_0\|_X) + p_1(\|U_0\|_X)\}, \qquad 0 < t < \infty.$$

Therefore, (2.16) is obtained in view of (2.13).

As a consequence of (2.13) and (2.16), we have

$$\|w'(t)\|_{L^\infty} \le C\|w'(t)\|_{H^{2\eta}} \le (1 + t^{-1-\eta}) p(\|U_0\|_X), \qquad 0 < t < \infty.$$

Then, (2.14) is observed directly from

$$u''(t) = \beta\delta w'(t) - \gamma'(v(t)) v'(t) u(t) - (\gamma(v(t)) + f) u'(t), \qquad 0 < t < \infty.$$

□

We conclude this section with describing the dynamical system determined by the Cauchy problem (2.1). For any $U_0 \in K$, let $U(t; U_0)$ be the global solution of (2.1). We set $S(t)U_0 = U(t; U_0)$ for every $0 \leq t < \infty$. Then $S(t)$ defines a nonlinear semigroup acting on K. According to [7, Proposition 5.3], the semigroup is continuous on K in the sense that the mapping $(t, U_0) \in [0, \infty) \times K \to K$ is continuous. Therefore, the set of all trajectories $S(t)U_0$ defines a dynamical system in X with phase space K which is denoted by $(S(t), K, X)$.

According to [7, Theorem 6.1], there exists an invariant and absorbing set $\mathcal{X}$ for $S(t)$ which is a bounded subset of $\mathcal{D}(A)$, namely,

$$\mathcal{X} \subset \left\{ \begin{pmatrix} u \\ v \\ w \end{pmatrix} ;\ 0 \leqslant u,\ v \in L^\infty(\Omega) \text{ and } 0 \leqslant w \in H^2_D(\Omega) \right.$$
$$\left. \text{with } \|u\|_{L^\infty} + \|v\|_{L^\infty} + \|w\|_{H^2} \leqslant C_{\mathcal{X}} \right\}$$

with some constant $0 < C_{\mathcal{X}} < \infty$. Therefore, $(S(t), \mathcal{X}, X)$ is also a dynamical system and the asymptotic behavior of trajectories of $(S(t), K, X)$ is reduced to that of $(S(t), \mathcal{X}, X)$.

3 Lyapunov function In this section we shall construct a Lyapunov function $\Psi(U)$ for the dynamical system $(S(t), K, X)$ and shall establish some results concerning the asymptotic behavior of trajectories $S(t)U_0$.

Let $U_0 \in K$ and let $S(t)U_0 = U(t) = {}^t(u(t), v(t), w(t))$ for $0 \leq t < \infty$. Put $\varphi(t) = fu(t) - hv(t)$, $0 \leqslant t < \infty$. From the first and second equations of (1.1) it is easily observed that

$$\frac{\partial \varphi}{\partial t} = f\beta\delta w - [\gamma(v) + f + h]\varphi - h[\gamma(v)v + fv], \qquad 0 < t < \infty.$$

Multiply this by $\varphi(t) = \frac{\partial v}{\partial t}$ and integrate the product in Ω. Then,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \varphi^2 dx + h\frac{d}{dt}\int_\Omega \Gamma(v)dx - f\beta\delta\int_\Omega \frac{\partial v}{\partial t} w\, dx = -\int_\Omega [\gamma(v) + f + h]\left(\frac{\partial v}{\partial t}\right)^2 dx, \tag{3.1}$$

where $\Gamma(v) = \int_0^v [\gamma(v)v + fv]dv$.

While, multiplying the third equation of (1.1) by $\frac{\partial w}{\partial t}$ and integrating the product in Ω, we obtain that

$$\frac{d}{2}\frac{d}{dt}\int_\Omega |\nabla w|^2 dx + \frac{\beta}{2}\frac{d}{dt}\int_\Omega w^2 dx - \alpha\int_\Omega v\frac{\partial w}{\partial t}\, dx = -\int_\Omega \left(\frac{\partial w}{\partial t}\right)^2 dx. \tag{3.2}$$

These two energy equalities (3.1) and (3.2) then provide that

$$\begin{aligned} \frac{d}{dt}\int_\Omega &\Big[\frac{\alpha}{2}\varphi^2 + \frac{df\beta\delta}{2}|\nabla w|^2 + h\alpha\Gamma(v) + \frac{f\beta^2\delta}{2}w^2 - (f\alpha\beta\delta)vw\Big]dx \\ &= -\int_\Omega \Big[\alpha\{\gamma(v) + f + h\}\left(\frac{\partial v}{\partial t}\right)^2 + f\beta\delta\left(\frac{\partial w}{\partial t}\right)^2\Big]dx \leqslant 0, \qquad 0 < t < \infty. \end{aligned} \tag{3.3}$$

Note that

$$\frac{\alpha}{2}(fu - hv)^2 + \frac{df\beta\delta}{2}|\nabla w|^2 + h\alpha\Gamma(v) + \frac{f\beta^2\delta}{2}w^2 - (f\alpha\beta\delta)vw \geqslant -C$$

with some constant C independent of U. This shows that the functional

$$(3.4)\quad \varPsi(U)=\int_\Omega\Big[\frac{\alpha}{2}(fu-hv)^2+\frac{df\beta\delta}{2}|\nabla w|^2+h\alpha\varGamma(v)$$
$$+\frac{f\beta^2\delta}{2}w^2-(f\alpha\beta\delta)vw\Big]dx,\qquad U\in\mathcal{D}(A^{\frac12})$$

is a Lyapunov function for the present dynamical system $(S(t),K,X)$.

From these arguments we obtain the following energy estimates.

Theorem 3.1. *For any trajectory $S(t)U_0=U(t)$, we have*

$$(3.5)\qquad \int_1^\infty\left\|\frac{dU}{dt}(t)\right\|_{L^2}^2dt<\infty.$$

Proof. Integrate both the sides of (3.3) in t on an interval $[1,T]$. Then,

$$\int_1^T\int_\Omega\Big[\alpha\{\gamma(v)+f+h\}\left(\frac{\partial v}{\partial t}\right)^2+f\beta\delta\left(\frac{\partial w}{\partial t}\right)^2\Big]dxdt$$
$$\le\int_\Omega\Big[\frac{\alpha}{2}\varphi(1)\frac{df\beta\delta}{2}|\nabla w(1)|^2+h\alpha\varGamma(v(1))+\frac{f\beta^2\delta}{2}w(1)^2+f\alpha\beta\delta v(T)w(T)\Big]dx.$$

Due to (2.8) and (2.10),

$$(3.6)\qquad \int_1^\infty\int_\Omega\Big[\alpha\{\gamma(v)+f+h\}\left(\frac{\partial v}{\partial t}\right)^2+f\beta\delta\left(\frac{\partial w}{\partial t}\right)^2\Big]dxdt<\infty.$$

Differentiating both the sides of the first equations of (1.1), we have

$$\frac{\partial^2u}{\partial t^2}=\beta\delta\frac{\partial w}{\partial t}-(\gamma(v)+f)\frac{\partial u}{\partial t}-2au(v-b)\frac{\partial v}{\partial t},\qquad 0<t<\infty.$$

Multiply this by $\frac{\partial u}{\partial t}$ and integrate the product in Ω. Then,

$$\frac12\frac{d}{dt}\int_\Omega\left(\frac{\partial u}{\partial t}\right)^2dx=\int_\Omega\left(\beta\delta\frac{\partial w}{\partial t}-2au(v-b)\frac{\partial v}{\partial t}\right)\frac{\partial u}{\partial t}dx-\int_\Omega(\gamma(v)+f)\left(\frac{\partial u}{\partial t}\right)^2dx$$
$$\leqslant Cp(\|U_0\|_X)\int_\Omega\Big[\left(\frac{\partial v}{\partial t}\right)^2+\left(\frac{\partial w}{\partial t}\right)^2\Big]dx-\frac{f}{2}\int_\Omega\left(\frac{\partial u}{\partial t}\right)^2dx.$$

Integrating both the sides in t, we obtain that

$$\frac{f}{2}\int_1^T\int_\Omega\left(\frac{\partial u}{\partial t}\right)^2dxdt\le\int_\Omega\left(\frac{\partial u}{\partial t}(1)\right)^2dx+Cp(\|U_0\|_X)\int_1^T\int_\Omega\Big[\left(\frac{\partial v}{\partial t}\right)^2+\left(\frac{\partial w}{\partial t}\right)^2\Big]dxdt.$$

Therefore, in view of (3.6), we conclude that

$$\int_1^\infty\int_\Omega\left(\frac{\partial u}{\partial t}\right)^2dxdt<\infty.$$

This together with (3.6) then yields the desired estimate (3.5). □

Theorem 3.2. *For any trajectory $S(t)U_0 = U(t)$, as $t \to \infty$, the derivative $\frac{dU}{dt}(t)$ tends to 0 in the L^2 topology.*

Proof. We prove the assertion of theorem by contradiction. Suppose that $\frac{dU}{dt}(t)$ might not converge to 0 in $L^2(\Omega)$ as $t \to \infty$. Then there would exist a number $\varepsilon > 0$ and a time sequence $\{t_n\}$ tending to ∞ such that

$$\left\| \frac{dU}{dt}(t_n) \right\|_{L^2}^2 \geq \varepsilon, \qquad n = 1, 2, 3, \dots .$$

In the meantime, by Propositions 2.2 and 2.3, we have

$$\left| \frac{d}{dt} \left\| \frac{dU}{dt}(t) \right\|_{L^2}^2 \right| = 2 \left| \left(\frac{d^2U}{dt^2}(t), \frac{dU}{dt}(t) \right) \right| \leq M, \qquad 1 \leq t < \infty$$

with some constant M. Consequently, by the mean-value theorem,

$$\left\| \frac{dU}{dt}(t) \right\|_{L^2}^2 \geq \begin{cases} M(t - t_n + \frac{\varepsilon}{M}), & t_n - \frac{\varepsilon}{M} \leq t \leq t_n, \\ -M(t - t_n - \frac{\varepsilon}{M}), & t_n \leq t \leq t_n + \frac{\varepsilon}{M}. \end{cases}$$

This is a contradiction to the fact that $\|\frac{dU}{dt}(t)\|_{L^2}^2$ is integrable in $(1, \infty)$, i.e., (3.5). □

4 ω-limit sets

In this section, we shall introduce three types of ω-limit sets, namely, $\omega(U_0)$, L^2-$\omega(U_0)$ and w*-$\omega(U_0)$, and shall investigate their relations.

As well known, the (usual) ω-limit set of $S(t)U_0$, $U_0 \in K$, is defined by

$$\omega(U_0) = \bigcap_{t \geqslant 0} \overline{\{S(\tau)U_0;\ t \leqslant \tau < \infty\}} \qquad \text{(closure in the topology of } X\text{)},$$

namely, $\overline{U} \in \omega(U_0)$ if and only if there exists a time sequence $\{t_n\}$ tending to ∞ such that $S(t_n)U_0 \to \overline{U}$ in the topology of X. As explained in [7, Introduction], we have an evidence which suggests that there exists a trajectory which starts from a continuous initial functions $U_0 = {}^t(u_0(x), v_0(x), w_0(x)) \in K$ but, as $t \to \infty$, converges to a discontinuous stationary solution $\overline{U} = {}^t(\overline{u}(x), \overline{v}(x), \overline{w}(x))$. If this phenomenon is true, then any sequence $S(t_n)U_0$ cannot converge to $\overline{U}$ in the topology of X, namely, it is possible that $\omega(U_0) = \emptyset$.

We define the L^2 topology of X as follows. A sequence $\{{}^t(u_n, v_n, w_n)\}$ in X is said to be L^2 convergent to ${}^t(u_0, v_0, w_0) \in X$ as $n \to \infty$, if

$$\begin{cases} u_n \to u_0 & \text{strongly in } L^2(\Omega), \\ v_n \to v_0 & \text{strongly in } L^2(\Omega), \\ w_n \to w_0 & \text{strongly in } L^2(\Omega). \end{cases}$$

Then, using this topology we define the L^2-ω-limit set of $S(t)U_0$, $U_0 \in K$, by

$$L^2\text{-}\omega(U_0) = \bigcap_{t \geqslant 0} \overline{\{S(\tau)U_0;\ t \leqslant \tau < \infty\}} \qquad \text{(closure in the } L^2 \text{ topology of } X\text{)}. \tag{4.1}$$

In addition, we may equip X with the weak* topology. A sequence $\{{}^t(u_n, v_n, w_n)\}$ in X is said to be weak* convergent to ${}^t(u_0, v_0, w_0) \in X$ as $n \to \infty$, if

$$\begin{cases} u_n \to u_0 & \text{weak* in } L^\infty(\Omega), \\ v_n \to v_0 & \text{weak* in } L^\infty(\Omega), \\ w_n \to w_0 & \text{strongly in } L^2(\Omega). \end{cases}$$

Using this topology, we define the w*-ω-limit set of $S(t)U_0$, $U_0 \in K$, by

$$\text{w}^*\text{-}\omega(U_0) = \bigcap_{t \geqslant 0} \overline{\{S(\tau)U_0;\ t \leqslant \tau < \infty\}} \quad \text{(closure in the weak* topology of } X). \tag{4.2}$$

According to [7, Theorem 6.3], it is already known that w*-$\omega(U_0) \neq \emptyset$ for any initial data $U_0 \in K$.

In general we observe the following relations.

Theorem 4.1. *For each $U_0 \in K$, $\omega(U_0) \subset L^2$-$\omega(U_0) \subset$ w*-$\omega(U_0)$.*

Proof. The first relation $\omega(U_0) \subset L^2$-$\omega(U_0)$ is obvious by the definition.

Let $\overline{U} = (\overline{u}, \overline{v}, \overline{w}) \in L^2$-$\omega(U_0)$. Then, there exists a sequence $\{t_n\}$ tending to ∞ such that $S(t_n)U_0 = (u(t_n), v(t_n), w(t_n)) \to \overline{U}$ in the L^2 topology of X. Let $\varphi \in L^1(\Omega)$. For any $f \in L^2(\Omega)$,

$$\left| \int_\Omega \varphi\{u(t_n) - \overline{u}\} dx \right| \le \|\varphi - f\|_{L^1} \|u(t_n) - \overline{u}\|_{L^\infty} + \left| \int_\Omega f\{u(t_n) - \overline{u}\} dx \right|.$$

Since $L^2(\Omega)$ is dense in $L^1(\Omega)$ and since (2.7) is valid, we verify that, as $t_n \to \infty$,

$$\left| \int_\Omega \varphi\{u(t_n) - \overline{u}\} dx \right| \to 0.$$

Hence, $u(t_n) \to \overline{u}$ in the weak* topology of $L^\infty(\Omega)$. Due to (2.8), it is the same for the weak* convergence of $v(t_n)$ to $\overline{v}$. Thus we have $\overline{U} \in$ w*-$\omega(U_0)$. □

We do not know whether the converse relation w*-$\omega(U_0) \subset L^2$-$\omega(U_0)$ is true in general or not. We can however prove some weak result.

Theorem 4.2. *For $U_0 \in K$, let there exist a sequence $\{t_n\}$ tending to ∞ such that $S(t_n)U_0 = {}^t(u(t_n), v(t_n), w(t_n))$ converges to a triplet of functions $\overline{U} = {}^t(\overline{u}, \overline{v}, \overline{w}) \in X$ almost everywhere in Ω. Then, $\overline{U} \in L^2$-$\omega(U_0)$.*

Proof. By virtue of (2.7), (2.8) and (2.10), the almost everywhere convergence implies L^2 convergence for each sequence of $u(t_n)$, $v(t_n)$ and $w(t_n)$. Hence, $\overline{U} \in L^2$-$\omega(U_0)$. □

The rest of this section is devoted to proving some structural results for the ω-limit sets under specific conditions assumed to hold for the coefficients of equations in (1.1).

Theorem 4.3. *Assume that $h > \frac{f\alpha\delta}{c+f}$. Then, $\omega(U_0) = L^2$-$\omega(U_0) =$ w*-$\omega(U_0) = \{{}^t(0, 0, 0)\}$ for every $U_0 \in K$.*

Proof. Let $U_0 = {}^t(u_0, v_0, w_0) \in K$ and let $S(t)U_0 = {}^t(u(t), v(t), w(t))$ be the global solution. Multiply the first equation of (1.1) by $2(c+f)u$ and integrate the product in Ω. Then,

$$(4.3)\quad (c+f)\frac{d}{dt}\int_\Omega u^2dx + 2(c+f)^2\int_\Omega u^2dx - 2(c+f)\beta\delta\int_\Omega wudx \\ = -2a(c+f)\int_\Omega (v-b)^2u^2dx \leqslant 0, \qquad 0<t<\infty.$$

Similarly, multiply the second equation of (1.1) by $\frac{2(c+f)\alpha\delta}{f}v$ and integrate the product in Ω. Then,

$$(4.4)\quad \frac{(c+f)\alpha\delta}{f}\frac{d}{dt}\int_\Omega v^2dx + 2(\alpha\delta)^2\int_\Omega v^2dx - 2(c+f)\alpha\delta\int_\Omega uvdx \\ + \frac{2(c+f)\alpha\delta}{f}\Big(h - \frac{f\alpha\delta}{c+f}\Big)\int_\Omega v^2dx = 0, \qquad 0<t<\infty.$$

Multiply the third equation of (1.1) by $2\beta\delta^2 w$ and integrate the product in Ω. Then,

$$(4.5)\quad \beta\delta^2\frac{d}{dt}\int_\Omega w^2dx + 2(\beta\delta)^2\int_\Omega w^2dx - 2\alpha\beta\delta^2\int_\Omega vwdx \\ = -2d\beta\delta^2\int_\Omega |\nabla w|^2dx \leqslant 0, \qquad 0<t<\infty.$$

Summing up (4.3), (4.4) and (4.5), we obtain that

$$\frac{d}{dt}\int_\Omega \Big((c+f)u^2 + \frac{(c+f)\alpha\delta}{f}v^2 + \beta\delta^2w^2\Big)dx + 2\int_\Omega\{((c+f)u)^2 + (\alpha\delta v)^2 + (\beta\delta w)^2\}dx \\ - 2\int_\Omega\{(c+f)u\alpha\delta v + \alpha\delta v\beta\delta w + \beta\delta w(c+f)u\}dx + 3\int_\Omega \varepsilon v^2dx \leqslant 0,$$

where $\varepsilon = \frac{2(c+f)\alpha\delta}{3f}\Big(h - \frac{f\alpha\delta}{c+f}\Big) > 0$. We here notice that

$$2\big[((c+f)u)^2 + (\alpha\delta v)^2 + (\beta\delta w)^2 - (c+f)u\alpha\delta v - \alpha\delta v\beta\delta w - \beta\delta w(c+f)u\big] + 3\varepsilon v^2 \\ = \Big[\frac{\big((c+f)\alpha\delta\big)^2}{\alpha^2\delta^2+\varepsilon}u^2 - 2(c+f)u\alpha\delta v + (\alpha^2\delta^2+\varepsilon)v^2\Big] \\ + \Big[(\alpha^2\delta^2+\varepsilon)v^2 - 2\alpha\delta v\beta\delta w + \frac{(\alpha\delta)^2(\beta\delta)^2}{\alpha^2\delta^2+\varepsilon}w^2\Big] + \big[\beta\delta w - (c+f)u\big]^2 \\ + \varepsilon\Big[\frac{(c+f)^2}{\alpha^2\delta^2+\varepsilon}u^2 + v^2 + \frac{(\beta\delta)^2}{\alpha^2\delta^2+\varepsilon}w^2\Big].$$

Therefore, with an appropriate exponent $\rho > 0$ and appropriate constants $C_i > 0$, $i = 1, 2, 3$,

$$\frac{d}{dt}\int_\Omega (C_1u^2 + C_2v^2 + C_3w^2)dx + \rho\int_\Omega (C_1u^2 + C_2v^2 + C_3w^2)dx \leqslant 0.$$

We thus conclude that

$$C_1\|u(t)\|^2_{L^2} + C_2\|v(t)\|^2_{L^2} + C_3\|w(t)\|^2_{L^2} \\ \leqslant e^{-\rho t}(C_1\|u_0\|^2_{L^2} + C_2\|v_0\|^2_{L^2} + C_3\|w_0\|^2_{L^2}), \qquad 0<t<\infty.$$

As a result, as $t \to \infty$, $S(t)U_0$ converges to ${}^t(0,0,0)$ in the L^2 topology. More strongly, since $\|w(t)\|_{L^\infty} \le C_\varepsilon \|w(t)\|_{H^{1+\varepsilon}} \le C_\varepsilon \|w(t)\|_{L^2}^{(1-\varepsilon)/2} \|w(t)\|_{H^2}^{(1+\varepsilon)/2}$, we deduce from the L^2 convergence of $w(t)$ that in the L^∞ topology (due to (2.13)). Furthermore, from the formulae (2.4) and (2.5), this implies convergence of $u(t)$ and $v(t)$ to 0 in the L^∞ topology. In this way, we ultimately conclude that, as $t \to \infty$, $S(t)U_0$ converges to $(0,0,0)$ in the L^∞ topology. From this the assertion of theorem follows immediately. □

Theorem 4.4. *Assume that* $ab^2 < 3(c+f)$. *Then,* L^2-$\omega(U_0) = \mathrm{w}^*$-$\omega(U_0)$ *for every* $U_0 \in K$.

Proof. Let $S(t)U_0 = U(t) = {}^t(u(t), v(t), w(t))$. Consider any time sequence $\{t_n\}$ which tends to ∞ as $n \to \infty$. By (2.9), $\|w(t_n)\|_{H^2}$ is a bounded sequence; so, we can choose a subsequence $\{t_{n'}\}$ for which $\{w(t_{n'})\}$ is convergent to $\overline{w}$ in $H^{1+\varepsilon}(\Omega)$ and hence in $L^\infty(\Omega)$. From the first and second equations of (2.1) it is easily observed that

$$(4.6) \qquad [\gamma(v(t_{n'})) + f]v(t_{n'}) = \frac{f}{h}\left[\beta\delta w(t_{n'}) - \frac{du}{dt}(t_{n'}) - \frac{\gamma(v(t_{n'})) + f}{f}\frac{dv}{dt}(t_{n'})\right].$$

Here, we introduce the cubic function

$$P(v) \equiv (\gamma(v) + f)v = av^3 - 2abv^2 + (ab^2 + c + f)v, \qquad -\infty < v < \infty.$$

It is easy to see the following property.

Lemma 4.1. *When* $ab^2 < 3(c+f)$, $w = P(v)$ *is a monotone increasing function for* $v \in (-\infty, \infty)$. *Its inverse function* $P^{-1}(w)$ *is a single-valued smooth function for* w *with uniformly bounded derivative in the whole real axis* $w \in (-\infty, \infty)$.

Proof of lemma. Obviously we have

$$P'(v) = 3av^2 - 4abv + (ab^2 + c + f) = 3a\left(v - \frac{2b}{3}\right)^2 - \frac{ab^2 - 3(c+f)}{3} > 0.$$

Therefore, the assertion of lemma is clear. □

Using $P^{-1}(w)$, we can write

$$v(t_{n'}) = P^{-1}\left(\frac{f}{h}\left\{\beta\delta w(t_{n'}) - \frac{du}{dt}(t_{n'}) - \frac{\gamma(v(t_{n'})) + f}{f}\frac{dv}{dt}(t_{n'})\right\}\right).$$

Since $w(t_{n'}) \to \overline{w}$ in $L^\infty(\Omega)$ and since Theorem 3.2 is true, we conclude that $v(t_{n'})$ converges to $\overline{v} = P^{-1}(\frac{f\beta\delta}{h}\overline{w})$ in $L^2(\Omega)$. Since Theorem 3.2 provides in particular that, as $t \to \infty$, $fu(t) - hv(t) \to 0$ in $L^2(\Omega)$, we conclude also that $u(t_{n'})$ converges to $\frac{h}{f}\overline{v}$ in $L^2(\Omega)$. Thus we have shown that ${}^t(u(t_{n'}), v(t_{n'}), w(t_{n'})) \to {}^t(\overline{u}, \overline{v}, \overline{w})$ in $L^2(\Omega)$.

We now know that any sequence $(u(t_n), v(t_n), w(t_n))$ has a subsequence which converges to some vector of X in the L^2 topology. Hence, the relation w^*-$\omega(U_0) \subset L^2$-$\omega(U_0)$ is proved, cf., Proof of Theorem 4.1. □

5 Constituents of L^2 ω-limit sets In this the section, we shall show that every L^2 ω-limit set consists of stationary solutions of (2.1). For this end, we begin with verifying the following Proposition.

Proposition 5.1. *For each $U_0 \in K$, L^2-$\omega(U_0)$ is an invariant set of $S(t)$, i.e.,*

$$S(t)\Big(L^2\text{-}\omega(U_0)\Big) \subset L^2\text{-}\omega(U_0), \qquad t > 0.$$

Proof. In the proof of this proposition, it is essential to show that $S(t)$ is continuous from K into itself in the L^2 topology.

To see this, consider two initial values $U_{01} = {}^t(u_{01}, v_{01}, w_{01})$ and $U_{02} = {}^t(u_{02}, v_{02}, w_{02})$ in K, and let ${}^t(u_1(t), v_1(t), w_1(t))$ and ${}^t(u_2(t), v_2(t), w_2(t))$ be the solutions to (2.1) with the initial value U_{01} and U_{02}, respectively. Let $T > 0$ be arbitrarily fixed time, and let t varies in the bounded interval $[0, T]$.

Then, from (2.4),

$$u_i(t) = e^{-\int_0^t \{\gamma(v_i)+f\}ds} u_{0i} + \beta\delta \int_0^t e^{-\int_\tau^t \{\gamma(v_i)+f\}ds} w_i(\tau)d\tau, \qquad i = 1,\, 2.$$

Consequently,

$$\begin{aligned} u_2(t) - u_1(t) = {} & e^{-\int_0^t \{\gamma(v_1)+f\}ds}\big(e^{-\int_0^t \{\gamma(v_2)-\gamma(v_1)\}ds} - 1\big)u_{01} \\ & + e^{-\int_0^t \{\gamma(v_2)+f\}ds}(u_{02} - u_{01}) + \beta\delta \int_0^t e^{-\int_\tau^t \{\gamma(v_2)+f\}ds}(w_2(\tau) - w_1(\tau))d\tau \\ & + \beta\delta \int_0^t e^{-\int_\tau^t \{\gamma(v_1)+f\}ds}\big(e^{-\int_\tau^t \{\gamma(v_2)-\gamma(v_1)\}ds} - 1\big)w_1(\tau)d\tau. \end{aligned}$$

In view of (2.7), (2.8) and (2.10), we obtain that

$$\begin{aligned} & \|u_2(t) - u_1(t)\|_{L^2} \leqslant \|u_{02} - u_{01}\|_{L^2} \\ & \quad + Cp(\|U_{01}\|_X + \|U_{02}\|_X)\Big\{ \Big\|e^{-\int_0^t \{\gamma(v_2)-\gamma(v_1)\}ds} - 1\Big\|_{L^2} + \int_0^t \|w_2(\tau) - w_1(\tau)\|_{L^2}d\tau \\ & \quad + \int_0^t \Big\|e^{-\int_\tau^t \{\gamma(v_2)-\gamma(v_1)\}ds} - 1\Big\|_{L^2} \tau^{-(1+\varepsilon)/2}d\tau\Big\}, \qquad 0 \le t \le T. \end{aligned}$$

For any $R > 0$, there exists a constant $C_R > 0$ such that $|e^\xi - 1| \le C_R|\xi|$ holds for all $|\xi| \le R$. Using this estimate, we verify that

$$\Big\|e^{-\int_0^t \{\gamma(v_2)-\gamma(v_1)\}ds} - 1\Big\|_{L^2} \le Cp(\|U_{01}\|_X + \|U_{02}\|_X) \int_0^t \|v_2(\tau) - v_1(\tau)\|_{L^2}d\tau.$$

Similarly,

$$\begin{aligned} & \int_0^t \Big\|e^{-\int_\tau^t \{\gamma(v_2)-\gamma(v_1)\}ds} - 1\Big\|_{L^2} \tau^{-(1+\varepsilon)/2}d\tau \\ & \qquad \le Cp(\|U_{01}\|_X + \|U_{02}\|_X) \int_0^t \int_\tau^t \|v_2(s) - v_1(s)\|_{L^2}\tau^{-(1+\varepsilon)/2}dsd\tau \\ & \qquad \le Cp(\|U_{01}\|_X + \|U_{02}\|_X) \int_0^t \|v_2(s) - v_1(s)\|_{L^2}ds. \end{aligned}$$

Hence,

$$(5.1)\quad \|u_2(t)-u_1(t)\|_{L^2} \leqslant \|u_{02}-u_{01}\|_{L^2} + Cp(\|U_{01}\|_X+\|U_{02}\|_X)\int_0^t [\|v_2(\tau)-v_1(\tau)\|_{L^2}+\|w_2(\tau)-w_1(\tau)\|_{L^2}]d\tau, \qquad 0\le t\le T.$$

In a similar way, from (2.5) it follows that

$$(5.2)\qquad \|v_2(t)-v_1(t)\|_{L^2} \leqslant \|v_{02}-v_{01}\|_{L^2} + C\int_0^t \|u_2(\tau)-u_1(\tau)\|_{L^2}d\tau, \qquad 0\le t\le T.$$

Finally, from (2.6) we have

$$w_2(t)-w_1(t) = e^{-tA}(w_{02}-w_{01}) + \alpha\int_0^t e^{-(t-\tau)A}[v_2(\tau)-v_1(\tau)]d\tau.$$

Therefore,

$$(5.3)\qquad \|w_2(t)-w_1(t)\|_{L^2} \leqslant \|w_{02}-w_{01}\|_{L^2} + \alpha\int_0^t \|v_2(\tau)-v_1(\tau)\|_{L^2}d\tau, \qquad 0\le t\le T.$$

Summing up (5.1), (5.2) and (5.3) and using Gronwall's inequality, we conclude that

$$\|u_2(t)-u_1(t)\|_{L^2}+\|v_2(t)-v_1(t)\|_{L^2}+\|w_2(t)-w_1(t)\|_{L^2} \leqslant \|U_{02}-U_{01}\|_{L^2}e^{Cp(\|U_{01}\|_X+\|U_{02}\|_X)t}, \qquad 0\le t\le T.$$

This shows that, for $0\le t\le T$, the semigroup $S(t)$ is continuous in the L^2 topology. But, as $T>0$ is arbitrary, it is the same for any $0\le t<\infty$.

It is now immediate to prove the assertion of theorem. Let $\overline{U}\in L^2\text{-}\omega(U_0)$. By definition there exists a sequence t_n tending to ∞ such that $S(t_n)U_0\to\overline{U}$ in the L^2 topology. By the L^2 continuity proved above, we have $S(t_n+t)U_0=S(t)S(t_n)U_0\to S(t)\overline{U}$ in L^2. Therefore, $S(t)\overline{U}\in L^2\text{-}\omega(U_0)$. □

Theorem 5.1. *For any $U_0\in K$, $L^2\text{-}\omega(U_0)$ consists of equilibria of the dynamical system.*

Proof. Let $\overline{U}={}^t(\overline{u},\overline{v},\overline{w})\in L^2\text{-}\omega(U_0)$. There exists a sequence $t_n\to\infty$ such that $S(t_n)U_0=U(t_n)\to\overline{U}$ in the L^2 topology. Since $w(t_n)$ is a bounded sequence in $H^2(\Omega)$, we can take a subsequence $\{w(t_{n'})\}$ of $\{w(t_n)\}$ such that $w(t_{n'})\to\overline{w}'$ strongly in $H^1(\Omega)$. It is then easy to see that $\overline{w}=\overline{w}'$. Meanwhile, in view of (2.7) and (2.8), $u(t_n)\to\overline{u}$ and $v(t_n)\to\overline{v}$ in any L^p topology with finite p such that $2\le p<\infty$.

By these facts we conclude that the Lyapunov function $\Psi(U(t_{n'}))$ given by (3.4) is convergent to $\Psi(\overline{U})$ as $t_{n'}\to\infty$. That is,

$$\Psi(\overline{U}) = \lim_{n'\to\infty}\Psi(U(t_{n'})) = \inf_{0\le t<\infty}\Psi(S(t)U_0)\equiv\Psi_\infty.$$

This means that $\Psi(\overline{U})\equiv\Psi_\infty$ for all $\overline{U}$'s of vectors in $L^2\text{-}\omega(U_0)$. By Proposition 5.1, $S(t)\overline{U}\in L^2\text{-}\omega(U_0)$ for every $t>0$. Hence,

$$\Psi(S(t)\overline{U})\equiv\Psi_\infty, \qquad 0<t<\infty,\ \overline{U}\in L^2\text{-}\omega(U_0).$$

Furthermore, let $S(t)\overline{U} = \overline{U}(t) = {}^t(\overline{u}(t), \overline{v}(t), \overline{w}(t))$; then, by (3.3), we have

$$\frac{d}{dt}\Psi(\overline{U}(t)) = -\int_\Omega \left\{\alpha[\gamma(\overline{v}) + f + h]\left(\frac{\partial \overline{v}}{\partial t}\right)^2 + f\beta\delta\left(\frac{\partial \overline{w}}{\partial t}\right)^2\right\}dx \equiv 0, \quad 0 < t < \infty.$$

Hence, $\frac{\partial \overline{v}}{\partial t}(t) \equiv \frac{\partial \overline{w}}{\partial t}(t) \equiv 0$ for $0 < t < \infty$. In addition, from the second equation of (2.1), it follows that $f\overline{u}(t) \equiv h\overline{v}(t)$; hence, $\frac{\partial \overline{u}}{\partial t}(t) \equiv 0$ for $0 < t < \infty$. Thus, it has been shown that $S(t)\overline{U} \equiv \overline{U}$ for every $0 < t < \infty$, namely, $\overline{U}$ must be an equilibrium. □

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