WEAK CONVERGENCE THEOREMS FOR FINDING COMMON ELEMENTS OF FINITE SETS IN BANACH SPACES

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Abstract. In this paper, we deal with the problem for finding a common element of finite sets in a Banach space. We first prove that an operator given by a convex combination of sunny generalized nonexpansive retractions in a Banach space is asymptotically regular. Using this result, we obtain a weak convergence theorem which is connected with the problem of image recovery. Further, using another weak convergence theorem, we prove a weak convergence theorem of Mann's type for finding a common element of finite sets in a Banach space.

1. Introduction

Let H be a Hilbert space, let $C_1, C_2, \dots, C_r$ be nonempty closed convex subsets of H and let I be the identity operator on H. Then the problem of image recovery in a Hilbert space setting may be stated as follows: The original (unknown) image z is known a priori to belong to intersection C_0 of r well-defined sets $C_1, C_2, \dots, C_r$ in a Hilbert space; given only the metric projections P_i of H onto C_i $(i = 1, 2, \dots, r)$, recover z by an iterative scheme.

In 1991, Crombez [2] proved the following: Let $T = \alpha_0 I + \sum_{i=1}^{r} \alpha_i T_i$ with $T_i = I + \lambda_i(P_i - I)$ for all i, $0 < \lambda_i < 2$, $\alpha_i > 0$, $i = 0, 1, 2, \dots, r$, $\sum_{i=0}^{r} \alpha_i = 1$, where each P_i is the metric projection of H onto C_i and $C_0 = \cap_{i=1}^{r} C_i$ is nonempty. Then starting from an arbitrary element x of H, the sequence $\{T^n x\}$ converges weakly to an element of C_0. Later, Kitahara and Takahashi [7] and Takahashi and Tamura [18] dealt with the problem of image recovery by convex combinations of nonexpansive retractions in a uniformly convex Banach space. Recently, Ibaraki and Takahashi [4, 5] proved some results for generalized nonexpansive mappings and sunny generalized nonexpansive retractions in a smooth Banach space which generalize nonexpansive mappings and metric projections in a Hilbert space, respectively. In particular, they [5] considered the iteration method of Mann's type for finding a fixed point of a generalized nonexpansive mapping T in a Banach space E as follows: $x_1 \in E$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad n = 1, 2, \dots, \tag{1.1}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$. Further, they obtained that the sequence (1.1) converges weakly to a fixed point of T under $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$ and $F(T) = \hat{F}(T)$, where $F(T)$ is the set of fixed points of T and $\hat{F}(T)$ is the set of asymptotic fixed points of T.

In this paper, we deal with the problem of image recovery in a Banach space. We first prove that an operator given by a convex combination of sunny generalized nonexpansive retractions in a smooth and uniformly convex Banach space is asymptotically regular. Using this result, we obtain a weak convergence theorem of Crombez's type which is connected with the problem of image recovery. Moreover, using Ibaraki and Takahashi's result for

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a sequence generated by (1.1), we prove a weak convergence theorem of Mann's type for finding a common element of finite sets in a smooth and uniformly convex Banach space.

2. Preliminaries

Let E be a real Banach space with its dual E^*. We write $x_n \rightharpoonup x_0$ to indicate that the sequence $\{x_n\}$ converges weakly to x_0. Similarly, $x_n \to x_0$ will symbolize the strong convergence. A Banach space E is said to be strictly convex if

$$\|x\| = \|y\| = 1, \quad x \neq y \Rightarrow \left\| \frac{x+y}{2} \right\| < 1.$$

Also, E is said to be uniformly convex if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that

$$\|x\| = \|y\| = 1, \quad \|x - y\| \geq \varepsilon \Rightarrow \left\| \frac{x+y}{2} \right\| \leq 1 - \delta.$$

We know that if E is uniformly convex then E is reflexive and strictly convex. The following result was proved by Xu [19].

Lemma 2.1 ([19])**.** *Let $r_0 > 0$ and let E be a uniformly convex Banach space. Then exists a continuous, strictly increasing, and convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x - y\|) \tag{2.1}$$

for all $x, y \in B_{r_0} := \{z \in E : \|z\| \leq r_0\}$ and λ with $0 \leq \lambda \leq 1$.

A Banach space E is said to be smooth if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.2}$$

exists for each $x, y \in \{z \in E : \|z\| = 1\} (=: S(E))$. A Banach space E is said to be uniformly smooth if the limit (2.2) is attained uniformly for $x, y \in S(E)$.

The normalized duality mapping J from E into E^* is defined by

$$J(x) := \left\{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\right\}, \quad \forall x \in E.$$

We know that if E is strictly convex then J is one-to-one and satisfies that $\langle x - y, x^* - y^* \rangle > 0$ for each $x, y \in E$ $(x \neq y)$, $x^* \in Jx$ and $y^* \in Jy$. E is smooth if and only if J is single-valued.

Let E be a smooth Banach space and consider the following function studied in Alber[1] and Kamimura and Takahashi[6]:

$$V(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$$

for each $x, y \in E$. We also know that for each $x, y, z \in E$,

$$V(x, y) = V(x, z) + V(z, y) + 2\langle x - z, Jz - Jy \rangle. \tag{2.3}$$

It is obvious from the definition of V that

$$(\|x\| - \|y\|)^2 \leq V(x, y) \leq (\|x\| + \|y\|)^2 \tag{2.4}$$

for each $x, y \in E$(see [6]). It is also easy to see that if E is additionally assumed to be strictly convex, then

$$V(x, y) = 0 \iff x = y. \tag{2.5}$$

See [11] for more details. The following lemma is well-known.

Lemma 2.2 ([6])**.** *Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty} V(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Let C be a nonempty closed convex subset of E and let T be a mapping from C into itself. We denote by $F(T)$ the set of fixed points of T. A point p in C is said to be an asymptotic fixed point of T [14] if C contains a sequence $\{x_n\}$ which converges weakly to p and the strong $\lim_{n\to\infty}(x_n - Tx_n) = 0$. The set of asymptotic fixed points of T is denoted by $\hat{F}(T)$. A mapping T is called generalized nonexpansive [3, 4] if $F(T) \neq \emptyset$ and $V(Tx, p) \leq V(x, p)$ for each $x \in C$ and $p \in F(T)$. A mapping T is called asymptotically regular if, for each $x \in C$, $T^{n+1}x - T^n x$ converges strongly to 0.

Let D be a nonempty subset of E. A mapping $R : E \to D$ is said to be sunny if

$$R(Rx + t(x - Rx)) = Rx, \ \forall x \in E, \ \forall t \geq 0.$$

A mapping $R : E \to D$ is said to be a retraction if $Rx = x$, $\forall x \in D$. If E is smooth and strictly convex, then a sunny generalized nonexpansive retraction of E onto D is uniquely decided (see [3, 4]). Then, if E is a smooth and strictly convex, a sunny generalized nonexpansive retraction of E onto D is denoted by R_D. A subset D of E is said to be a sunny generalized nonexpansive retract (resp. a generalized nonexpansive retract) of E if there exists a sunny generalized nonexpansive retraction (resp. a generalized nonexpansive retraction) of E onto D (see [3, 4] for more details). The set of fixed points of such a sunny generalized nonexpansive retraction is D.

We know the following results for generalized nonexpansive mappings and sunny generalized nonexpansive retractions.

Lemma 2.3 ([3, 4]). *Let D be a nonempty subset of a smooth and strictly convex Banach space E. Let R_D be a retraction of E onto D. Then R_D is sunny and generalized nonexpansive if and only if*

$$\langle x - R_D x, JR_D x - Jy \rangle \geq 0$$

for each $x \in E$ and $y \in D$.

Lemma 2.4 ([4, 5]). *Let D be a nonempty subset of a reflexive, strictly convex, and smooth Banach space E. If R is a sunny generalized nonexpansive retraction of E onto D, then*

$$V(x, Rx) + V(Rx, u) \leq V(x, u) \tag{2.6}$$

for each $x \in E$ and $u \in D$.

Lemma 2.5 ([5]). *Let E be a reflexive, strictly convex, and smooth Banach space and let D be a nonempty weakly closed subset of E. If R is a sunny generalized nonexpansive retraction of E onto D, then $\hat{F}(R) = F(R)$.*

Theorem 2.6 ([5]). *Let E be a smooth and uniformly convex Banach space, let C be a nonempty closed convex subset of E, let T be a generalized nonexpansive mapping from C into itself, and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. Suppose $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n = 1, 2, \dots .$$

If $F(T) = \hat{F}(T)$, then the sequence $\{x_n\}$ converges weakly to an element of $F(T)$.

3. Weak convergence theorem of Crombez's type

In this section, we consider the problem of image recovery for finding a common element of finite sets in a Banach space. Throughout this section, we denote by I the identity operator. To obtain our result, we need four lemmas. Compare these lemmas with the results in Crombez [2] and Kitahara and Takahashi [7].

Lemma 3.1. *Let E be a reflexive, strictly convex, and smooth Banach space and let D be a nonempty subset of E. Let R be a sunny generalized nonexpansive retraction of E onto D and $x \in E$. If $V(Rx,p) = V(x,p)$ for some $p \in D$, then $Rx = x$.*

Proof. Let $x \in E$ and $p \in D$ with $V(Rx,p) = V(x,p)$. By Lemma 2.4, we have

$$V(x,p) \geq V(x,Rx) + V(Rx,p) = V(x,Rx) + V(x,p)$$

and hence $V(x,Rx) \leq 0$. Therefore $V(x,Rx) = 0$. This implies that $Rx = x$. ∎

Lemma 3.2. *Let E be a smooth and uniformly convex Banach space and let S be an operator on E given by $S = \beta_0 I + \sum_{i=1}^r \beta_i S_i$, $0 < \beta_i < 1$ for $i = 0,1,\dots,r$, $\sum_{i=0}^r \beta_i = 1$, such that each S_i is a generalized nonexpansive mapping from E into itself and $\cap_{i=1}^r F(S_i)$ is nonempty. Then S is asymptotically regular.*

Proof. Let $x \in E$ and $p \in \cap_{i=1}^r F(S_i)$. Putting $x_n = S^n x$ for each $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned} V(x_{n+1},p) &= V(Sx_n,p) \\ &= V\left(\beta_0 x_n + \sum_{i=1}^r \beta_i S_i x_n, p\right) \\ &\leq \beta_0 V(x_n,p) + \sum_{i=1}^r \beta_i V(S_i x_n,p) \\ &\leq \beta_0 V(x_n,p) + \sum_{i=1}^r \beta_i V(x_n,p) \\ &= V(x_n,p). \end{aligned}$$

Hence, $\lim_{n\to\infty} V(x_n,p)$ exists and in particular, $V(x_n,p)$ is bounded. So, by (2.4), $\{x_n\}$ is bounded. This implies that $\{S_i x_n\}$ is also bounded for each $i = 1,2,\dots,r$. Let

$$z_n := \frac{1}{1-\beta_0}\sum_{i=1}^r \beta_i S_i x_n$$

for each $n \in \mathbb{N} \cup \{0\}$. Since $\{S_i x_n\}$ is bounded, $\{z_n\}$ is also bounded. Put $r_0 = \sup_{n\in\mathbb{N}\cup\{0\}} \{\|x_n\|, \|z_n\|\}$. Then, by Lemma 2.1, there exists a continuous, strictly increasing, and convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ satisfying (2.1), where $B_{r_0} = \{x \in E : \|x\| \leq r_0\}$. We have

$$\begin{aligned} V(x_{n+1},p) &= V\left(\beta_0 x_n + \sum_{i=1}^r \beta_i S_i x_n, p\right) = V(\beta_0 x_n + (1-\beta_0) z_n, p) \\ &= \|\beta_0 x_n + (1-\beta_0) z_n\|^2 - 2\langle \beta_0 x_n + (1-\beta_0) z_n, Jp\rangle + \|p\|^2 \\ &\leq \beta_0 \|x_n\|^2 + (1-\beta_0)\|z_n\|^2 - \beta_0(1-\beta_0) g(\|x_n - z_n\|) \\ &\quad -2\beta_0 \langle x_n, Jp\rangle - 2(1-\beta_0)\langle z_n, Jp\rangle + \|p\|^2 \\ &= \beta_0\Big(\|x_n\|^2 - 2\langle x_n, Jp\rangle + \|p\|^2\Big) + (1-\beta_0)\Big(\|z_n\|^2 - 2\langle z_n, Jp\rangle + \|p\|^2\Big) \\ &\quad -\beta_0(1-\beta_0) g(\|x_n - z_n\|) \\ &= \beta_0 V(x_n,p) + (1-\beta_0) V(z_n,p) - \beta_0(1-\beta_0) g(\|x_n - z_n\|) \\ &= \beta_0 V(x_n,p) + (1-\beta_0) V\left(\frac{1}{(1-\beta_0)}\sum_{i=1}^r \beta_i S_i x_n, p\right) \\ &\quad -\beta_0(1-\beta_0) g(\|x_n - z_n\|) \end{aligned}$$

$$\begin{aligned} &\leq \beta_0 V(x_n, p) + \sum_{i=1}^{r} \beta_i V(S_i x_n, p) - \beta_0(1-\beta_0) g(\|x_n - z_n\|) \\ &\leq \beta_0 V(x_n, p) + \sum_{i=1}^{r} \beta_i V(x_n, p) - \beta_0(1-\beta_0) g(\|x_n - z_n\|) \\ &= V(x_n, p) - \beta_0(1-\beta_0) g(\|x_n - z_n\|) \end{aligned}$$

and hence

$$g(\|x_n - z_n\|) \leq \frac{1}{\beta_0(1-\beta_0)}\Big(V(x_n, p) - V(x_{n+1}, p)\Big).$$

Since $\{V(x_n, p)\}$ converges, it follows that

$$\lim_{n\to\infty} g(\|x_n - z_n\|) = 0.$$

Then the properties of g yield that

$$\lim_{n\to\infty} \|x_n - z_n\| = 0.$$

and hence

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|\beta_0 x_n + (1-\beta_0) z_n - x_n\| \\ &= (1-\beta_0)\|x_n - z_n\| \end{aligned}$$

Therefore we have that $\lim_{n\to\infty} \|Sx_n - x_n\| = \lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. This implies that S is asymptotically regular. □

Lemma 3.3. *Let E be a reflexive, strictly convex, and smooth Banach space and let $D_1, D_2, \ldots, D_r$ be nonempty sunny generalized nonexpansive retracts of E such that $\cap_{i=1}^{r} D_i$ is nonempty. Let S be an operator on E given by $S = \sum_{i=1}^{r} \alpha_i S_i$, $0 < \alpha_i < 1$, $i = 1, 2, \ldots, r$, $\sum_{i=1}^{r} \alpha_i = 1$, such that for each i, $S_i = (1-\lambda_i)I + \lambda_i R_i$, $0 < \lambda_i < 1$, where each R_i is a sunny generalized nonexpansive retraction of E onto D_i. Then $F(S) = \cap_{i=1}^{r} D_i$.*

Proof. It is obvious that $\cap_{i=1}^{r} D_i \subset F(S)$. Conversely, let $x \in F(S)$ and $p \in \cap_{i=1}^{r} D_i$. From the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned} V(x,p) &= V(Sx,p) \\ &\leq \sum_{i=1}^{r} \alpha_i V(S_i x, p) \\ &\leq \sum_{i=1}^{r} \alpha_i \Big((1-\lambda_i) V(x,p) + \lambda_i V(R_i x, p)\Big) \\ &\leq \sum_{i=1}^{r} \alpha_i \Big((1-\lambda_i) V(x,p) + \lambda_i V(x, p)\Big) \\ &= \sum_{i=1}^{r} \alpha_i V(x,p) = V(x,p) \end{aligned}$$

and hence $V(x,p) = V(S_i x, p)$. So, for each i,

$$\begin{aligned} V(x,p) &= V(S_i x, p) \\ &\leq (1-\lambda_i) V(x,p) + \lambda_i V(R_i x, p) \\ &\leq V(x,p) \end{aligned}$$

and hence $V(x,p) = V(R_i x, p)$. So, it follows from Lemma 3.1 that $R_i x = x$. This implies that $x \in D_i$ for each i. So, we have $x \in \cap_{i=1}^{r} D_i$. Therefore, we have $F(S) \subset \cap_{i=1}^{r} D_i$. This implies $F(S) = \cap_{i=1}^{r} D_i$. □

Lemma 3.4. *Let E be a smooth and uniformly convex Banach space and let $D_1, D_2, \ldots, D_r$ be nonempty weakly closed sunny generalized nonexpansive retracts of E such that $\cap_{i=1}^r D_i$ is nonempty. Let S be an operator on E given by $S = \sum_{i=1}^r \alpha_i S_i$, $0 < \alpha_i < 1$, $i = 1, 2, \ldots, r$, $\sum_{i=1}^r \alpha_i = 1$, such that for each i, $S_i = (1 - \lambda_i)I + \lambda_i R_i$, $0 < \lambda_i < 1$, where each R_i is a sunny generalized nonexpansive retraction of E onto D_i. Then $\hat{F}(S) = F(S)$.*

Proof. It is obvious that $F(S) \subset \hat{F}(S)$. From Lemmas 2.5 and 3.3, we have $\cap_{i=1}^r \hat{F}(R_i) = \cap_{i=1}^r D_i = F(S)$. Therefore, to show this Lemma, it is sufficient to prove that $\hat{F}(S) \subset \cap_{i=1}^r \hat{F}(R_i)$.

Let $v \in \hat{F}(S)$ and $p \in \cap_{i=1}^r D_i$. There exists a sequence $\{x_n\} \subset E$ such that $x_n \rightharpoonup v$ and $\|x_n - Sx_n\| \to 0$. Then, $\{x_n\}$ is bounded. So, by (2.4), we have $\{V(x_n, p)\}$ is bounded and hence $\limsup_{n\to\infty} V(x_n, p) < +\infty$. Since $\|x_n - Sx_n\|$ converges to 0, it follows that

$$\limsup_{n\to\infty} V(x_n, p) = \limsup_{n\to\infty} V(Sx_n, p).$$

Then, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\lim_{k\to\infty} V(x_{n_k}, p) = \limsup_{n\to\infty} V(x_n, p).$$

Put $\alpha = \lim_{k\to\infty} V(x_{n_k}, p)$. We have, for each i,

$$V(Sx_{n_k}, p) \le (1 - \alpha_i)V(x_{n_k}, p) + \alpha_i V(S_i x_{n_k}, p)$$

and hence

$$\frac{1}{\alpha_i}\Big(V(Sx_{n_k}, p) - (1 - \alpha_i)V(x_{n_k}, p)\Big) \le V(S_i x_{n_k}, p) \le V(x_{n_k}, p).$$

So, we have $\alpha \le \liminf_{k\to\infty} V(S_i x_{n_k}, p) \le \limsup_{k\to\infty} V(S_i x_{n_k}, p) \le \alpha$ and hence $\alpha = \lim_{k\to\infty} V(x_{n_k}, p) = \lim_{k\to\infty} V(S_i x_{n_k}, p)$. We also have, for each i

$$\begin{aligned} V(S_i x_{n_k}, p) &= V((1 - \lambda_i)x_{n_k} + \lambda_i R_i x_{n_k}, p) \\ &\le (1 - \lambda_i)V(x_{n_k}, p) + \lambda_i V(R_i x_{n_k}, p) \\ &\le V(x_{n_k}, p) \end{aligned}$$

and hence $\alpha = \lim_{k\to\infty} V(x_{n_k}, p) = \lim_{k\to\infty} V(R_i x_{n_k}, p)$. From Lemma 2.4, we get, for each i,

$$V(x_{n_k}, R_i x_{n_k}) \le V(x_{n_k}, p) - V(R_i x_{n_k}, p)$$

and hence $V(x_{n_k}, R_i x_{n_k}) \to 0$. From Lemma 2.2, we have $\|x_{n_k} - R_i x_{n_k}\| \to 0$. Since $\{x_n\}$ is converges weakly to v, it follows that $v \in \hat{F}(R_i)$ for each $i = 1, 2, \ldots, r$ and hence $v \in \cap_{i=1}^r \hat{F}(R_i)$. So, we have that $\hat{F}(S) \subset \cap_{i=1}^r \hat{F}(R_i)$. This implies that $\hat{F}(S) = F(S)$. □

Now, we prove the following theorem which is one of our main results in this paper.

Theorem 3.5. *Let E be a smooth and uniformly convex Banach space and $D_1, D_2, \ldots, D_r$ be nonempty weakly closed sunny generalized nonexpansive retracts of E such that $\cap_{i=1}^r D_i$ is nonempty. Let S be an operator on E given by $S = \sum_{i=1}^r \alpha_i S_i$, $0 < \alpha_i < 1$, $i = 1, 2, \ldots, r$, $\sum_{i=1}^r \alpha_i = 1$, such that for each i, $S_i = (1 - \lambda_i)I + \lambda_i R_i$, $0 < \lambda_i < 1$, where each R_i is a sunny generalized nonexpansive retraction of E onto D_i. Then, for each $x \in E$, $\{S^n x\}$ converges weakly to an element of $F(S) = \cap_{i=1}^r D_i$.*

Proof. Choose β_1 such that $\alpha_1\lambda_1 < \beta_1 < \alpha_1$ and set $\mu_1 = \alpha_1\lambda_1/\beta_1$. Then $0 < \beta_1 < 1$, $0 < \mu_1 < 1$ and

$$\begin{aligned}\alpha_1 S_1 &= \alpha_1\{(1-\lambda_1)I + \lambda_1 R_1\}\\ &= \alpha_1 I - \alpha_1\lambda_1 I + \alpha_1\lambda_1 R_1\\ &= (\alpha_1-\beta_1)I + \beta_1 I - \beta_1\mu_1 I + \beta_1\mu_1 R_1\\ &= (\alpha_1-\beta_1)I + \beta_1\Big\{(1-\mu_1)I + \mu_1 R_1\Big\}.\end{aligned}$$

Putting $S_1^{'} = (1-\mu_1)I+\mu_1 R_1$, we have that $S_1^{'}$ is generalized nonexpansive and $F(S_1^{'}) = D_1$. Now, we have $S = (\alpha_1-\beta_1)I + \beta_1 S_1^{'} + \sum_{i=2}^{r}\alpha_i S_i$. By Lemma 3.2, S is asymptotically regular.

Let $x \in E$ and $p \in F(S)$. Putting $x_n = S^n x$ for each $n \in \mathbb{N}\cup\{0\}$, we have

$$\begin{aligned}V(x_{n+1},p) &= V(Sx_n,p)\\ &\le \sum_{i=1}^{r}\alpha_i V(S_i x_n,p)\\ &\le \sum_{i=1}^{r}\alpha_i V(x_n,p)\\ &= V(x_n,p).\end{aligned}$$

Therefore $\lim_{n\to\infty} V(x_n,p)$ exists and in particular, $\{V(x_n,p)\}$ is bounded. So, by (2.4), $\{x_n\}$ is bounded. Since E is reflexive, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup v$ for some $v \in E$. Since S is asymptotically regular, by Lemma 3.4, v is a fixed point of S.

Let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be two subsequences of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v_1$ and $x_{n_j} \rightharpoonup v_2$. As above, we have $v_1, v_2 \in F(S)$. Put

$$a = \lim_{n\to\infty}\Big(V(x_n,v_1) - V(x_n,v_2)\Big).$$

Note that

$$V(x_n,v_1) - V(x_n,v_2) = 2\langle x_n, Jv_2 - Jv_1\rangle + \|v_1\|^2 - \|v_2\|^2, \quad n = 0,1,2,\dots.$$

From $x_{n_i} \rightharpoonup v_1$ and $x_{n_j} \rightharpoonup v_2$, we have

$$a = 2\langle v_1, Jv_2 - Jv_1\rangle + \|v_1\|^2 - \|v_2\|^2 \tag{3.1}$$

and

$$a = 2\langle v_2, Jv_2 - Jv_1\rangle + \|v_1\|^2 - \|v_2\|^2. \tag{3.2}$$

Combining (3.1) and (3.2), we obtain

$$\langle v_1 - v_2, Jv_1 - Jv_2\rangle = 0$$

Since E is strictly convex, from the property of J, it follows that $v_1 = v_2$. Therefore $\{x_n\}$ converges weakly to an element of $F(S) = \cap_{i=1}^{r} D_i$. □

4. Weak convergence theorem of Mann's type

In this section, we prove a weak convergence theorem of Mann's type for finding a common element of finite sunny generalized nonexpansive retracts. As in Reich [14], we first prove the following lemmas.

Lemma 4.1. *Let E be a smooth and uniformly convex Banach space, let C be a nonempty closed convex subset of E and let $T_1, T_2, \dots, T_m$ be generalized nonexpansive mappings from C into itself such that for each $i = 1, 2, \dots, m$, $F(T_i) = \hat{F}(T_i)$ and*

$$V(x, T_i x) + V(T_i x, u) \leq V(x, u), \quad \forall x \in C, \ \forall u \in F(T_i). \tag{4.1}$$

If $\cap_{i=1}^m F(T_i)$ is nonempty, then $\hat{F}(T_m T_{m-1} \cdots T_1) = F(T_m T_{m-1} \cdots T_1) = \cap_{i=1}^m F(T_i)$.

Proof. Put $T := T_m T_{m-1} \cdots T_1$. Then, it is obvious that $\cap_{i=1}^m F(T_i) \subset F(T) \subset \hat{F}(T)$. To show this lemma, it is sufficient to prove that $\hat{F}(T) \subset \cap_{i=1}^m \hat{F}(T_i)$.

Let $z \in \hat{F}(T)$ and let $u \in \cap_{i=1}^m F(T_i)$. There exists a sequence $\{x_n\} \subset C$ such that $x_n \rightharpoonup z$ and $\|x_n - Tx_n\| \to 0$. Put $y_n := T_{m-1} T_{m-2} \cdots T_1 x_n$. Then $T_m y_n = Tx_n$. From (4.1), we have

$$\begin{aligned} V(y_n, T_m y_n) &\leq V(y_n, u) - V(T_m y_n, u) \\ &\leq V(x_n, u) - V(Tx_n, u) \\ &= \|x_n\|^2 - \|Tx_n\|^2 - 2\langle x_n - Tx_n, Ju\rangle \\ &\leq \Big(\|x_n\| + \|Tx_n\|\Big)\Big(\|x_n\| - \|Tx_n\|\Big) + 2\|x_n - Tx_n\|\|u\| \\ &\leq \Big(\|x_n\| + \|Tx_n\|\Big)\|x_n - Tx_n\| + 2\|x_n - Tx_n\|\|u\| \end{aligned}$$

and hence $V(y_n, T_m y_n) \to 0$ as $n \to \infty$. We have from Lemma 2.2 that $\|y_n - T_m y_n\| \to 0$ as $n \to \infty$. Moreover, we have

$$\begin{aligned} \|x_n - y_n\| &= \|x_n - Tx_n + Tx_n - y_n\| \\ &\leq \|x_n - Tx_n\| + \|T_m y_n - y_n\| \end{aligned}$$

and hence $\|x_n - y_n\| \to 0$. This implies that $y_n \rightharpoonup z$. Since $\|y_n - T_m y_n\| \to 0$, we have $z \in \hat{F}(T_m)$. From the definition of y_n, we have that $\|x_n - T_{m-1} T_{m-2} \cdots T_1 x_n\| \to 0$ and $T_{m-1} T_{m-2} \cdots T_1 x_n \rightharpoonup z$. Hence, $z \in \hat{F}(T_{m-1} T_{m-2} \cdots T_1)$. Similarly, putting $z_n := T_{m-2} T_{m-3} \cdots T_1 x_n$, we have

$$\begin{aligned} V(z_n, T_{m-1} z_n) &\leq V(z_n, u) - V(T_{m-1} z_n, u) \\ &\leq V(x_n, u) - V(y_n, u) \\ &= \|x_n\|^2 - \|y_n\|^2 - 2\langle x_n - y_n, Ju\rangle \\ &\leq \Big(\|x_n\| + \|y_n\|\Big)\|x_n - y_n\| + 2\|x_n - y_n\|\|u\| \end{aligned}$$

and hence $V(z_n, T_{m-1} z_n) \to 0$ as $n \to \infty$. So, we have that $\|z_n - T_{m-1} z_n\| \to 0$ and $z_n \rightharpoonup z$. Therefore, we have that $z \in \hat{F}(T_{m-1})$, $\|x_n - T_{m-2} T_{m-3} \cdots T_1 x_n\| \to 0$ and $z \in \hat{F}(T_{m-2} T_{m-3} \cdots T_1)$. By such a method, we have that $z \in \hat{F}(T_i)$ for each $i = m-2, m-3, \dots, 1$. This implies that $z \in \cap_{i=1}^m \hat{F}(T_i)$. So, we have $\hat{F}(T) \subset \cap_{i=1}^m \hat{F}(T_i)$. Therefore, $F(T) = \hat{F}(T) = \cap_{i=1}^m F(T_i)$. □

Lemma 4.2. *Let E be a smooth and uniformly convex Banach space, let C be a nonempty closed convex subset of E and let $T_1, T_2, \dots, T_m$ be generalized nonexpansive mappings from C into itself such that for each $i = 1, 2, \dots, m$, $F(T_i) = \hat{F}(T_i)$ and*

$$V(x, T_i x) + V(T_i x, u) \leq V(x, u), \quad \forall x \in C, \ \forall u \in F(T_i). \tag{4.2}$$

If $\cap_{i=1}^m F(T_i)$ is nonempty, then $T_m T_{m-1} \cdots T_1$ is a generalized nonexpansive mapping from C into itself.

Proof. Put $T = T_m T_{m-1} \cdots T_1$. From Lemma 4.1, we have $F(T) = \cap_{i=1}^m F(T_i)$. For $x \in C$ and $u \in F(T) = \cap_{i=1}^m F(T_i)$, we have $V(Tx, u) \le V(x, u)$. Therefore, $T_m T_{m-1} \cdots T_1$ is a generalized nonexpansive mapping from C into itself. □

Using Theorem 2.6, Lemmas 4.1 and 4.2, we can prove the following result.

Theorem 4.3. *Let E be a smooth and uniformly convex Banach space and let C be a nonempty closed convex subset of E. Let $T_1, T_2, \dots, T_m$ be generalized nonexpansive mappings of C into itself such that for each $i = 1, 2, \dots, m$, $F(T_i) = \hat{F}(T_i)$, and*

$$V(x, T_i x) + V(T_i x, u) \le V(x, u), \quad \forall x \in C, \ \forall u \in F(T_i). \tag{4.3}$$

Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \le \alpha_n \le 1$ for each $n = 1, 2, \dots,$ and $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. Suppose that $\cap_{i=1}^m F(T_i)$ is nonempty and $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T_{m-1} T_{m-2} \cdots T_1 x_n, \quad n = 1, 2, \dots .$$

Then the sequence $\{x_n\}$ converges weakly to an element of $\cap_{i=1}^m F(T_i)$.

Proof. From Lemmas 4.1 and 4.2, we have that $T_m T_{m-1} \cdots T_1$ is a generalized nonexpansive mapping with $F(T_m T_{m-1} \cdots T_1) = \hat{F}(T_m T_{m-1} \cdots T_1) = \cap_{i=1}^m F(T_i)$. Therefore, by Theorem 2.6, $\{x_n\}$ converges weakly to an element of $\cap_{i=1}^m F(T_i)$. □

Finally, we obtain the following weak convergence theorem which is connected with the problem for image recovery.

Theorem 4.4. *Let E be a smooth and uniformly convex Banach space, let $D_1, D_2, \dots, D_m$ be nonempty weakly closed sunny generalized nonexpansive retracts of E such that $\cap_{i=1}^m D_i$ is nonempty, and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \le \alpha_n \le 1$ for each $n = 1, 2, \dots,$ and $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. Suppose $\{x_n\}$ is the sequence generated by $x_1 = x \in E$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) R_m R_{m-1} \cdots R_1 x_n, \quad n = 1, 2, \dots ,$$

where each R_i is a sunny generalized nonexpansive retraction of E onto D_i. Then the sequence $\{x_n\}$ converges weakly to an element of $\cap_{i=1}^m D_i$.

Proof. From Lemmas 2.4 and 2.5, we have that for each $i = 1, 2, \dots, m$, $\hat{F}(R_i) = F(R_i)$ and

$$V(x, R_i x) + V(R_i x, u) \le V(x, u), \quad \forall x \in E, \ \forall u \in D_i. \tag{4.4}$$

We recall that $F(R_i) = D_i$ for each $i = 1, 2, \dots, m$. Using Theorem 4.3, we have that $\{x_n\}$ converges weakly to an element of $\cap_{i=1}^m D_i$. □

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