A GENERALIZATION OF CONTRACTION PRINCIPLE IN A WEAK LEFT SMALL SELF DISTANCE SPACE

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Abstract. In this paper we generalize multivalued contraction theorem due to S. B. Nadler [8]. As corollaries we obtain fixed point theorems for a multivalued operator in complete dislocated metric spaces and complete partial metric space.

1 Introduction A generalization of contraction principle was introduced in [8]. Hitzler and Seda (2000) introduced the dislocated metrics space in [3,4] as a generalization of metrics where self-distances need not be zero. Also dislocated metrics were studied under the name of metric domains in context of domain theory in [1]. The notion of dislocated metric is useful in the context of electronic engineering (see [3]). In 1985 [6], S. G. Matthews introduced a generalization of Banach contraction principle. In 2000, Hitzler and Seda [4] introduced an alternative proof of this result in dislocated metric spaces. The plan of this paper is as follows. In Section 2, some preliminaries are presented. In Section 3, we define a weak left small self distance space and give a new topology on a distance space different than the distance topology due to P. Waszkiewicz [9,10]. In section 4, we generalize multivalued contraction theorem due to S. B. Nadler [8]. As corollaries of this result we obtain fixed point theorems for a multivalued operator in complete dislocated metric space and complete partial metric space.

2 Preliminaries In this section we give some preliminaries.

Definition 2.1 [9,10]. A distance on a set X is a map $d : X \times X \to [0, \infty)$. A pair (X, d) is called a distance space.
Let $\epsilon > 0$ and $x \in X$. Then $B_d(x, \epsilon) = \{y \in X | d(x, y) < d(x, x) + \epsilon\}$ and $N_x = \{A \subseteq X | \exists$ some $\epsilon > 0$ such that $B_d(x, \epsilon) \subseteq A\}$.The family $\tau_d = \{A \subseteq X | \forall x \in A, A \in N_x\}$ is a topology on X and is called the distance topology, the topology τ_d is called the distance topology. If $d(x, y)$ is replaced by $d(y, x)$ it is called the dual distance topology.
We need the following conditions for a distance operator:

(d_1) $\forall x \in X, d(x, x) = 0$,
(d_2) $\forall x, y \in X, d(x, y) = d(y, x) = 0 \Rightarrow x = y$,
(d_3) $\forall x, y \in X, d(x, y) = d(y, x)$,
(d_4) $\forall x, y, z \in X, d(x, y) \leq d(x, z) + d(z, y)$,
(d_5) $\forall\, x, y \in X.\ d(x, x) \leq \min\{d(x, y), d(y, x)\}$,
(d_6) $\forall x, y, z \in X, d(x, y) \leq d(x, z) + d(z, y) - d(z, z)$,

for all $x, y, z \in X$. If d satisfies conditions $(d_1) - (d_4)$, then (X, d) is called a metric space. If it satisfies conditions $(d_2) - (d_4)$, then (X, d) is called a dislocated metric space (d-metric space for short). If d satisfies conditions (d_2),(d_3),(d_5) and (d_6) then (X, d) is called a partial metric space [7].

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One can deduce that every partial metric space is a dislocated metric space.

Definition 2.2. Let (X, d) be a distance space. A multivalued operator f on $A \subseteq X$ is a operator
$f : A \to 2^A - \{\phi\}$.

Theorem 2.1 [3,6]. Let (X, d) be a complete d-metric space and let $f : X \to X$ be a Banach contraction operator. Then f has a unique fixed point.

Theorem 2.2 [8]. Let (X, d) be a complete metric space and $A \subseteq X$. Suppose that f is a multivalued operator on A, A is closed, $f(x)$ is closed for all $x \in A$, D is the Hausdorff metric, and Multivalued k-contraction condition

$$D(f(x), f(y)) \leq kd(x, y)$$

is satisfied for all $x, y \in A$ and fixed $k \in [0, 1)$. Then f has a fixed point, i.e. $\exists\, x \in A$ such that $x \in f(x)$.

Definition 2.3 [2,5]. Let (X, d) be a metric space. A mapping T from X to itself is said to be asymptotically regular if and only if
$\lim_{n\to\infty} d(T^n(x), T^{n+1}(x)) = 0,\ \forall\, x \in X$.

3 On distance spaces

Definition 3.1. Let (X, d) be a distance space and let $A \subseteq X$.
(1) The left $d-$closure of A is denoted and defined by
$d_l - cl(A) = \{x \in X|$ and $\exists$ a sequence (x_n) in X
s.t $\lim_{n\to\infty} d(x_n, A) = \lim_{n\to\infty} d(x_n, x) = 0\}, if\ A \neq \phi$, and $d_l - cl(\phi) = \phi$.
(2) A is called left$-d-$closed (d_l-closed for short) iff $d_l - cl(A) \subseteq A$,
(3) Let $a \in X$, $B \subseteq X$. Define $d(a, B) = inf_{b\in B} d(a, b)$ and $d(B, a) = inf_{b\in B} d(b, a)$,
(4) Let $A \subseteq X$, $B \subseteq X$. Define $d(A, B) = \inf\{d(a, b)|a \in A, b \in B\}$,
(5) The Hausdorff distance D on X is defined by

$$D(A, B) = \max\{sup_{a\in A} d(a, B), sup_{b\in B} d(A, b)\},$$

where $A \subseteq X$, $B \subseteq X$.

Note. In Theorem 2.2 [8] above the definition of Hausdorff metric coincides with the definition of Hausdorff distance iff d is a metric.

Definitions 3.2. A sequence (x_n) in a distance space (X, d) is said to be left convergent to an $x \in X$ (resp. asymptotically regular) iff $\lim_{n\to\infty} d(x_n, x) = 0$. (resp. $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$). If $\lim_{n\to\infty} d(x_n, x) = 0$, then x is called a left limit of (x_n).

Note. If (X, d) is a metric space and T is a mapping from X to itself, then T is asymptotically regular iff $\forall\, x \in X$, the sequence $T^n(x)$ is left asymptotically regular.

Note. It is clear that every left Cauchy sequence (see Definition 4.2 below) is left asymptotically regular. The converse is not true even in metric spaces (E.g. Let $X = R$ and $d(x, y) = |x - y|\ \forall\, x, y \in X$, then (x_n) in X defined by $x_n = \sum_{k=1}^{n} \frac{1}{k}$ is asymptotically regular but not Cauchy).

Definition 3.3. A distance space is called complete left asymptotically regular distance iff every left asymptotically regular sequence (x_n) in X, left converges to some point in X.

Theorem 3.1. Let (X, d) be a distance space. Then

$$\tau_{d_l} = \{A | A \in 2^X \quad and \quad A^c \quad is \quad d_l - closed\}$$

is a topology on X, where A^c is the complement of A.

Proof. (i) Since $d_l - cl(\phi) = \phi \subseteq \phi$, then ϕ is d_l-closed and so $X \in \tau_{d_l}$. Since $d_l - cl(X) \subseteq X$, then X is d_l-closed and so $\phi \in \tau_{d_l}$.

(ii) Let $A, B \in \tau_{d_l}$. $x \in d_l - cl(A^c \cup B^c) \Rightarrow \exists$ a sequence (x_n) such that $\lim_{n\to\infty} d(x_n, A^c \cup B^c) = \lim_{n\to\infty} d(x_n, x) = 0$, this implies that ether $\lim_{n\to\infty} d(x_n, A^c) = \lim_{n\to\infty} d(x_n, x) = 0$ or $\lim_{n\to\infty} d(x_n, B^c) = \lim_{n\to\infty} d(x_n, x) = 0 \Rightarrow x \in d_l - cl(A^c)$ or $x \in d_l - cl(B^c) \Rightarrow x \in A^c \cup B^c = (A \cap B)^c$.

(iii) Let $\{A_j | j \in J\} \subseteq \tau_{d_l}$. $x \in d_l - cl(\cap_{j\in J} A_j^c) \Rightarrow \exists$ a sequence (x_n) such that $\lim_{n\to\infty} d(x_n, \cap_{j\in J} A_j^c) = \lim_{n\to\infty} d(x_n, x) = 0 \Rightarrow \lim_{n\to\infty} d(x_n, A_j^c) = \lim_{n\to\infty} d(x_n, x) = 0 \; \forall j \in J \Rightarrow x \in d_l - cl(A_j^c) \; \forall j \in J \Rightarrow x \in \cap_{j\in J} A_j^c = (\cup_{j\in J} A_j)^c$

Definition 3.4 [9,10]. A distance space (X, d) is called a small self-distance iff d satisfies the condition (d_5) above.

Definition 3.5. A distance space (X, d) is called a weak left small self-distance iff d satisfies the condition

$d(x, y) = 0 \Rightarrow d(x, x) = 0, \forall\, x, y \in X.$

Note. It is clear that any small self distance space is weak left small self-distance.
The following counterexamples illustrate that τ_{d_l} and the distance topology τ_d due to P. Wasztiewiecz are different concepts even in a weak left small self distance space.

Counterexample 1. Let $X = \{a, b, c\}$ and $d : X \times X \to [0, \infty)$ is defined by: $d(x, y) = 1 \; \forall\, x, y \in X$. It is clear that (X, d) is weak left small self distance space. since $\{a\} \notin \tau_d$ and $\tau_{d_l} = 2^X$, then $\tau_{d_l} \nsubseteq \tau_d$.

Counterexample 2. Let $X = \{a, b, c\}$ and $d : X \times X \to [0, \infty)$ is defined as follows:
$d(b, a) = d(b, b) = 0$, $d(a, a) = \frac{1}{2}$ and $d(x, y) = 2 \; \forall\, (x, y) \in X \times X - \{(b, a), (b, b), (a, a)\}$. Then (X, d) is weak left small self distance space. Since $\{a\} \in \tau_d$ and $\{a\} \notin \tau_{d_l}$, then $\tau_d \nsubseteq \tau_{d_l}$.

4 Main results The following counterexample illustrates that there exists a sequence in a weak left small self distance satisfies (d_4) which is left convergent but not left asymptotically regular.

Counterexample 3. Let $X = \{a, b, c\}$ and d is a distance operator defined by
$d(a, b) = d(b, a) = d(c, b) = d(c, a) = d(c, c) = 1$, $d(b, c) = d(a, c) = d(b, b) = d(a, a) = 0$.
The desired sequence is (x_n) given by:

$$x_n = \begin{cases} a & \text{if } n \text{ odd} \\ b & \text{if } n \text{ even} \end{cases}$$

Definition 4.1. Let (X, d) be a distance space and $A \subseteq X$. A is called left-asymptotically regular-closed iff if (x_n) is a left asymptotically regular sequence left convergent to $x \in X$, then $x \in A$.

Theorem 4.1. Let (X, d) be a complete left asymptotically regular weak left small self distance space and $A \subseteq X$. If f is a multivalued operator on A, A is left-asymptotically regular-closed, $f(x)$ is left-d-closed $\forall\, x \in A$ and f satisfies the following condition: $D(f(x), f(y)) \leq kd(x, y)\ \forall x, y \in A$ for fixed $k \in (0, 1)$, then f has a fixed point.

Proof. (I) Choose a $k_0 \in (k, 1)$ and an $x_0 \in A$. If there is not an element $x_1 \in f(x_0)$ s.t. $x_1 \neq x_0$, then since $f(x_0) \neq \phi$ we have $x_0 \in f(x_0)$. Thus x_0 is a fixed point of f. If there exists $x \in f(x_0)$ and $x \neq x_0$ s.t. $d(x_0, x) = 0$, then $d(x_0, f(x_0)) = 0$. Consider the sequence $(x_n)_{n \in N}$ where $x_n = x_0\ \forall\ n \in N$. Then $\lim_{n\to\infty} d(x_n, f(x_0)) = \lim_{n\to\infty} d(x_n, x_0) = 0$, because (X, d) is weak left small self distance space, i.e. since $d(x_0, x) = 0$ we have $d(x_0, x_0) = 0$. Thus $x_0 \in d_l - cl(f(x_0)) \subseteq f(x_0)$. Hence x_0 is a fixed point. For the otherwise case choose $x_1 \in f(x_0)$ s.t. $x_1 \neq x_0$ and $d(x_0, x_1) > 0$. Now, $d(x_1, f(x_1)) \leq D(f(x_0), f(x_1)) < k_0 d(x_0, x_1)$. Then there exists $x_2 \in f(x_1)$ s.t. $d(x_1, x_2) < k_0 d(x_0, x_1)$. By repeating this process one can construct a sequence (x_n) s.t. $x_{n+1} \in f(x_n)$, $d(x_n, x_{n+1}) < k_0^n d(x_0, x_1)$ for all $n \in N$.Thus $lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ and so (x_n) is a left asymptotically regular sequence. Since (X, d) is complete left asymptotically regular distance space, then exists $x \in X$ s.t. $\lim_{n\to\infty} d(x_n, x) = 0$. Since A is left-asymptotically regular-closed, then $x \in A$.
(II) Since $d(x_{n+1}, f(x)) \leq D(f(x_n), f(x)) \leq kd(x_n, x)\ \forall\ n \in N$, then
$\lim_{n\to\infty} d(x_{n+1}, f(x)) \leq k \lim_{n\to\infty} d(x_n, x) = 0$,
i.e. $\lim_{n\to\infty} d(x_{n+1}, f(x)) = 0$. Now, $\lim_{n\to\infty} d(x_{n+1}, f(x)) = \lim_{n\to\infty} d(x_{n+1}, x) = 0 \Rightarrow x \in d_l - cl(f(x)) \subseteq f(x)$ then $x \in f(x)$.

Proposition 4.1. Let (X, d) be a distance space and $A \subseteq X$. Let f be a multivalued operator from A to A, s.t.
(1) $D(f(x), f(y)) = 0\ \ \forall\, x, y \in A$,
(2) $\exists\, x \in A$ s.t.
  (a) $x \in f(\ell)$ for some $\ell \in A$,
  (b) $d(x, f(x)) = 0 \Rightarrow x \in f(x)$.
Then f has a fixed point.

Definition 4.2. Let (X, d) be a distance space. A sequence (x_n) in X is called left Cauchy iff $\lim_{n\to\infty} d(x_n, x_m) = 0\ \forall\ n, m \in N$ s.t. $m > n$.

Definition 4.3. A distance space (X, d) is called left complete iff every left Cauchy sequence in X, left converges.

Definition 4.4. Let (X, d) be a distance space and $A \subseteq X$. A is called left-Cauchy-closed iff if (x_n) is a left Cauchy sequence left convergent to $x \in X$, then $x \in A$.

One can prove that the sequence (x_n) defined in (I) in the proof of Theorem 4.1 is left Cauchy if the triangular inequality hold. So one can have the following theorem.

Theorem 4.2. Let (X, d) be a left complete left small self distance space s.t. d satisfies (d_4) and let $A \subseteq X$. If f is a multivalued operator from A to A, A is left-Cauchy-closed, $f(x)$

is left-d-closed $\forall\ x \in A$ and f satisfies the following condition: $D(f(x), f(y)) \leq kd(x, y)\ \forall\ x, y \in A$ for fixed $k \in (0, 1)$, then f has a fixed point.

Notes. (1) If d satisfies (d_3) the term "left" in each concept is omitted.
(2) Let (X, d) be a distance space satisfies (d_3). If A is closed; i.e. if every convergent sequence in A converges in A; then A is left-asymptotically regular-closed.
(3) If (X, d) satisfies (d_3) and (d_2), then (X, d) is weak small self distance.

From Theorem 4.2 and the above notes one can have the following corollaries.

Corollary 4.1. Let (X, d) be a complete d-metric space and $A \subset X$. If f is a multivalued operator from A to A, s.t. $\forall x \in A$, A is closed, $f(x)$ is d-closed and f satisfies the condition mentioned in Theorem 4.2, then f has a fixed point.

Corollary 4.2. If (X, d) is a complete partial metric space and $A \subseteq X$. If f is a multivalued operator from A to A, s.t. $\forall\ x \in A$, A is closed, $f(x)$ is d-closed and f satisfies the condition mentioned in Theorem 4.1, then f has a fixed point.

The following corollary illustrates that Theorem 4.2 and Proposition 4.1 generalize the generalized k-contraction fixed point Theorem due to S. B. Nadler.

Corollary 4.3[8]. Let (X, d) be a complete metric space and $A \subseteq X$. Suppose that (i) $f : A \to 2^A - \{\phi\}$ is a multivalued map such that A is closed, $f(x)$ is closed for all $x \in A$, and Multivalued k-contraction condition

$$D(f(x), f(y)) \leq kd(x, y)$$

is satisfied for all $x, y \in A$ and fixed $k \in [0, 1)$. Then f has a fixed point.
Proof. The proof follows from the above notes and the following fact:
In metric spaces, the topology τ_{d_l} coincide with the usual metric topology.

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