DUAL BCK-ALGEBRA AND MV-ALGEBRA

Kyung Ho Kim and Yong Ho Yon

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Abstract. The aim of this paper is to study the properties of dual BCK-algebra and to prove that the MV-algebra is equvalent to the bounded commutative dual BCK-algebra.

1 Introduction Let $(L, \leq)$ be a *poset*. The greatest element of L is called the *top* element of L if it exists. Similarly, the least element of L is called the *bottom* element of L if it exists. We denote the top and the bottom element as 1 and 0, respectively. Let S be a subset of L and $u \in L$. u is called an *upper bound* of S if $s \leq u$ for all $s \in S$, and u is called the *join* of S if u is the least upper bound of S. Dually, we defnine a *lower bound* of S and the *meet* of S. A poset L is an *upper semilattice* if $\sup\{x, y\}$ exists for all $x, y \in L$ and a poset L is a *lower semilattice* if $\inf\{x, y\}$ exists for all $x, y \in L$ and a poset L is a *lattice* if it is an upper and lower semilattice. A lattice L is said to be *bounded* if there exists a bottom 0 and a top 1 in L([5, 3]).

BCK-algebras were introduced in 1966 by Iséki [4]. It is an algebraic formulation of the BCK-propositional calculus system of C. A. Meredith [7], and generalize the notion of implicative algebras. The notion of MV-algebra, originally introduced by C.C. Chang [2], is an attempt at developing a theory of algebraic systems that would correspond to the $\aleph_0$-valued propositional calculus; the axioms for this calculus are known as the Łukasiewicz axioms.

The purpose of this note is to study the relation between the MV-algebra and the dual concept of BCK-algebra. we will introduce some properties of dual BCK-algebras and MV-algebras, and prove that the MV-algebra is equvalent to the bounded commutative dual BCK-algebra.

2 Preliminaries In this section, we introduce the defninitions and some properties of a BCK-algebra and a MV-algebra.

Definition 2.1. [6] An algebra $(X, *, 0)$ of type $(2, 0)$ is called a *BCK-algebra* if it satisfies:

(1) $((x * y) * (x * z)) * (z * y)=0$,

(2) $(x * (x * y)) * y = 0$,

(3) $x * x = 0$,

(4) $x * y = 0$ and $y * x = 0$ imply $x = y$,

(5) $0 * x = 0$,

for all $x, y, z \in X$.

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From the above defninition, we have the defninition of dual BCK-algebra as following defninition.

Definition 2.2. A dual BCK-algebra is an algebra $(X, \circ, 1)$ of type $(2, 0)$ satisfying:

(1) $(x \circ y) \circ ((y \circ z) \circ (x \circ z)) = 1$.

(2) $x \circ ((x \circ y) \circ y) = 1$.

(3) $x \circ x = 1$.

(4) $x \circ y = 1$ and $y \circ x = 1$ imply $x = y$,

(5) $x \circ 1 = 1$.

for all $x, y, z \in X$.

Let $(X, \circ, 1)$ be a dual BCK-algebra. Then we can defnine a binary relation "$\leq$" on X as the following :

$$x \leq y \quad \text{if and only if} \quad x \circ y = 1$$

for $x, y \in X$, and has the following Lemma form the defninition of dual BCK-algebra.

Lemma 2.3. Let $(X, \circ, 1)$ be a dual BCK-algebra. Then for all $x, y, z \in X$,

(1) $x \circ y \leq (y \circ z) \circ (x \circ z)$,

(2) $x \leq (x \circ y) \circ y$,

(3) $x \leq x$,

(4) $x \leq y$ and $y \leq x$ imply $x = y$,

(5) $x \leq 1$.

Theorem 2.4. Let $(X, \circ, 1)$ be a dual BCK-algebra. Then for any $x, y, z \in X$ the following hold

(1) $x \leq y$ implies $y \circ z \leq x \circ z$,

(2) $x \leq y$ and $y \leq z$ imply $x \leq z$.

Proof. (1) Let $x \leq y$. Then $1 = x \circ y \leq (y \circ z) \circ (x \circ z)$ by 2.3 (1). Hence $(y \circ z) \circ (x \circ z) = 1$ by 2.3 (5), and $y \circ z \leq x \circ z$.

(2) If $x \leq y$ and $y \leq z$, then $1 = y \circ z \leq x \circ z$ by this lemma (1), hence $x \circ z = 1$ and $x \leq z$. □

A dual BCK-algebra $(X, \circ, 1)$ is a poset with the ordering $\leq$ from 2.3 and 2.4, and the element 1 in X is the top element with respect to the order relation $\leq$.

Theorem 2.5. Let $(X, \circ, 1)$ is a dual BCK-algebra and $x, y, z \in X$. Then

(1) $x \circ (y \circ z) = y \circ (x \circ z)$,

(2) $x \leq y \circ z$ imply $y \leq x \circ z$,

(3) $x \circ y \leq (z \circ x) \circ (z \circ y)$,

(4) $x \leq y$ imply $z \circ x \leq z \circ y$,

(5) $y \le x \circ y$,

(6) $1 \circ x = x$.

Proof. (1) ($\le$) Since $y \le (y \circ z) \circ z$ by 2.3(2), $((y \circ z) \circ z) \circ (x \circ z) \le y \circ (x \circ z)$ by 2.4(1). And $x \circ (y \circ z) \le ((y \circ z) \circ z) \circ (x \circ z)$ by 2.3 (1), hence $x \circ (y \circ z) \le y \circ (x \circ z)$. ($\ge$) Interchanging the role of x and y in ($\le$), we get $y \circ (x \circ z) \le x \circ (y \circ z)$.

(2) Let $x \le y \circ z$. Then $1 = x \circ (y \circ z) = y \circ (x \circ z)$, hence $y \le x \circ z$.

(3) Since $z \circ x \le (x \circ y) \circ (z \circ y)$ by 2.3 (1), $x \circ y \le (z \circ x) \circ (z \circ y)$ by the above (2).

(4) Let $x \le y$. Then $1 = x \circ y \le (z \circ x) \circ (z \circ y)$ by the above (3), hence $(z \circ x) \circ (z \circ y) = 1$. It follows that $z \circ x \le z \circ y$.

(5) From the above (1), $y \circ (x \circ y) = x \circ (y \circ y) = x \circ 1 = 1$, hence $y \le x \circ y$.

(6) Since $1 \le (1 \circ x) \circ x$ by 2.3 (2), $(1 \circ x) \circ x = 1$, hence $1 \circ x \le x$. Conversely, $x \circ (1 \circ x) = 1 \circ (x \circ x) = 1 \circ 1 = 1$, hence $x \le 1 \circ x$. □

Let $(X, \circ, 1)$ be a dual BCK-algebra. We defnine a binary operation "+" on X as the following : for any $x, y \in X$

$$x + y = (x \circ y) \circ y.$$

Remark 2.6. 1) Let $(X, \circ, 1)$ be a dual BCK-algebra and $x, y \in X$. Then $x + y$ is an upper bound of x and y, since $x \le (x \circ y) \circ y$ and $y \le (x \circ y) \circ y$ by 2.3(2) and 2.5(5).

2) Let $(X, \circ, 1)$ be a dual BCK-algebra and $x \in X$. Then $x + 1 = (x \circ 1) \circ 1 = 1 \circ 1 = 1$, $1 + x = (1 \circ x) \circ x = x \circ x = 1$, and $x + x = (x \circ x) \circ x = 1 \circ x = x$, by 2.2 and 2.5(6).

Lemma 2.7. If $(X, \circ, 1)$ is a dual BCK-algebra, then $(x + y) \circ y = x \circ y$ for all $x, y \in X$.

Proof. ($\le$) Since $x \le x + y$, $(x + y) \circ y \le x \circ y$ by 2.4 (1). ($\ge$) From 2.3 (2), $x \circ y \le ((x \circ y) \circ y) \circ y = (x + y) \circ y$. □

3 Bounded Commutative Dual BCK-algebras and MV-algebras

Definition 3.1. A dual BCK-algebra $(X, \circ, 1)$ is said to be *bounded* if there exists an element 0 in X such that $0 \circ x = 1$ for all $x \in X$.

The element 0 is the bottom element in X, since $0 \le x$ for all $x \in X$ from the defninition of the order $\le$.

Definition 3.2. Let $(X, \circ, 1, 0)$ be a bounded dual BCK-algebra and $x \in X$. $x \circ 0$ is called a *pseudocomplement* of x and we write $x^* = x \circ 0$ and $x^{**} = (x^*)^*$.

Theorem 3.3. Let $(X, \circ, 1, 0)$ be a bounded dual BCK-algebra and $x, y \in X$. Then

(1) $1^* = 0$ and $0^* = 1$,

(2) $x \le x^{**}$,

(3) $x \circ y \ \le \ y^* \circ x^*$,

(4) $x \le y$ implies $y^* \le x^*$,

(5) $x \circ y^* = y \circ x^*$,

(6) $x^{***} = x^*$.

Proof. (1) $1^* = 1 \circ 0 = 0$ by 2.5(6), and $0^* = 0 \circ 0 = 1$ by 2.2(3).
(2) $x \le (x \circ 0) \circ 0 = x^{**}$ by 2.3(2).
(3) $x \circ y \le (y \circ 0) \circ (x \circ 0) = y^* \circ x^*$ by 2.3(1).
(4) $x \le y$ implies $y^* = y \circ 0 \le x \circ 0 = x^*$ by 2.4 (1).
(5) $x \circ y^* = x \circ (y \circ 0) = y \circ (x \circ 0) = y \circ x^*$ by 2.5(1).
(6) $x^{***} = ((x \circ 0) \circ 0) \circ 0 = (x + 0) \circ 0 = x \circ 0 = x^*$ by 2.7. □

Definition 3.4. An element x in a bounded dual BCK-algebra X is said to be *regular* if $x^{**} = x$.

Definition 3.5. A dual BCK-algebra $(X, \circ, 1)$ is said to be *commutative* if $x + y = y + x$ for all $x, y \in X$.

Theorem 3.6. Let $(X, \circ, 1)$ be a dual BCK-algebra. Then the following conditions are equivalent :

(1) $x + y \le y + x$ for all $x, y \in X$,

(2) X is commutative,

(3) $y \le x$ implies $x = (x \circ y) \circ y$ for $x, y \in X$.

Proof. (1) $\Rightarrow$ (2). Interchanging the role of x and y, it is trivial.
(2) $\Rightarrow$ (3). Suppose X is commutative and $y \le x$ for $x, y \in X$, then $x = 1 \circ x = (y \circ x) \circ x = y + x = x + y = (x \circ y) \circ y$.
(3) $\Rightarrow$ (1). Suppose that $y \le x$ implies $x = (x \circ y) \circ y$ for $x, y \in X$. Since $y \le y + x$, $y + x = ((y + x) \circ y) \circ y$, hence $(x + y) \circ (y + x) = (x + y) \circ (((y + x) \circ y) \circ y) = ((y + x) \circ y) \circ ((x + y) \circ y) = ((y + x) \circ y) \circ (x \circ y)$ by 2.5(1) and 2.7, and since $x \le y + x$, $1 = x \circ (y + x) \le ((y + x) \circ y) \circ (x \circ y)$ by 2.3 (1). Hence $(x + y) \circ (y + x) = 1$ and $x + y \le y + x$. □

Theorem 3.7. A commutative dual BCK-algebra is an upper semilattice.

Proof. Let $(X, \circ, 1)$ be a commutative dual BCK-algebra and $x, y \in X$. Then $x + y$ is an upper bound of x and y by 2.6(1). We shall show that the $x + y$ is the least upper bound of x and y. Suppose z is an upper bound of x and y. Then $x \circ z = y \circ z = 1$. (i): $z = 1 \circ z = (x \circ z) \circ z = (z \circ x) \circ x$ by the commutativity, and (ii): $z = 1 \circ z = (y \circ z) \circ z = (z \circ y) \circ y$. (iii): $z = (z \circ x) \circ x = (((z \circ y) \circ y) \circ x) \circ x$ from (i) and (ii).
Set $u = (z \circ y) \circ y$, then $z = (u \circ x) \circ x$ from (iii). Since $y \le u$ by 2.5(5), $u \circ x \le y \circ x$ by 2.4(1), and then $(y \circ x) \circ x \le (u \circ x) \circ x = z$. Hence $x + y = (x \circ y) \circ y = (y \circ x) \circ x \le z$. Therefore, $x + y$ is the least upper bound of x and y and X is an upper semilattice with the join, say $x + y$, of any two elements x and y. □

We defnine a binary operation "$\cdot$" on a bounded dual BCK-algebra X by the following: $x \cdot y = (x^* + y^*)^*$ for each $x, y \in X$.

Theorem 3.8. Every element in a bounded commutative dual BCK-algebra X is a regular.

Proof. Since $0 \le x$, $x = (x \circ 0) \circ 0 = x^{**}$ for all $x \in X$ by 3.6(3). □

Theorem 3.9. Let $(X, \circ, 1, 0)$ be a bounded commutative dual BCK-algebra and $x, y \in X$. Then

(1) $x^* \cdot y^* = (x + y)^*$,

(2) $x^* + y^* = (x \cdot y)^*$,

(3) $y^* \circ x^* = x \circ y$.

Proof. (1) $x^* \cdot y^* = (x^{**} + y^{**})^* = (x + y)^*$ by the defninition of the binary operation $\cdot$ and 3.8.
(2) $x^* + y^* = (x^* + y^*)^{**} = (x^{**} \cdot y^{**})^* = (x \cdot y)^*$ by the above (1).
(3) $y^* \circ x^* = x \circ y^{**} = x \circ y$ from 3.3(5). □

Theorem 3.10. A bounded commutative dual BCK-algebra is a lower semilattice.

Proof. Let $(X, \circ, 1, 0)$ be a bounded commutative dual BCK-algebra and $x, y \in X$. Since $x^* + y^*$ is an upper bound of x^* and y^*, $(x^* + y^*)^* \leq x^{**} = x$ and $(x^* + y^*)^* \leq y^{**} = y$ by 3.3(4). Hence $x \cdot y = (x^* + y^*)^*$ is a lower bound of x and y.

Next, we shall show that $x \cdot y = (x^* + y^*)^*$ is the greatest lower bound of x and y. Suppose that z is a lower bound of x and y. Then $x^* \leq z^*$ and $y^* \leq z^*$ by 3.3(4), hence z^* is an upper bound of x^* and y^*. Since $x^* + y^*$ is the least upper bound of x^* and y^* by the proof of 3.7, $x^* + y^* \leq z^*$, and then $z = (z^*)^* \leq (x^* + y^*)^* = x \cdot y$. Hence $x \cdot y$ is the greatest lower bound of x and y and X is a lower semilattice. □

Theorem 3.11. A bounded commutative dual BCK-algebra is a bounded lattice.

Proof. From 3.7 and 3.10, it follows immediately. □

We note that $x + y$ and $x \cdot y$ are the join and the meet, respectively, of any elements x and y in a bounded commutative dual BCK-algebra X.

Definition 3.12. [1] An MV-algebra is an algebra $(A, \oplus, ', 0)$ of type $(2, 1, 0)$ satisfying the following equations:

(1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,

(2) $x \oplus y = y \oplus x$,

(3) $x \oplus 0 = x$,

(4) $x'' = x$,

(5) $x \oplus 0' = 0'$,

(6) $(x' \oplus y)' \oplus y = (y' \oplus x)' \oplus x$.

On a MV-algebra A, we defnine the constant 1 and the operations "$\odot$" and "$\ominus$" as follows: $1 = 0'$, $x \odot y = (x' \oplus y')'$ and $x \ominus y = x \cdot y' = (x' \oplus y)'$.

Lemma 3.13. [1] For $x, y \in A$, the following conditions are equvalent:

(1) $x' \oplus y = 1$,

(2) $x \odot y' = 0$,

(3) $y = x \oplus (y \ominus x)$,

(4) there is an element $z \in A$ such that $x \oplus z = y$.

We defnine the binary relation "$\leq$" on a MV-algebra A as follows: $x \leq y$ if and only if x and y satisfy one of the equvalent axioms 1) - 4) in the above lemma. The relation $\leq$ is a partial ordered relation on A.

Theorem 3.14. [1] Let A be a MV-algebra and $x, y, z \in A$. Then

(1) $1' = 0$,

(2) $x \oplus y = (x' \odot y')'$,

(3) $x \oplus 1 = 1$,

(4) $(x \ominus y) \oplus y = (y \ominus x) \oplus x$,

(5) $x \oplus x' = 1$,

(6) $x \ominus 0 = x, 0 \ominus x = 0, x \ominus x = 0, 1 \ominus x = x', x \ominus 1 = 0$,

(7) $x \oplus x = x$ iff $x \odot x = x$,

(8) $x \leq y$ iff $y' \leq x'$,

(9) if $x \leq y$, then $x \oplus z \leq y \oplus z$ and $x \odot z \leq y \odot z$,

(10) if $x \leq y$, then $x \ominus z \leq y \ominus z$ and $z \ominus y \leq z \ominus x$,

(11) $x \ominus y \leq x$, $x \ominus y \leq y'$,

(12) $(x \oplus y) \ominus x \leq y$,

(13) $x \odot z \leq y$ iff $z \leq x' \oplus y$,

(14) $x \oplus y \oplus x \odot y = x \oplus y$.

Theorem 3.15. A bounded commutative dual BCK-algebra $(X, \circ, 1, 0)$ is a MV-algebra $(X, \oplus, ', 0)$ with the operations "$\oplus$" and "$'$" defnined as following:

$$x \oplus y = x^* \circ y \quad \text{and} \quad x' = x^*$$

for all $x, y \in X$.

Proof. For any $x, y, z \in X$, we have $x \oplus (y \oplus z) = x^* \circ (y^* \circ z) = x^* \circ (z^* \circ y) = z^* \circ (x^* \circ y) = (x^* \circ y)^* \circ z = (x \oplus y) \oplus z$, $x \oplus y = x^* \circ y = y^* \circ x = y \oplus x$, $x \oplus 0 = x^* \circ 0 = x^{**} = x$, $x'' = x^{**} = x$ and $x \oplus 0' = x^* \circ 0^* = x^* \circ 1 = 1 = 0^* = 0'$. Thus we get the properties (1), (2), (3), (4) and (5) of the defninition of MV-algebra. Next we will prove the property (6) of the MV-algebra. $(x' \oplus y)' \oplus y = (x \circ y)^* \oplus y = (x \circ y)^{**} \circ y = (y \circ x) \circ x = (y \circ x)^* \oplus x = (X^* \circ y^*)^* \oplus x = (y' \oplus x)' \oplus x$. □

Theorem 3.16. A MV-algebra $(X, \oplus, ', 0)$ is a bounded commutative dual BCK-algebra with the operation "$\circ$" and the top element 1 defnined as following:

$$x \circ y = x' \oplus y \quad \text{and} \quad 1 = 0'$$

for all $x, y \in X$.

Proof. 1) $x \circ 1 = x' \oplus 1 = 1$. 2) $x \circ x = x' \oplus x = 1$. 3) $x \circ ((x \circ y) \circ y) = x' \oplus ((x' \oplus y)' \oplus y) = (x' \oplus y)' \oplus (x' \oplus y) = 1$. 4) If $x \circ y = 1$ and $y \circ x = 1$, then $x' \oplus y = 1$ and $y' \oplus x = 1$. From the defninition of the order $\leq$ in MV-algebra, $x \leq y$ and $y \leq x$, hence $x = y$.

5) Let $x, y, z \in X$. Then we have

$$
\begin{aligned}
&(x \circ y) \circ ((y \circ z) \circ (x \circ z)) \\
&\quad = (x \circ y)' \oplus ((y \circ z)' \oplus (x \circ z)) \\
&\quad = (x \circ y)' \oplus ((y' \oplus z)' \oplus (x' \oplus z)) \\
&\quad = (x \circ y)' \oplus (((y' \oplus z)' \oplus z) \oplus x') \\
&\quad = (x \circ y)' \oplus (((z' \oplus y)' \oplus y) \oplus x') \quad \text{(by 3.12(6))} \\
&\quad = (x \circ y)' \oplus ((z' \oplus y)' \oplus (x' \oplus y)) \\
&\quad = (x \circ y)' \oplus ((z' \oplus y)' \oplus (x \circ y)) \\
&\quad = ((x \circ y)' \oplus (x \circ y)) \oplus (z' \oplus y)' \\
&\quad = 1 \oplus (z' \oplus y)' \\
&\quad = 1.
\end{aligned}
$$

□

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* Department of Mathematics,
Chungju National University,
Chungju 380-702, Korea
e-mail : ghkim@cjnu.ac.kr

**Department of Mathematics,
Chungbuk National University,
Cheongju 361-763, Korea
E-mail address : yhyonkr@hanmail.net